Associative functions and abstract semigroups

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Introduction

In a previous paper [7] we studied the question of characterizing a class of associative functions on the unit square which had arisen in our work on statistical metric spaces ([4], [5]) and which we called *t-norms* (see Definition 1 below). By making use of some beautiful results of J. Aczél (see [1], p. 176—189), we showed that a large and important subclass of t-norms, the *strict* t-norms (see Definition 2 below), could be characterized completely in terms of ordinary real functions — i. e., functions of a single real variable. The converses to these characterization theorems then enabled us to construct strict t-norms at will; in particular, we showed how any strict t-norm could be obtained from any other strict t-norm by means of a simple, well-determined transformation (see (1. 4)). We further pointed out that these latter results were closely related to certain abstract theorems of A. C. Climescu [3] on transformations of semigroups into semigroups; that these results could be extended to yield characterizations for a much wider class of t-norms than the strict t-norms; and that this extension in turn led to various abstract generalizations of Climescu's results which are of considerable interest in their own right.

This paper is devoted to the detailed discussion of the above mentioned extensions and generalizations. It is divided into two parts — the first concrete, the second abstract.

The first part begins with a theorem (Theorem 1 below) which was motivated by a result of Climescu ([3], Theorem II) and is simultaneously a generalization of Theorem 6 of [7] and of a concrete version of Climescu's Theorem II. This theorem enables us to extend various results of [7] to arbitrary, i. e., not necessarily strict, t-norms and to thereby greatly enlarge the class of t-norms which can be characterized in terms of monotone real functions. It should be noted, however, that we still cannot characterize all t-norms in this way; indeed, it seems unlikely to us that any such universal characterization exists.

The second part of this paper is devoted to abstract semigroups. We begin with two theorems (Theorems 4 and 5) which are successive generalisations of Climescu's Theorem II. The first is the abstract analogue of Theorem 1; the second, which is more recondite, requires the concept of a *right-subinverse* of a mapping which we introduced in a different connection in [6]. Both of these theorems provide methods for generating semigroups from other semigroups. These methods are related to, but quite different in effect from, standard homomorphism theory. In particular, their application can yield a semigroup which is „larger", rather than „smaller", than the original semigroup. The connection with standard homomorphism theory is brought out in Theorem 7.

The paper concludes with an appendix in which we give another method of constructing t-norms from t-norms. This method is not directly related to the methods considered earlier in the paper; but it too is motivated by, and connected with a theorem of CLIMESCU ([3], Theorem V). It is included in this paper because it leads to a class of t-norms which seem certain to play an important role in the further study of statistical metric spaces.

We conclude this introduction with some definitions and notations which will be used in the body of the paper.

Definition 1. A *triangular norm* (briefly, a *t-norm*) is a 2-place function from the closed unit square $[0, 1]\times[0, 1]$ to the closed unit interval $[0, 1]$ which satisfies the following conditions:

(0. 1) $T(0, 0)=0,\ T(a, 1)=T(1, a)=a.$ *(Boundary Conditions)*

(0. 2) $T(a, b)\leqq T(c, d)$ *whenever* $a\leqq c,\ b\leqq d.$ *(Monotonicity)*

(0. 3) $T(a, b)=T(b, a)$ *(Symmetry)*

(0. 4) $T(T(a, b), c)=T(a, T(b, c)).$ *(Associativity)*

Definition 2. A *strict* t-norm is a t-norm which is continuous and strictly increasing in both places, i. e., satisfies the conditions:

(0. 5) $$T(a, b)=\lim_{c\to a} T(c, b)=\lim_{d\to b} T(a, d).$$

(0. 6) $$T(a, b)<T(c, b) \quad for \quad 0\leqq a<c\leqq 1,\ b>0.$$
$$T(a, b)<T(a, d) \quad for \quad 0\leqq b<d\leqq 1,\ a>0.$$

Of particular importance are the t-norms T_W, T_m, Prod, and Min, defined respectively as follows:

$$T_W(a, b)=\begin{cases} a, & b=1, \\ b, & a=1, \\ 0, & otherwise; \end{cases}$$

$$T_m(a, b) = \max(a+b-1, 0);$$

$$\mathrm{Prod}(a, b)=a\cdot b;$$

$$\mathrm{Min}(a, b) = \begin{cases} a, & a\leqq b, \\ b, & b\leqq a. \end{cases}$$

Of these, only Prod is a strict t-norm. Moreover, as shown in [4], every t-norm T satisfies the inequality

(0. 7) $$T_W(a, b)\leqq T(a, b)\leqq \mathrm{Min}(a, b).$$

Definition 3. A (2-dimensional) *copula* is a continuous 2-place function T from $[0, 1]\times[0, 1]$ to $[0, 1]$ satisfying (0. 1), (0. 2) and the following condition:

$$T(a, d)+T(c, b)\leqq T(a, b)+T(c, d), \text{ whenever } a\leqq c, b\leqq d. \tag{0. 8}$$

Every copula T satisfies the inequality

$$T_m(a, b)\leqq T(a, b)\leqq \mathrm{Min}\,(a, b).$$

Thus Min, Prod and T_m are copulas, whereas T_W is not.

For any function f, we shall denote the domain of f by Dom f and the range of f by Ran f. If f and g are one-place functions, we shall often denote the composite of f and g by fg. Similarly, we shall sometimes find it convenient to denote the value of f at x, i. e., $f(x)$, simply by fx. The letter j will denote the identity function on the reals, i. e., the function defined by: $j(x)=x$ for any real number x. Correspondingly, for any set A (whose elements may or may not be real numbers) j_A is the identity function on A, i. e., $j_A(x)=x$ for any element x in A.

1. Construction of t-norms from t-norms

In our previous paper on t-norms [7] it was shown that if h is a one-place function which is defined, continuous and strictly increasing on the closed unit interval $[0, 1]$ with $h(0)=0$ and $h(1)=1$, and if h^{-1} is the inverse of h, then the 2-place function T given by

$$T(a, b)=h^{-1}(h(a)\cdot h(b))=h^{-1}(\mathrm{Prod}\,(h(a), h(b))) \tag{1. 0}$$

is a strict t-norm. The function h was called a *multiplicative generator* of the t-norm T. It was further shown that *a 2-place function T is a strict t-norm if and only if it is derivable from the strict t-norm* Prod *via* (1. 0) *through the intermediary of a multiplicative generator h.*

Our first aim in this section is to extend the „if" part of the preceding italicized statement to arbitrary t-norms and to a more general class of functions h. We begin with two lemmas.

Lemma 1. *Let T be a t-norm, and let I_0 be the closed interval $[0, a_0]$, where $0\leqq a_0\leqq 1$. If one of the two numbers a, b is in I_0, and the other is in $[0, 1]$, then $T(a, b)$ is in I_0.*

Proof. In view of (0. 7), $T(a, b)\leqq a_0\in I_0$.

Lemma 2. *Let a_0 be a number such that $0\leqq a_0<1$, I_0 the interval $[0, a_0]$ and I_1 the interval $[a_0, 1]$. Let h be a continuous, strictly increasing function from $[0, 1]$ onto I_1 (whence $h(0)=a_0$ and $h(1)=1$) and let h^* be the function defined by*

$$h^*(x)=\begin{cases}0, & x\in I_0,\\ h^{-1}(x), & x\in I_1,\end{cases} \tag{1. 1}$$

where h^{-1} is the inverse of h. Then

$$h^*(h(x))=x, \text{ for every } x\in[0, 1], \tag{1. 2}$$

and

$$h(h^*(x)) = \begin{cases} a_0, & x \in I_0, \\ x, & x \in I_1. \end{cases} \tag{1. 3}$$

Proof. For every $x \in [0, 1]$ we have $h(x) \in I_1$. Hence

$$h^*(h(x)) = h^{-1}(h(x)) = x.$$

If $x \in I_0$, then $h^*(x) = 0$ and $h(h^*(x)) = h(0) = a_0$; if $x \in I_1$, then $h(h^*(x)) = h(h^{-1}(x)) = x$.

Theorem 1. *If S is a t-norm and I_0, I_1, h and h^* are defined as in Lemma 2, then the 2-place function T defined on $[0, 1] \times [0, 1]$ by*

$$T(a, b) = h^*(S(h(a), h(b))) \tag{1. 4}$$

is a t-norm.

Proof. The symmetry (0. 3) of T is an immediate consequence of that of S. Similarly, the monotonicity (0. 2) of T follows directly from the monotonicity of S, h and h^*. As for the boundary conditions (0. 1), we have

$$T(0, 0) = h^*(S(h(0), h(0))) = h^*(S(a_0, a_0)) = 0,$$

since by Lemma 1, $S(a_0, a_0)$ is in I_0; and

$$T(a, 1) = T(1, a) = h^*(S(h(1), h(a))) = h^*(S(1, h(a))) = h^*(h(a)) = a.$$

by Lemma 2.

To prove that T satisfies the associativity condition (0. 4) we first note that for any a, b, c in Dom h we have either

$$(A): S(h(a), h(b)) \in I_1, \quad \text{or} \quad (B): S(h(a), h(b)) \in I_0;$$

and either

$$(C): \ S(h(b), h(c)) \in I_1, \quad \text{or} \quad (D): \ S(h(b), h(c) \in I_0;$$

and that consequently we have four cases to consider — namely $(A \& C)$, $(A \& D)$, $(B \& C)$ and $(B \& D)$.

$(A \& C)$: In this case, using (1. 3), the associativity of S, and omitting superfluous parentheses, we have

$$T(T(a, b), c) = h^* S(hT(a, b), hc) = h^* S(hh^* S(ha, hb), hc) = h^* S(S(ha, hb), hc) =$$
$$= h^* S(ha, S(hb, hc)) = h^* S(ha, hh^* S(hb, hc)) = h^* S(ha, hT(b, c)) = T(a, T(b, c)).$$

$(A \& D)$: As in the previous case, we obtain

$$T(T(a, b), c) = h^* S(ha, S(hb, hc)) = h^* S(ha, t).$$

But now $t = S(hb, hc) \in I_0$, so that by Lemma 1, $S(ha, t) \in I_0$. Hence by the definition of h^*, $h^* S(ha, t) = T(T(a, b), c) = 0$. Next,

$$T(b, c) = h^* S(hb, hc) = h^* t = 0.$$

Hence,

$$T(a,\, T(b,\, c)) = T(a,\, 0) = h^* S(ha,\, h0) = h^* S(ha,\, a_0) = 0,$$

since again, by Lemma 1, $S(ha, a_0) \in I_0$.

$(B \,\&\, C)$: This follows on reversing the roles of $T(T(a, b), c)$ and $T(a, T(b,c))$ and arguing as in $(A \,\&\, D)$.

$(B \,\&\, D)$: In this case $T(a, b) = T(b, c) = 0$ and the second half of the argument in $(A \,\&\, D)$ yields $T(a, T(b, c)) = T(T(a, b), c) = 0$.

Thus (0. 4) holds in all cases and the theorem is proved.

Corollary. Let f be a continuous, strictly decreasing function on [0, 1] such that $f(0) = b_0 > 0$ (we permit b_0 to be infinite) and $f(1) = 0$. Let f^* be the function defined on the extended half-line $[0, \infty]$ by

$$f^*(x) = \begin{cases} f^{-1}(x), & x \in [0, b_0], \\ 0, & x \in [b_0, \infty], \end{cases} \tag{1. 5}$$

where f^{-1} is the inverse of f. Then the function T defined on $[0, 1] \times [0, 1]$ by

$$T(a, b) = f^*(f(a) + f(b)) \tag{1. 6}$$

is a t-norm.

Proof. Let h and h^* be the functions defined by

$$h = \exp(-f) = e^{-f}, \quad h^* = f^*(-\log); \tag{1. 7}$$

and let $I_0 = [0, e^{-b_0}]$, $I_1 = [e^{-b_0}, 1]$. Then h, h^*, I_0 and I_1 satisfy the hypotheses of Theorem 1. Moreover we have

$$f = -\log h, \quad f^* = h^*(e^{-j}) = h^* \exp(-j). \tag{1. 8}$$

Thus (1. 6) may be written as

$$T(a, b) = h^* \exp\{-(-\log h(a) - \log h(b))\} = \\ = h^*(h(a), h(b)) = h^*(\mathrm{Prod}\,(h(a), h(b))). \tag{1. 9}$$

Since Prod is a t-norm, all the conditions of Theorem 1 are satisfied and the conclusion follows.

In conformity with and as an extension of the terminology of [7], the function f appearing in (1. 6) will be called an *additive generator* of the t-norm T. Correspondingly, the function h appearing in (1. 9) is a *multiplicative generator* of T. Any t-norm possessing an additive generator also possesses a multiplicative generator and conversely, the two being connected by (1. 7) and (1. 8).

In [7] (Theorems 9 and 10) it was shown that if T is a strict t-norm and f an additive generator of T, then T is a copula if and only if f is convex. With the aid of the preceding corollary this result may be extended as follows:

Theorem 2. *If a t-norm T has an additive generator f, then T is a copula if and only if f is convex.*

PROOF. With a little care, the proof of Theorem 9 of [7] can be taken over verbatim. The care is necessitated by the fact that, when b_0 is finite, f^* is a proper extension of f^{-1} rather than simply f^{-1} itself. However, it follows from the definition of f^* that the convexity of f always implies the convexity of f^*, that the composite $f^*f=j_{[0,1]}$, and that, on Ran f, $ff^*=j_{\text{Ran} f}$. With these facts in mind, the proof of Theorem 9 given in [7] can be carried over to the present case without change.

Before proceeding to some applications of these theorems, we call attention to a result which shows that the transformation (1. 4) need not always yield something new.

Theorem 3. *The t-norms T_W and* Min *are invariant under the transformation (1. 4), i. e., for every pair of functions h, h^* satisfying the conditions of Theorem* 1, *we have*

$$h^* T_W(ha, hb) = T_W(a, b),$$

$$h^* \text{ Min } (ha, hb) = \text{Min } (a, b).$$

PROOF. Consider first T_W and let $T(a, b) = h^* T_W(ha, hb)$. By Theorem 1, T is a t-norm. Therefore, if $a=1$, then $T(a, b)=b=T_W(a, b)$. Similarly, if $b=1$, then $T(a, b)=a=T_W(a, b)$. Finally, if neither a nor b is 1 then neither ha nor hb can be 1, whence $T_W(ha, hb)=0$ and

$$T(a, b) = h^*0 = 0 = T_W(a, b).$$

As for Min, because of the monotonicity of h we have Min $(h(a), h(b)) = h(\text{Min } (a, b))$. Consequently by (1. 2),

$$h^* \text{ Min } (h(a), h(b)) = h^* h \text{ Min } (a, b) = \text{Min } (a, b).$$

As an example showing how the results of this section extend and improve those obtained in [7], consider the one-parameter family of functions h_p defined by:

$$h_p(x) = \exp\left(\frac{1}{p}(1-x^{-p})\right), \qquad p \neq 0, \quad 0 \leqq x \leqq 1,$$

$$h_0(x) = x = \lim_{p \to 0} h_p(x), \qquad 0 \leqq x \leqq 1,$$

where it should be noted that p is now allowed to range over the set of all real numbers rather than only the set of all positive numbers as in [7] (p. 175, Example (d)). It is readily verified that the functions h_p satisfy the conditions of Theorem 1 and that the functions h_p^* are given by:

$$h_p^*(x) = (1 - p \log x)^{-1/p}, \qquad p > 0,$$

$$h_0^*(x) = x = \lim_{p \to 0} h_p^*(x),$$

$$h_p^*(x) = \begin{cases} (1 - p \log x)^{-1/p}, & x \geqq e^{1/p}, \\ 0, & x \leqq e^{1/p}, \end{cases} \qquad p < 0,$$

Now, taking $S = \text{Prod}$ in (1. 4), we obtain the following family of t-norms T_p:

$$T_p(a, b) = (a^{-p} + b^{-p} - 1)^{-1/p} \qquad p > 0,$$

$$T_0(a, b) = \lim_{p \to 0} T_p(a, b) = a \cdot b = \text{Prod}\,(a, b)$$

$$T_p(a, b) = \begin{cases} (a^{-p} + b^{-p} - 1)^{-1/p}, & a^{-p} + b^{-p} \geqq 1 \\ 0, & a^{-p} + b^{-p} \leqq 1 \end{cases} \qquad p < 0.$$

For $p \geqq 0$, the t-norms are strict — this follows from the results of [7] quoted at the beginning of this section. However, for $p < 0$, they are definitely not strict. In particular $T_{-1} = T_m$. Thus T_m possesses multiplicative generators and, by virtue of (1. 8), additive generators as well.

Additive generators f_p corresponding to the multiplicative generators h_p are readily found by use of (1. 8). We have:

$$f_p(a) = \frac{1}{p}(a^{-p} - 1), \qquad p \neq 0, \quad 0 \leqq a \leqq 1,$$

$$f_0(a) = -\log a, \qquad 0 \leqq a \leqq 1;$$

$$f_p^*(a) = (1 + pa)^{-1/p}, \qquad p > 0, \; a \geqq 0,$$

$$f_0^*(a) = e^{-a}, \qquad a \geqq 0,$$

$$f_p^*(a) = \left.\begin{cases} (1 + pa)^{-1/p}, & 0 \leqq a \leqq -1/p, \\ 0, & a \geqq -1/p, \end{cases}\right\} \; p < 0.$$

By differentiating f_p twice, we easily verify the fact that f_p is convex if and only if $p \geqq -1$. Consequently, by Theorem 2 the t-norms T_p are copulas if and only if $p \geqq -1$.

One final note about the t-norms T_p. We have

$$\lim_{p \to \infty} T_p(a, b) = \text{Min}\,(a, b),$$

$$\lim_{p \to -\infty} T_p(a, b) = T_W(a, b),$$

so each of the t-norms T_W, T_m, Prod, Min is either a member of the family $\{T_p\}$ or a limit of members of this family.

2. Transformations of abstract semigroups

To extend some of the results of the preceding section to abstract semigroups we shall have to consider several semigroups at the same time. It is therefore convenient to designate semigroups by ordered pairs, e. g. (B, T), where B denotes the set of elements of the semigroup and T is the semigroup operation, i. e., T is a mapping from $B \times B$ into B which satisfies the associativity condition (0. 4). We also need the concept of an *ideal* (cf. [2], p. 5) which is defined as follows:

Definition 4. An ideal I_0 of a semigroup (A, S) is a (possibly empty) subset of A such that for any element a in I_0 and any element b in A, both $S(a, b)$ and $S(b, a)$ are in I_0.

Note that in this terminology Lemma 1 can be simply restated to read: The interval $I_0=[0, a_0]$ is an ideal of the semigroup $([0, 1], T)$.

Lemma 3. *Let A be a non-empty set and I_0 a subset of A. If I_0 is empty, let a_0 be any element of A; otherwise, let a_0 be an element of I_0. In either case, let I_1 be the set $(A-I_0)\cup\{a_0\}$. Let h be an invertible (i. e., one-one) mapping from a set B onto I_1, and α_0 that element of B for which $h(\alpha_0)=a_0$. Finally let h^* be the mapping with domain A and range B defined by:*

$$h^*(x)=\begin{cases}\alpha_0=h^{-1}(a_0), & x\in I_0,\\ h^{-1}(x), & x\in I_1,\end{cases} \tag{2. 1}$$

where h^{-1} is the inverse of h. Then

$$h^*(h(x))=x, \quad \text{for every} \quad x\in B, \tag{2. 2}$$

$$h(h^*(x))=\begin{cases}a_0, & x\in I_0,\\ x, & x\in I_1.\end{cases} \tag{2. 3}$$

PROOF. The same as the proof of Lemma 2, with the interval [0, 1] replaced by the set B, and the number 0 by the element α_0.

Theorem 4. *Let (A, S) be a semigroup and I_0 an ideal of (A, S). Let a_0, I_1, B, h and h^* be defined as in Lemma 3. Then the mapping T from $B\times B$ into B defined by*

$$T(a, b)=h^*(S(ha, hb)) \tag{2. 4}$$

is associative and (B, T) is a semigroup.

PROOF. The same as the second part of the proof of Theorem 1 (i. e., the proof of the associativity condition (0. 4)) with the phrase „using (1. 3)" replaced by the phrase „using (2. 3)" and the phrase „Lemma 1" replaced by the phrase „the fact that I_0 is an ideal of (A, S)".

Corollary. If $B=I_1$ and $h=j_{I_1}$, then $T(a, b)=h^*(S(a, b))$ and (I_1, T) is a semigroup.

Note that (I_1, T) is *not* in general a subsemigroup of (A, S).

Theorem 4 is thus the abstract analogue of Theorem 1. It is also a direct generalization of the following theorem of CLIMESCU ([3], Theorem II): *If (A, S) is a semigroup, if h is a one-one mapping from A into A, and if T is defined on $A\times A$ by (2. 4), then (A, T) is a semigroup.*

CLIMESCU's Theorem II is the special case of Theorem 4 in which the ideal I_0 is either empty or has only one element; for it is in precisely these cases that I_1 coincides with A and h^* with h^{-1}. Correspondingly, since h^* is definitely not invertible when I_0 contains more than one element, Theorem 4 is a proper generalization of Theorem II, a generalization in which an invertible function is replaced by a function which is the union of an invertible and a constant function. We can generalize further: namely we can show that the conclusion of Theorem 4 does not depend on the assumption that the mapping h from B to I_1 is invertible. To do this we need the concept of a *right—subinverse* [6].

Definition 5. A right-subinverse of a mapping f is a mapping g such that

$$\text{(2. 5)} \qquad \operatorname{Dom} g = \operatorname{Ran} f, \quad \operatorname{Ran} g \subseteq \operatorname{Dom} f, \quad \text{and} \quad fg = j_{\operatorname{Ran} f},$$

i. e., $f(g(x)) = x$ for every $x \in \operatorname{Ran} f$.

Any mapping, whether invertible (one-one) or not, has at least one right-subinverse. This follows from and (as is shown in [8], p. 108 and [10]) is in fact equivalent to the Axiom of Choice. Moreover, if g is a right-subinverse of f, then g is invertible and the inverse of g is the (generally proper) restriction of f to $\operatorname{Ran} g$. This latter restriction can be written as $fj_{\operatorname{Ran} g}$. Accordingly, we have

$$\text{(2. 6)} \qquad gfj_{\operatorname{Ran} g} = j_{\operatorname{Ran} g}.$$

We also note in passing that by virtue of (1. 3) and (2. 2) the function h in Theorems 1 and 4 is a right-subinverse of h^*; and that, correspondingly the function f of the corollary to Theorem 1 is a right-subinverse of f^*.

Lemma 4. *Let A, I_0, a_0, and I_1 be as in Lemma* 3. *Let h be an arbitrary* (*not necessarily invertible*) *mapping from a set B onto I_1, and let h^* be a mapping with domain A and range a subset of B, defined as follows:*

$$h^*(x) = \begin{cases} g(x), & x \in I_1, \\ g(a_0) = \alpha_0, & x \in I_0, \end{cases}$$

where g is any right- subinverse of h. Then the mappings h and h^* so defined satisfy (2. 3) *and, in addition, are relative inverses* [2], *i. e., satisfy*

$$\text{(2. 7)} \qquad h^*hh^*(x) = h^*(x) \quad \text{for every } x \text{ in } A,$$

$$\text{(2. 8)} \qquad hh^*h(x) = h(x) \quad \text{for every } x \text{ in } B.$$

Proof. By Definition 5, for any x in I_1 we have

$$hh^*(x) = j_{\operatorname{Ran} h}(x) = j_{I_1}(x) = x.$$

In particular then, $h(\alpha_0) = hh^*(a_0) = a_0$. Hence for x in I_0, we have $hh^*(x) = h(\alpha_0) = a_0$. This yields (2. 3). To obtain (2. 7), apply h to both sides of (2. 3). If x is in I_1, the result is immediate; if x is in I_0, (2. 7) follows from the fact that $h^*(x) = \alpha_0$. Lastly, (2. 8) follows from equation (2. 3) and the fact that $h(x)$ is in I_1.

Theorem 5. *Let (A, S), I_0, a_0 and I_1 be as in Theorem 4, and let B, h and h^* be as defined in Lemma* 4. *Then the mapping T from $B \times B$ into B defined by* (2. 4) *is associative and (B, T) is a semigroup.*

Proof. Once we have observed that, by virtue of Lemma 4, the mappings h and h^* satisfy (2. 3), the proof of the theorem is *verbatim* the same as the proof of Theorem 4.

Theorem 5 shows how, by working with right-subinverses, it is possible to prove a result which heretofore would have required some sort of invertibility assumption. In this respect it is a typical illustration of the usefulness of the general concept of right-subinverse.[1]) However, as far as the central problem of Section 1 — the construction of t-norms from t-norms — is concerned, Theorem 5, despite the fact that it is a proper generalization, does not lead to any results beyond those of Theorem 4. The reasons for this are to be found in the following lemma and theorem, which are of interest in their own right.

Lemma 5. *Let (A, S), (B, T), h and h^* be as in Theorem 5. If there is an element u_R of (B, T) such that $T(a, u_R)=a$ for all a in B, or an element u_L of (B, T) such that $T(u_L, a)=a$ for all a in B, then $\operatorname{Ran} h^* = B = \operatorname{Dom} h$.*

PROOF. Assume that u_R exists. Then for any a in B, we have

$$a = T(a, u_R) = h^* S(ha, hu_R).$$

Thus a is a value of the mapping h^* and this means that $\operatorname{Ran} h^* = B$. The same conclusion also follows from the existence of u_L.

N. B. The element u_R is a *right-unit*, and u_L a *left-unit*, of the semigroup (B, T). Right and left units need not be unique, but it is a standard theorem that if a semigroup has both a right-unit u_R and a left-unit u_L, then $u_L = u_R$ and the semigroup has a unique *unit*. The t-norms of Definition 1 all have the unit 1 by virtue of (0. 1).

Theorem 6. *Let (A, S), I_0, I_1, (B, T), h and h^* be as in Theorem 5. If the semigroup (B, T) has either a right-unit u_R or a left-unit u_L, then the mapping h is invertible, and its inverse is $h^* j_{I_1}$, the restriction of h^* to the set I_1.*

PROOF. The mapping $h^* j_{I_1}$ is a right-subinverse of h. It is therefore invertible and, by virtue of the remarks following Definition 5, its inverse is the mapping $h j_{\operatorname{Ran} h^* j_{I_1}}$. Now by the definition of h^*, $\operatorname{Ran} h^* j_{I_1} = \operatorname{Ran} h^*$, and by Lemma 5 we have $\operatorname{Ran} h^* = \operatorname{Dom} h$. Hence the inverse of $h^* j_{I_1}$ is $h j_{\operatorname{Dom} h} = h$. Since the inverse of an invertible mapping is itself invertible, its inverse being the original mapping, the conclusion of the theorem follows.

Corollary. If the ideal I_0 is empty or has only one element, then $h(u_R)$ is a right-unit of (A, S) and $h(u_L)$ is a left-unit of (A, S).

PROOF. In this case $I_1 = A$.

Theorem 6 and its corollary thus account for the invertibility of the functions h and f of Theorem 1 and its corollary, since as remarked above, all t-norms have units. We also note in passing that if, in Lemma 2, h is assumed to be increasing but not necessarily strictly increasing and if h^* is defined to agree with some right-subinverse of h on I_1 and to have the constant value 0 on I_0, then it follows from Theorem 5 that the function T defined by (1. 4) has all the properties of a t-norm except: $T(a, 1) = T(1, a) = a$.

Finally, we turn to a connection between these extensions of CLIMESCU's theorem and homomorphism theory. In our notation, the standard definitions (cf. [2], p. 9) take the following form: (A, S) is a homomorph of (B, T) if there is a mapping h, with domain B and range A, such that

$$h(T(a, b)) = S(ha, hb) \tag{2. 9}$$

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[1]) Right-subinverses play a very important role in our study of the algebra of functions [6]. It is also worth observing that any antiderivative is a right-subinverse of the derivative operator; and that any branch of the complex logarithm is a right-subinverse of the complex exponential function.

for all a, b in B. The mapping h is a *homomorphism*. An invertible homomorphism is an *isomorphism*, and two semigroups (or more generally, two groupoids) connected via (2. 9) by an isomorphism are *isomorphic*. We then have:

Theorem 7. *Let* (A, S), I_0, I_1, (B, T), h, h^* *and* a_0 *be as in Theorem 5. Define a mapping* S_1 *with domain* $I_1 \times I_1$ *by*

$$S_1 = hh^* S,$$

(i. e., $S_1(x, y) = hh^* S(x, y)$ *for all* x, y in I_1). *Then* (I_1, S_1) *is a semigroup which is a homomorph of* (B, T) *under the homomorphism* h. *Furthermore,* T *and* S_1 *are connected by* (2. 4), *i. e., for every* a, b *in* B, *we have*

$$T(a, b) = h^* S_1(ha, hb). \tag{2. 10}$$

PROOF. Applying h to both sides of (2. 4), we obtain

$$hT(a, b) = hh^* S(ha, hb) = S_1(ha, hb), \tag{2. 11}$$

whence, by (2. 9), (I_1, S_1) is a homomorph of (B, T); and since it is a standard fact that a homomorphic image of a semigroup is a semigroup, (I_1, S_1) is a semigroup. Next, applying h^* to (2. 11), we obtain

$$h^* hT(a, b) = h^* S_1(ha, hb). \tag{2. 12}$$

Now using (2. 4) and (2. 7), we have

$$\begin{aligned} h^* hT(a, b) &= h^* hh^* S(ha, hb) \\ &= h^* S(ha, hb) \\ &= T(a, b). \end{aligned}$$

Combining this with (2. 12), we obtain (2. 10) and the theorem is proved.

Corollary. If h is invertible (in particular, if (B, T) has a right-unit or a left-unit), then (B, T) and (I_1, S_1) are isomorphic.

It should again be noted that, although I_1 is a subset of A, (I_1, S_1) need not be a subsemigroup of (A, S).

Appendix: Another construction of t-norms from t-norms

In his paper [3] CLIMESCU also proved the following:

Theorem V. Let (A, F) *and* (B, G) *be semigroups. If the sets* A *and* B *are disjoint and if* U *is the mapping defined on* $(A \cup B) \times (A \cup B)$ *by*

$$U(x, y) = \begin{cases} F(x, y), & x \in A, y \in A, \\ x, & x \in A, y \in B, \\ y, & x \in B, y \in A, \\ G(x, y), & x \in B, y \in B, \end{cases}$$

then $(A \cup B, U)$ *is a semigroup.*

This theorem leads quite readily to another construction for t-norms from t-norms which proceeds by piecing together three t-norms rather than transforming one.

Theorem 8. *Let T_1 and T_2 be t-norms and λ a number in the open interval $(0, 1)$. Let U_1 be a mapping on the square $[0, \lambda)\times[0, \lambda)$ defined by*

$$U_1(a, b) = \lambda T_1(a/\lambda, b/\lambda),$$

and let U_2 be a mapping on the square $[\lambda, 1]\times[\lambda, 1]$ defined by

$$U_2(a, b) = \lambda + (1-\lambda)\, T_2\left(\frac{a-\lambda}{1-\lambda}, \frac{b-\lambda}{1-\lambda}\right).$$

Then the mapping T defined on the closed unit square $[0, 1]\times[0, 1]$ by:

$$T(a, b) = \begin{cases} U_1(a, b), & a\in[0, \lambda),\ b\in[0, \lambda), \\ a = \mathrm{Min}\,(a, b), & a\in[0, \lambda),\ b\in[\lambda, 1], \\ b = \mathrm{Min}\,(a, b), & a\in[\lambda, 1],\ b\in[0, \lambda), \\ U_2(a, b), & a\in[\lambda, 1],\ b\in[\lambda, 1], \end{cases}$$

is a t-norm.

Proof. The verification of the boundary, monotonicity and symmetry conditions (0. 1), (0. 2) and (0. 3) is immediate. To prove that T satisfies the associativity condition (0. 4), we have, in view of Theorem V and the properties of Min, only to show that $([0, \lambda), U_1)$ and $([\lambda, 1], U_2)$ are semigroups. Now, since T_1 is associative, for any a, b, c in $[0, \lambda)$, we have

$$U_1(U_1(a, b), c) = T_1\left(\frac{1}{\lambda} U_1(a, b), \frac{c}{\lambda}\right) = T_1\left(T_1\left(\frac{a}{\lambda}, \frac{b}{\lambda}\right), \frac{c}{\lambda}\right) =$$
$$= T_1\left(\frac{a}{\lambda}, T_1\left(\frac{b}{\lambda}, \frac{c}{\lambda}\right)\right) = U_1(a, U_1(b, c)).$$

Similarly, for any a, b, c in $[\lambda, 1]$, we have

$$U_2(U_2(a, b), c) = \lambda + (1-\lambda)\, T_2\left(\frac{U_2(a, b)-\lambda}{1-\lambda}, \frac{c-\lambda}{1-\lambda}\right) =$$
$$= \lambda + (1-\lambda)\, T_2\left(T_2\left(\frac{a-\lambda}{1-\lambda}, \frac{b-\lambda}{1-\lambda}\right), \frac{c-\lambda}{1-\lambda}\right) =$$
$$= \lambda + (1-\lambda)\, T_2\left(\frac{a-\lambda}{1-\lambda}, T_2\left(\frac{b-\lambda}{1-\lambda}, \frac{c-\lambda}{1-\lambda}\right)\right) = U_2(a, U_2(b, c)).$$

Thus U_1 and U_2 are associative. Moreover, Ran $U_1 \subseteqq [0, \lambda)$ and Ran $U_2 = [\lambda, 1]$. Thus $([0, \lambda), U_1)$ and $([\lambda, 1], U_2)$ are semigroups, the hypotheses of Theorem V are satisfied, whence $([0, 1], T)$ is a semigroup and the theorem is proved.

Remark. It is readily seen that, because of the boundary conditions (0. 1), we can extend U_1 to the closed unit square $[0, \lambda]\times[0, \lambda]$ without affecting the conclusion of Theorem 8 in any way.

The construction in Theorem 8 can evidently be iterated within each of the squares $[0, \lambda]\times[0, \lambda]$ and $[\lambda, 1]\times[\lambda, 1]$. The graph of a t-norm obtained in this way may be crudely but effectively described by saying that it differs from the graph of Min only in having a series of larger or smaller square notches cut out along the diagonal from $(0, 0, 0)$ to $(1, 1, 1)$.

A particularly interesting class of t-norms is obtained by taking $T_1 = T_W$, $T_2 = \text{Min}$, and λ close to 1. The result is a t-norm which coincides with the minimal t-norm T_W over most of the unit squaie, but which, unlike T_W, is continuous on the boundary of the unit square. Thus Theorems 7. 2, 8. 1 and 8. 2 of [4], as well as the metrization theorems of [5] and [9], are applicable in any statistical metric space which is a Menger space (see [4]) under such a t-norm.

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