The quasi-series decomposition of two-terminal graphs

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To Professor Béla Szőkefalvi-Nagy on his 50th birthday

§ 1. Introduction

The present paper contains a graph-theoretical theorem and deals with the application of this theorem to the problem of the non-repeating realizability of truth functions.

B. TRACHTENBROT has arrived at partial results in elucidating the following question: in an irreducible 2-graph, to which pairs of edges k_1, k_2 does a path exist which contains both k_1 and k_2. There are some situations of edges in which the non-existence of a path of this character is obvious. In certain other situations, TRACHTENBROT has proved the existence of such a path; he left the problem open if the edges are non-adjacent inner ones. In our Corollary, the problem is solved completely: it is shown that the obvious situations are the only cases when a path of the desired character does not exist. These „exceptional" cases are: both of k_1, k_2 are incident to the same terminal of the graph; k_1 and k_2 form a so-called separating pair, i. e., by deleting them, P and Q lose their connectedness.

In § 3, we study some graph-theoretical concepts in which the notion of separating pair plays a central rôle. After the lemmas exposed in § 4, we give in § 5 the solution of the graph-theoretical problem for the 2-graphs generally.[1]) In the definitions of § 6, it is discussed whether certain graph-theoretical facts are reflected in the behavior of a truth function or not.[2]) There are defined three classes of truth functions irreducible for repetition-free superposition:

functions for which the quasi-series decomposition is not defined,

functions indecomposable in quasi-series manner,

functions decomposable in quasi-series manner.

Theorem 2 states that no function of the first class admits a repetition-free realization; Theorem 3 reduces the realizability problem of the third class to the one of the second class.

[1]) Although the problem has interest chiefly for irreducible graphs, the method used in the proof makes it necessary to formulate the result more generally, while the intended particular case itself is exposed as a corollary. Our proof is rather lengthy as compared with the relative simplicity of the result; therefore it would be desirable to find a simpler, immediate proof for the Corollary. I did not succeed in doing this.

[2]) In studying these definitions, it is advantageous if the reader keeps throughout in view the analogy with the graph-theoretical notions.

If one joins the terminals P, Q by an additional edge, then Theorem 1 can be transformed into the solution of the following problem: in a general graph (without distinction of vertices), what is the necessary and sufficient condition in order that *three* edges can be completed into a *circuit*. The analogous problem for more than three edges seems to be interesting.

§ 2. Preliminaries

It is supposed that the terminology exposed in §§ 1—3 of [3] (on graphs) and in § 2 of [2] (on truth functions) is known to the reader. The considered 2-graphs are always strongly connected.

An edge incident to the beginning vertex P of a 2-graph is called a *beginning edge*. Similarly, an edge is a *final edge* if it is incident to the end vertex. Two edges are said to form a *beginning* resp. *final pair* if both of them are beginning resp. final. „*Terminal edge*", „*terminal pair*" is a common term for the beginning and final edges resp. pairs. An edge is called an *inner edge* if it is incident to no terminal of the graph.

Two edges are *adjacent* if there exists a vertex incident to both of them. Two edges k_1, k_2 of the 2-graph $\mathfrak{G}$ are said to be *completable* (or, more precisely, $\mathfrak{G}$-*completable*) if there exists a path in $\mathfrak{G}$ which contains both of k_1 and k_2.

Two 2-subgraphs of a 2-graph are said to be *disjoint* if they contain no edge in common. The difference $\mathfrak{H}_1 - \mathfrak{H}_2$ of two subgraphs $\mathfrak{H}_1$ and $\mathfrak{H}_2$ consists of those edges which are contained in $\mathfrak{H}_1$ and are not contained in $\mathfrak{H}_2$, and of the terminals of these edges. The intersection $\mathfrak{H}_1 \cap \mathfrak{H}_2$ is meant analogously.

Let $\mathfrak{H}$ be a 2-subgraph of the 2-graph $\mathfrak{G}$. Delete the edges and the inner vertices of $\mathfrak{H}$, and let the terminals of $\mathfrak{H}$ be joined by a new edge h. We denote the graph, originating by this construction, by $\mathfrak{G}/\mathfrak{H}$.[3]) If $\mathfrak{H}_1$, $\mathfrak{H}_2$ are disjoint 2-subgraphs of $\mathfrak{G}$, then $\mathfrak{G}/(\mathfrak{H}_1, \mathfrak{H}_2)$ is defined in an analogous manner. — Let a be a path of $\mathfrak{H}$ and b be a path of $\mathfrak{G}/\mathfrak{H}$ such that b contains h. Then we get evidently a path of $\mathfrak{G}$ if a is substituted for h in b.

In § 6, we shall consider such truth functions which are supposed to depend effectively and in a monotonically increasing way from each of their variables. For a function of this type, there exists a uniquely determined, easily presentable simplest disjunctive normal form, and the terms of this form are exactly the prime implicants of f. The concept of *prime implicatum* is in a duality relation with the prime implicant, i. e. the elementary disjunction $\mathfrak{A}$ is a prime implicatum of f if $f \to \mathfrak{A}$ is identically true, but this remains never valid if $\mathfrak{A}$ is replaced by a proper sub-disjunction of it. — A truth function is *irreducible* for superposition if it has no non-trivial separable subset of variables. — If a truth function f can be represented in the form $g \,\&\, h$ where the functions g and h have no variable in common, then the simplest disjunctive normal form of f can be formed from the corresponding forms of g, h *only* by applying the distributive law (without reduction).

The following known facts will be used without an explicit reference.

If $\mathfrak{H}_1$, $\mathfrak{H}_2$ are non-disjoint 2-subgraphs of a 2-graph, then either

[3]) In similar cases we shall denote the substituting edge and the substituted graph by the (Latin resp. German) variants of the *same* letter.

one of $\mathfrak{H}_1, \mathfrak{H}_2$ is included in the other, or

$\mathfrak{H}_1 - \mathfrak{H}_2, \mathfrak{H}_1 \cap \mathfrak{H}_2, \mathfrak{H}_2 - \mathfrak{H}_1$ join to each other by parallel composition, or

$\mathfrak{H}_1 - \mathfrak{H}_2, \mathfrak{H}_1 \cap \mathfrak{H}_2, \mathfrak{H}_2 - \mathfrak{H}_1$ join to each other by series composition. ([4], Theorem 3, p. 242.)

If we delete one edge of a non-trivial irreducible graph, then we get a strongly connected graph. ([4], remark after Lemma 4, p. 235.)

If we delete all the edges which are incident to an inner vertex of an irreducible graph, then we get a strongly connected graph. ([4], Lemma 3, p. 234.)

If one of the following suppositions holds for the edges k_1 and k_2 of an irreducible graph, then k_1, k_2 are completable:

exactly one of k_1, k_2 is a beginning edge,
exactly one of k_1, k_2 is a final edge,
k_1, k_2 are adjacent inner edges.

([4], Lemma 5 and Theorem 1, pp. 235—236.)

In an indecomposable graph, there exists a pair of disjoint paths. ([4], Theorem 4, p. 244; [3], Theorem 1, p. 384.)

The notion of repetition-free realization is defined e. g. in [4] (pp. 230—231) or in [1] (p. 208). It is well known that the prime implicants of the realized function correspond to the paths of the realizing graph. Let the relation ϱ_2 be true for two variables x_1, x_2 of a realizable (consequently, monotonically increasing) truth function f exactly when f has a prime implicatum containing both of x_1, x_2. The transitive extension ε_2 of ϱ_2 is true for x_1, x_2 if and only if the edges k_1, k_2 (corresponding to x_1, x_2 respectively) lie in a common series component of that 2-graph which realizes f. ([1], Theorem 3, p. 211.)

Sometimes in the proofs we do not mention such cases when the statement, to be proved instantly, is accessible by a simple idea. In particular, we shall speak on maximal or minimal graphs satisfying certain properties; the proof of the existence and unicity of such extreme graphs will be left to the reader.

§ 3. Quasi-series decomposition

Let $\mathfrak{G}_1, \mathfrak{G}_2$ be irreducible 2-graphs (disjoint from each other) with the terminals P_1 and Q_1, P_2 and Q_2, respectively. Suppose that exactly the edges $k_1(A_1Q_1)$ and $k_1'(A_1'Q_1)$ are incident to Q_1 in $\mathfrak{G}_1$, and exactly the edges $k_2(P_2A_2)$ and $k_2'(P_2A_2')$ are incident to P_2 in $\mathfrak{G}_2$. Let us form a graph $\mathfrak{G}$ in the following way: The vertices of $\mathfrak{G}$ are the vertices of $\mathfrak{G}_1$ and of $\mathfrak{G}_2$, excepting Q_1 and P_2. The beginning and inner edges of $\mathfrak{G}_1$ are edges in $\mathfrak{G}$ too, the inner and final edges of $\mathfrak{G}_2$ are edges in $\mathfrak{G}$ too. Moreover, let $\mathfrak{G}$ have two additional edges: $k(A_1A_2)$ and $k'(A_1'A_2')$. (We can say, presenting this construction illustratively, that we cut up Q_1, P_2, and we identify k_1 with k_2, k_1' with k_2'.) It is said that $\mathfrak{G}$ originates by the *quasi-series composition* of $\mathfrak{G}_1$ and $\mathfrak{G}_2$. Evidently, there exist two possibilities for composing $\mathfrak{G}_1, \mathfrak{G}_2$ in quasi-series manner; the two graphs resulting are not isomorphic in general. In consequence of the irreducibility of $\mathfrak{G}_1$ and $\mathfrak{G}_2$, $\mathfrak{G}$ is irreducible.

In the remaining part of this §, our aim is to investigate the decomposition which corresponds to the composition introduced above. Let $\mathfrak{G}$ be an irreducible

2-graph. We say that the inner edges k_1 and k_2 form a *separating pair* if each path of $\mathfrak{G}$ contains (at least) one of k_1, k_2. (The graph of Fig. 1 has three separating pairs: (k_1, k_2), (k_3, k_4), (k_5, k_6).)

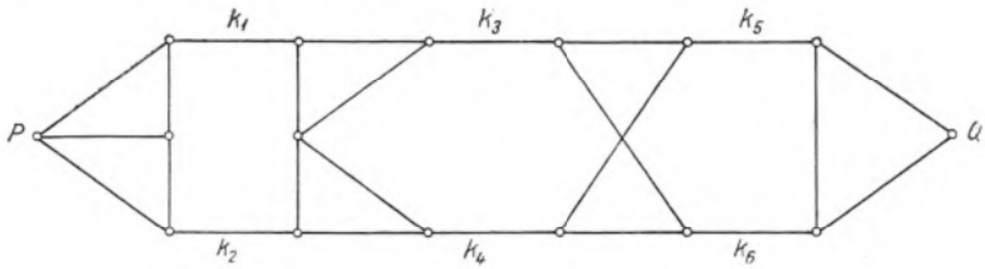

Figure 1

If we delete a separating pair (k_1, k_2) in $\mathfrak{G}$, then $\mathfrak{G}$ splits into two connected parts in such a manner that P and Q are in distinct parts. Since we have only two edges (namely k_1, k_2) between these parts $\mathfrak{G}_P, \mathfrak{G}_Q$, any path goes through only once from $\mathfrak{G}_P$ to $\mathfrak{G}_Q$. Hence, any path of $\mathfrak{G}$ contains *exactly one* of k_1, k_2.

Lemma 1. *Let (k_1, k_2) and (k_1, k_3) be two separating pairs of edges of the irreducible graph $\mathfrak{G}$. Then we have $k_2 = k_3$.*

Proof. Let us consider the connected parts $\mathfrak{G}_P, \mathfrak{G}_Q$ which result by deleting k_1, k_2. If k_3 differs from k_2, then we can suppose (by the symmetry) that k_3 is in $\mathfrak{G}_Q$. Let now $\mathfrak{G}'_P, \mathfrak{G}'_Q$ be the connected parts of $\mathfrak{G}$ resulting by deleting k_1 and k_3. Then k_2, k_3 and the edges being in both of $\mathfrak{G}_Q, \mathfrak{G}'_P$ form a 2-subgraph having at least two edges. This contradicts the irreducibility of $\mathfrak{G}$.

Lemma 2. *Let (k_1, k_2) and (k_3, k_4) be two different separating pairs of edges in the irreducible graph $\mathfrak{G}$. Let us consider the connected parts $\mathfrak{G}_P, \mathfrak{G}_Q$ which result after deleting k_1, k_2. Then k_3 and k_4 are in the same connected part.*

Proof. Assume that the lemma is not fulfilled. If we delete only k_2, then we get a strongly connected graph. This graph is series decomposable, k_1 is itself a series component in it. One of k_3, k_4 occurs in a component lying before k_1, the other of k_3, k_4 is in a component lying after k_1. So there exists a path which contains both k_3, k_4; this is a contradiction.

Lemmas 1, 2 make it possible to introduce a natural preceding relation in the set of the separating pairs of edges of an irreducible graph $\mathfrak{G}$. Let (k_1, k_2), (k_3, k_4), … …, (k_{2t-1}, k_{2t}) be (all) the separating pairs of $\mathfrak{G}$, enumerated according to this precedence. We are going to define the *quasi-series components* of $\mathfrak{G}$. In order to define the s^{th} component $(2 \leqq s \leqq t)$, let $\mathfrak{G}_P, \mathfrak{G}_Q$ and $\mathfrak{G}'_P, \mathfrak{G}'_Q$ denote the connected parts after deleting (k_{2s-3}, k_{2s-2}) and (k_{2s-1}, k_{2s}), respectively. Let the s^{th} quasi-series component consist of the common edges and vertices of $\mathfrak{G}_Q$ and $\mathfrak{G}'_P$, of the edges $k_{2s-3}, k_{2s-2}, k_{2s-1}, k_{2s}$, and of the additional vertices P_s, Q_s. The vertices, transferred from $\mathfrak{G}$, preserve the incidence relations to the edges; P_s is incident to k_{2s-3}, k_{2s-2}; Q_s is incident to k_{2s-1}, k_{2s}. — The definition of the first and $(t+1)^{\text{th}}$ quasi-series components is analogous, but somewhat simpler in details. Here only one separating pair and only one additional vertex (Q_1 resp. P_{t+1}) occurs.

§ 4. Some lemmas

We prove in the present § a number of lemmas which will play an auxiliary rôle in verifying Theorem 1.

Lemma 3. *Let A be an inner vertex of the irreducible graph $\mathfrak{G}$. Then $\mathfrak{G}$ has a chain $b(PA)$ which contains Q.*

PROOF. Let c_1, c_2 be disjoint paths of $\mathfrak{G}$, B an inner vertex of c_1. Let $d(AB)$ be an inner chain of the graph; denote by C the first vertex of d which is contained in c_1 or c_2. One of the chains $c_1 \cdot c_2^{-1}[QC] \cdot d^{-1}[CA]$ and $c_2 \cdot c_1^{-1}[QC] \cdot d^{-1}[CA]$ exists and satisfies the assertion.

Lemma 4. *Let the edges k_1, k_2 form a separating pair in the irreducible graph $\mathfrak{G}$. Let $A(\neq P)$ be a vertex of the part $\mathfrak{G}_P$ of $\mathfrak{G}$ (separated by k_1, k_2). Then $\mathfrak{G}$ has a chain $f(PA)$ such that f contains both of k_1 and k_2, but does not contain Q.*

PROOF. For a moment, let us consider $\mathfrak{G}_P$ as a two-terminal graph (by identifying those terminals of k_1, k_2 which lie in $\mathfrak{G}_Q$), and let us apply Lemma 3. The chain b splits into two non-connecting subchains b_1, b_2 if we consider the original graph $\mathfrak{G}$. Since $\mathfrak{G}$ is irreducible, the terminals of k_1, k_2 being in $\mathfrak{G}_Q$ can be joined in $\mathfrak{G}_Q$ by a chain c which does not contain Q. The chain $b_1 c b_2^{-1}$ fulfils the requirements.

Lemma 5. *Let k_1, k_2 be two final edges in the irreducible graph $\mathfrak{G}$. Then $\mathfrak{G}$ contains two paths b_1, b_2 disjoint from each other, such that k_1 occurs in b_1, k_2 occurs in b_2.*

PROOF. Delete all those final edges which differ from k_1, k_2. The remaining graph is strongly connected and indecomposable, consequently, it has a pair of disjoint paths.

Lemma 6. *Let the edges k_1, k_2 form either a terminal or a separating pair in the irreducible graph $\mathfrak{G}$, and let A be an inner vertex. Then $\mathfrak{G}$ has a terminal R and a chain $b(RA)$ such that b contains both of k_1, k_2.*

PROOF. If k_1, k_2 are separating, then Lemma 4 ensures the statement. — Let now k_1, k_2 form a final pair. Let us consider the paths b_1, b_2 occuring in Lemma 5, and denote by B an inner vertex of b_1. There exists an inner chain $d(AB)$ in $\mathfrak{G}$. The further proof coincides with proving Lemma 3. — If k_1, k_2 form a beginning pair, the proof is symmetrical.

Lemma 7. *Let $k(AB)$ be an inner edge of the indecomposable graph $\mathfrak{G}$. Assume that $\mathfrak{G}$ has no proper 2-subgraph $\mathfrak{G}^*$ such that Q is a terminal of $\mathfrak{G}^*$ and $\mathfrak{G}^*$ contains k. Then $\mathfrak{G}$ has a chain $b(PA)$ which contains neither B nor Q.*

PROOF. The validity of the lemma for irreducible graphs is an immediate consequence of the fact that one gets a strongly connected graph from $\mathfrak{G}$ by deleting the edges incident to B. The statement can be extended for indecomposable graphs by an easy induction.

Lemma 8. *Let A be an inner vertex, k an edge of the indecomposable or series decomposable graph $\mathfrak{G}$. Then $\mathfrak{G}$ has a terminal R and a chain $b(RA)$ such that b contains k and it does not contain the other terminal of $\mathfrak{G}$.*

PROOF. Let c be a path containing k, C an inner vertex of c, $d(AC)$ aninner chain of $\mathfrak{G}$. Denote by B the first vertex of d which lies on c. Then both $c[PB]\cdot\cdot d^{-1}[BA]$, $c^{-1}[QB]\cdot d^{-1}[BA]$ do exist, and one of them contains k.

Lemma 9. *Let the edges k_1 and $k_2(AB)$ form a terminal or separating pair in the irreducible graph $\mathfrak{G}$. Then there exists a terminal R and a chain $b(AR)$ in $\mathfrak{G}$ such that b contains k_1, but it does not contain k_2.*

PROOF. If A is a terminal, then each path containing k_1 fulfils the assertion. — If B is a terminal, then $b=a(AC)\cdot k_1^{-1}(CB)$ is a chain of the desired property where a is an inner chain of $\mathfrak{G}$. — If k_1, k_2 form a separating pair, then denote by $\mathfrak{G}_A$ the separated part of $\mathfrak{G}$ which contains A, by C the terminal of k_1 lying in $\mathfrak{G}_A$, by f a path containing k_1. There exists a chain $d(AC)$ in $\mathfrak{G}_A$. The chain $b=d[AD]\cdot\cdot f[DR]$ satisfies the lemma where D is the first vertex of d lying on f and R is the terminal of $\mathfrak{G}$ which does not lie in $\mathfrak{G}_A$.

Lemma 10. *If A is an inner vertex, and k is an arbitrary edge of an indecomposable graph $\mathfrak{G}$, then $\mathfrak{G}$ has a chain $b(PA)$ which contains k.*[4])

PROOF.[5]) Let the vertices P and A be joined by an additional edge k'; let us denote the extended graph by $\mathfrak{G}^+$. It is well known that the relation which holds for two edges if they can be completed into a circuit is an equivalence. This relation is true for each pair of edges of $\mathfrak{G}^+$ (in the contrary case, $\mathfrak{G}^+$ would have an articulation vertex, hence either $\mathfrak{G}$ would not be strongly connected or $\mathfrak{G}$ would be series decomposable). If we delete k' from a circuit containing both k, k', then we get a chain with the desired property.

§ 5. Main theorem

Let k_1, k_2 be two edges of a (not necessarily irreducible) 2-graph $\mathfrak{G}$. Denote the minimal 2-subgraph of $\mathfrak{G}$ containing both of k_1, k_2 by $\mathfrak{G}^*$; further, in case if $\mathfrak{G}^*$ is indecomposable, the maximal proper subgraphs of $\mathfrak{G}^*$ which contains k_1 or k_2 by $\mathfrak{G}_1$ or $\mathfrak{G}_2$, respectively. (It is allowed that e. g. $\mathfrak{G}_1$ has the single edge k_1 only.) If the edges g_1 and g_2 form a terminal pair in $\mathfrak{G}^*/(\mathfrak{G}_1, \mathfrak{G}_2)$, then k_1, k_2 are called a *generally-terminal pair* of edges in $\mathfrak{G}$. Similarly, k_1 and k_2 are called a *generally-separating pair* of edges in $\mathfrak{G}$ if g_1, g_2 form a separating pair in $\mathfrak{G}^*/ /(\mathfrak{G}_1, \mathfrak{G}_2)$.[6]) We supplement the definition exposed above (concerning the case of the indecomposability of $\mathfrak{G}^*$) by the agrement: if $\mathfrak{G}^*$ is parallel decomposable, then k_1, k_2 are both generally-terminal and generally-separating; if $\mathfrak{G}^*$ is series decomposable, then k_1, k_2 are neither generally-terminal nor generally-separating.

Theorem 1. *Let k_1, k_2 be two edges of the 2-graph $\mathfrak{G}$. The following statements are equivalent for k_1, k_2:*

A) *There exists a path in $\mathfrak{G}$ which contains both of k_1, k_2.*

B) *k_1 and k_2 form neither a generally-terminal nor a generally-separating pair in $\mathfrak{G}$.*

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[4]) It is allowed that Q occurs in b.

[5]) This simple proof is due to Prof. T. GALLAI.

[6]) We remind the reader of the agreement in Footnote 3.

Next we expose the important particular case of Theorem 1 concerning irreducible graphs.

Corollary. *Let k_1, k_2 be two edges of the irreducible 2-graph $\mathfrak{G}$. Then the following two statements are equivalent for k_1, k_2:*

there exists a path in $\mathfrak{G}$ which contains both of k_1 and k_2,

k_1, k_2 form neither a terminal pair nor a separating pair in $\mathfrak{G}$.

PROOF of Theorem 1. First we enumerate some facts which are evidently true for $\mathfrak{G}$ and a 2-subgraph $\mathfrak{H}$ of $\mathfrak{G}$. Two edges of $\mathfrak{H}$ are $\mathfrak{G}$-completable exactly if they are $\mathfrak{H}$-completable. An edge k_1 of $\mathfrak{H}$ and an edge k_2 of $\mathfrak{G}$ beside $\mathfrak{H}$ are $\mathfrak{G}$-completable exactly if h and k_2 are $\mathfrak{G}/\mathfrak{H}$-completable. Two edges of $\mathfrak{G}$ beside $\mathfrak{H}$ are $\mathfrak{G}$-completable exactly if they are $\mathfrak{G}/\mathfrak{H}$-completable. Two edges of $\mathfrak{H}$ form a generally-terminal or a generally-separating pair in $\mathfrak{G}$ exactly if they form a generally-terminal or a generally-separating in pair $\mathfrak{H}$, respectively. Similarly, an edge k_1 of $\mathfrak{H}$ and an edge k_2 beside $\mathfrak{H}$ form a generally-terminal (or generally-separating) pair in $\mathfrak{G}$ exactly if h and k_2 form a pair of corresponding nature in $\mathfrak{G}/\mathfrak{H}$; further, two edges beside $\mathfrak{H}$ are generally-terminal (or generally-separating) in $\mathfrak{G}$ exactly if they are so in $\mathfrak{G}/\mathfrak{H}$.

It can be verified easily that the falsity of B) implies the falsity of A). The implication B) $\rightarrow$ A) will be proved inductively. Assume that it is valid for all graphs with $1, 2, \ldots, n-1$ edges. Let $\mathfrak{G}$ have n edges, and let k_1, k_2 be a pair of edges of $\mathfrak{G}$ which is not $\mathfrak{G}$-completable.

Among the following cases, the first three can overlap.

Case 1: $\mathfrak{G}$ has a proper 2-subgraph $\mathfrak{H}$ containing both of k_1, k_2. These edges are not $\mathfrak{H}$-completable, thus, by the induction hypothesis, they form a generally-terminal or generally-separating pair in $\mathfrak{H}$. Hence, they do not fulfil B) in $\mathfrak{G}$.

Case 2: $\mathfrak{G}$ contains a non-trivial 2-subgraph $\mathfrak{H}$ which contains exactly one of k_1, k_2, e. g. k_1. Then h and k_2 are not $\mathfrak{G}/\mathfrak{H}$-completable, thus, they do not satisfy B) in $\mathfrak{G}/\mathfrak{H}$, consequently, k_1 and k_2 do not fulfil B) in $\mathfrak{G}$.

Case 3: $\mathfrak{G}$ contains a non-trivial 2-subgraph $\mathfrak{H}$ which contains neither k_1 nor k_2. The inference is similar to the preceding cases.

Case 4: $\mathfrak{G}$ is irreducible. We are going to show the impossibility of the following situation: the (not $\mathfrak{G}$-completable) inner edges k_1, k_2 do not form a separating pair in $\mathfrak{G}$. The assumption that k_1, k_2 are not separating means that $\mathfrak{G}$ has a path b which contains neither k_1 nor k_2. Let $\mathfrak{G}_1$ be that graph which originates from $\mathfrak{G}$ by deleting a final edge $l(AQ)$ different from the last edge of b. $\mathfrak{G}_1$ is strongly connected, reducible in general. Since k_1, k_2 cannot be $\mathfrak{G}_1$-completable, they form a generally-terminal or a generally-separating pair in $\mathfrak{G}_1$ by the induction hypothesis.

Case 4/a: $\mathfrak{G}_1$ is irreducible. k_1 and k_2 form a separating pair in $\mathfrak{G}_1$. Since they form no separating pair in $\mathfrak{G}$, A lies in the first part $\mathfrak{G}_P$ (separated by them) of $\mathfrak{G}_1$. Lemma 4 ensures that $\mathfrak{G}_1$ has a chain between P and A which contains k_1 and k_2 and does not contain Q. So k_1, k_2 are $\mathfrak{G}$-completable, and this is a contradiction.

Case 4/b: $\mathfrak{G}_1$ is reducible. The irreducibility of $\mathfrak{G}$ implies some assertions concerning the situation of the non-trivial 2-subgraphs of $\mathfrak{G}_1$. Every such subgraph contains A as its inner vertex. $\mathfrak{G}_1$ has a uniquely determined maximal proper 2-subgraph $\mathfrak{G}_2$, moreover, either

$\mathfrak{G}_1/\mathfrak{G}_2$ is irreducible and g_2 is no final edge in it, or

$\mathfrak{G}_1/\mathfrak{G}_2$ consists of two series-connected edges such that g_2 is incident to P.

Case 4/b/α: $\mathfrak{G}_1/\mathfrak{G}_2$ is irreducible, and both of k_1, k_2 are beside $\mathfrak{G}_2$. Then k_1, k_2 form a separating pair in $\mathfrak{G}_1/\mathfrak{G}_2$ (because they are not $\mathfrak{G}_1/\mathfrak{G}_2$-completetable and they are not terminal edges in $\mathfrak{G}$, consequently neither in $\mathfrak{G}_1/\mathfrak{G}_2$) and g_2 lies in the first separated part of $\mathfrak{G}_1/\mathfrak{G}_2$. An inference, similar to Case 4/a, leads to a contradiction.

Case 4/b/β: $\mathfrak{G}_1/\mathfrak{G}_2$ is irreducible, both of k_1, k_2 are contained in $\mathfrak{G}_2$. Denote by $\mathfrak{H}$ the minimal 2-subgraph of $\mathfrak{G}_2$ which contains both k_1 and k_2. It is clear that one of the following five alternatives holds for the structure of $\mathfrak{H}$:

(i) $\mathfrak{H}$ is irreducible,

(ii) $\mathfrak{H}$ has a unique maximal proper 2-subgraph $\mathfrak{H}'$, $\mathfrak{H}/\mathfrak{H}'$ is irreducible,

(iii) $\mathfrak{H}$ has two series components such that one of them consists of a single edge,

(iv) $\mathfrak{H}$ has three series components such that only the middle one consists of two or more edges,

(v) $\mathfrak{H}$ has two parallel components such that one of them consists of a single edge, this single edge is one of k_1, k_2, e. g. k_1.

Since k_1, k_2 are supposed not to be $\mathfrak{H}$-completetable, the cases (iii), (iv) are at once contradicting. In the other cases, we are going to get a chain between P and A which contains k_1, k_2 and does not contain Q. The existence of such a chain is a contradiction, since it would be completed by the edge l into a path of $\mathfrak{G}$.

If (i) holds, k_1 and k_2 form a terminal pair or a separating pair in $\mathfrak{H}$. In both of these cases, the application of Lemmas 6, 7 in $\mathfrak{H}$, $\mathfrak{G}_1/\mathfrak{H}$, respectively, gives a chain with the desired properties. — If (ii) holds and none of k_1, k_2 is in $\mathfrak{H}'$, the procedure is similar. — If (ii) holds and one of k_1, k_2 — e. g. k_2 — lies in $\mathfrak{H}'$, then we can apply Lemma 8 in $\mathfrak{H}'$, Lemma 9 in $\mathfrak{H}/\mathfrak{H}'$ (for k_1 and h), and Lemma 7 in $\mathfrak{G}_1/\mathfrak{H}$. — If (v) holds, then we apply Lemma 8 in the non-trivial parallel component of $\mathfrak{H}$, Lemma 7 in $\mathfrak{G}_1/\mathfrak{H}$ (for h and a suitable terminal of it).

Case 4/b/γ: $\mathfrak{G}_1/\mathfrak{G}_2$ is irreducible, exactly one of k_1, k_2 — e. g. k_2 — is contained in $\mathfrak{G}_2$. k_1 and g_2 form a separating pair in $\mathfrak{G}_1/\mathfrak{G}_2$ (they cannot form a terminal pair since k_1 is an inner edge of $\mathfrak{G}$). Denote by P_1, Q_1 the terminals of g_2 such that P_1 precedes Q_1 in any path of $\mathfrak{G}_1/\mathfrak{G}_2$ which contains g_2. Lemma 4 ensures the existence of such a chain c_1 in $\mathfrak{G}_1/\mathfrak{G}_2$ whose first vertex is P, and whose last edge is g_2 towards P_1, which contains k_1 and does not contain Q. Since k_2 is not a first series component of $\mathfrak{G}_2$ (because k_1, k_2 are not separating in $\mathfrak{G}$), Lemma 10 ensures the existence of a chain c_2 in $\mathfrak{G}_2$ such that its first vertex is Q_1, its last vertex is A and it contains k_2. The chain, which results if c_2 is substituted for g_2 in c_1, and the edge l form together a path of $\mathfrak{G}$ which contains both k_1, k_2. Contradiction.

Case 4/b/δ: $\mathfrak{G}_1$ is series decomposable. We recall that it consists of two series components, the first one is $\mathfrak{G}_2$, the second one consists of a single edge. In this case, k_1, k_2 lie necessarily in $\mathfrak{G}_2$. Consider the graph $\mathfrak{G}_2$, supplemented by the additional edge l^* between A and the end vertex Q_2 of $\mathfrak{G}_2$. k_1 and k_2 are not completable in the resulting graph $\mathfrak{G}^*$ (since they are not $\mathfrak{G}$-completable). The number of edges of $\mathfrak{G}^*$ is $n-1$. Each 2-subgraph of $\mathfrak{G}^*$ (if there exits any) contains A as its inner vertex and Q_2 as its terminal. The induction hypothesis assures that k_1, k_2 form a generally-separating or generally-final pair in $\mathfrak{G}^*$. They cannot be adjacent in consequence of their non completability in $\mathfrak{G}$. There are two alternatives:

(i) $\mathfrak{G}^*$ has a proper 2-subgraph $\mathfrak{H}$ in which k_1, k_2 form a separating pair,

(ii) $\mathfrak{G}^*$ has a 2-subgraph $\mathfrak{H}$ such that e. g. k_1 is a final edge of $\mathfrak{H}$, k_2 lies in a 2-subgraph $\mathfrak{H}_1$ ($\subset \mathfrak{H}$).

According to the cases (i), (ii), Lemma 4 or Lemma 8 showes that k_1, k_2 are $\mathfrak{G}$-completable.

§ 6. Application for the realizability problem

Our next aim is to introduce the notion of those truth functions for which the quasi-series decomposition is defined. In order to characterize this somewhat complicated notion, we begin the treatment with three partial definitions. The concept itself will be precised after these partial definitions.

Let $f(x_1, x_2, \dots, x_n)$ be a monotonic increasing, effective truth function which is irreducible for the (repetition-free) superposition. The (ordered) set of variables $\{x_1, x_2, \dots, x_n\}$ will be denoted by Θ. We define the binary relation ϱ for two different elements x_i, x_j of Θ by the following rule: $\varrho(x_i, x_j)$ is true if and only if f has no prime implicant containing both x_i and x_j. If $x_i = x_j$, let ϱ be false. ϱ is clearly a symmetrical relation.

Partial definition I. If Θ has three different elements x_i, x_j, x_k such that $\varrho(x_i, x_j) = \varrho(x_j, x_k) = \uparrow$ and $\varrho(x_i, x_k) = \downarrow$, then the quasi-series decomposition is not defined for f.

Now assume that ϱ is transitive for triples of different elements. This means that Θ includes a subset Θ' and Θ' admits a partition π such that each class of Θ' mod π consists of at least two elements and

$$\varrho(x_i, x_j) = \uparrow \text{ if and only if } x_i \equiv x_j \ (\text{mod } \pi),\ x_i \neq x_j.$$

It is obvious that any prime implicant of f contains at most one element from each class of Θ' mod π.

Partial definition II. If there exists a prime implicant $\mathfrak{A}$ of f and a class Π of Θ' mod π such that $\mathfrak{A}$ contains no element of Π, then the quasi-series decomposition is not defined for f.

Suppose that each prime implicant of f contains exactly one element of any class. For a variable $x_i (\in \Theta')$, let f_{x_i} mean that truth function which is represented as the disjunction of those prime implicants of f which contain x_i. Denote by Π_{x_i}

the class of Θ' mod π containing x_i. For any $x_i(\in\Theta')$, let the equivalence relation σ_{x_i} be defined in the set $\Theta-\Pi_{x_i}$ as the transitive extension of that relation which holds for two elements of $\Theta-\Pi_{x_i}$ exactly if they occur in a prime implicatum of f_{x_i} in common.

Partial definition III. If there exist two elements x_i, x_j of Θ', belonging to a common class mod π, such that the relations $\sigma_{x_i}, \sigma_{x_j}$ do not coincide, then the quasi-series decomposition is not defined for f.

Definition. Assume that the suppositions of the partial definitions I—III are not fulfilled for the function f. We say that *the quasi-series decomposition is defined for f* (or, equivalently, that *f admits a quasi-series decomposition*) if and only if the elements of f can be partioned into sets

$$(*)\qquad \Pi^{(0)}, \Gamma^{(1)}, \Pi^{(1)}, \Gamma^{(2)}, \ldots, \Pi^{(t)}, \Gamma^{(t+1)}, \Pi^{(t+1)} \qquad (t\geqq 0)$$

such that the following statements hold:[7])

α) the sets $\Pi^{(0)}, \Pi^{(1)}, \ldots, \Pi^{(t+1)}$ coincide with the equivalence classes of Θ' mod π,

β)
$$\bar{\bar{\Pi}}^{(0)}\geqq 2, \quad \bar{\bar{\Pi}}^{(t+1)}\geqq 2,$$
$$\bar{\bar{\Pi}}^{(1)}=\bar{\bar{\Pi}}^{(2)}=\ldots=\bar{\bar{\Pi}}^{(t)}=2,$$
$$\bar{\bar{\Gamma}}^{(1)}\geqq 1,\ \bar{\bar{\Gamma}}^{(2)}\geqq 1,\ \ldots,\ \bar{\bar{\Gamma}}^{(t)}\geqq 1$$

hold for the cardinalities of the sets enumerated in the sequence $(*)$,

γ) if $x_i\in\Pi^{(s)}$ (s can be $0, 1, \ldots, t+1$), then the equivalence classes of $\Theta-\Pi^{(s)}$ mod σ_{x_i} are precisely

$$\Pi^{(0)}\cup\Gamma^{(1)}\cup\Pi^{(1)}\cup\Gamma^{(2)}\cup\ldots\cup\Pi^{(s-1)}\cup\Gamma^{(s)}$$

and

$$\Gamma^{(s+1)}\cup\Pi^{(s+1)}\cup\ldots\cup\Gamma^{(t+1)}\cup\Pi^{(t+1)}.$$

Definition. If f admits a quasi-series decomposition and $t=0$,[8]) then we say that f is *indecomposable in quasi-series manner*. If f admits a quasi-series decomposition and $t\geqq 1$, then f is said to be *decomposable in quasi-series manner*.

Definition. If f is indecomposable in quasi-series manner, then we say that f has a single *quasi-series component*, namely itself. If f is decomposable in quasi-series manner, then we define the s^{th} $(1\leqq s\leqq t+1)$ *quasi-series component* of f as that function f_s which is represented by the disjunction of the conjuctions of the form $\mathfrak{A}_{\Pi^{(s-1)}\cup\Gamma^{(s)}\cup\Pi^{(s)}}$ where $\mathfrak{A}$ runs through the prime implicants of f.

Proposition. *Every quasi-series component of f is irreducible for superposition and quasi-series indecomposable. Exactly the conjuctions of form* $\mathfrak{A}_{\Pi^{(s-1)}\cup\Gamma^{(s)}\cup\Pi^{(s)}}$

[7]) The sequence $(*)$ is evidently uniquely determined by f up to the conversion of the ordering.

[8]) I. e. Θ' splits into two classes mod π.

are the prime implicants of the quasi-series component f_s of f. A conjunction $\mathfrak{A}$ is a prime implicant of f if and only if $\mathfrak{A}$ can be represented in the form

$$(**) \qquad \mathfrak{B}^{(1)} \,\&\, \mathfrak{B}^{(2)} \,\&\, \dots \,\&\, \mathfrak{B}^{(t+1)}$$

where $\mathfrak{B}^{(s)}$ is a prime implicant of f_s $(1 \leqq s \leqq t+1)$ and $\mathfrak{B}^{(s)}, \mathfrak{B}^{(s+1)}$ contain a variable in common[9]) $(1 \leqq s \leqq t)$.

PROOF. We are going to prove the second assertion indirectly. Let $\mathfrak{A}_{\Pi^{(s-1)} \cup \Gamma^{(s)} \cup \Pi^{(s)}}$ be a proper partial conjuction of $\mathfrak{B}_{\Pi^{(s-1)} \cup \Gamma^{(s)} \cup \Pi^{(s)}}$ where $\mathfrak{A}, \mathfrak{B}$ are prime implicants of f. Denote by x_i, x_j the single variables of $\mathfrak{A}_{\Pi^{(s-1)}}, \mathfrak{A}_{\Pi^{(s)}}$, respectively. We have $\mathfrak{B}_{\Pi^{(s-1)}} = x_i$ and $\mathfrak{B}_{\Pi^{(s)}} = x_j$. f_{x_i}, f_{x_j} can be represented in the form

$$g^{(1)}[\Pi^{(0)} \cup \Gamma^{(1)} \cup \dots \cup \Gamma^{(s-1)}] \,\&\, x_i \,\&\, g^{(2)}[\Gamma^{(s)} \cup \Pi^{(s)} \cup \dots \cup \Pi^{(t+1)}],$$

$$h^{(1)}[\Pi^{(0)} \cup \Gamma^{(1)} \cup \dots \cup \Gamma^{(s)}] \,\&\, x_j \,\&\, h^{(2)}[\Gamma^{(s+1)} \cup \Pi^{(s+1)} \cup \dots \cup \Pi^{(t+1)}],$$

respectively. Moreover, the function $f_{[x_i, x_j]}$ defined as the disjunction of those prime implicants of f which contain *both* of x_i, x_j can be represented in the form

$$g^{(1)}[\Pi^{(0)} \cup \Gamma^{(1)} \cup \dots \cup \Gamma^{(s-1)}] \,\&\, x_i \,\&\, f^*[\Gamma^{(s)}] \,\&\, x_j \,\&\, h^{(2)}[\Gamma^{(s+1)} \cup \Pi^{(s+1)} \cup \dots \cup \Pi^{(t+1)}],$$

and this representation can be obtained also by seperating the variables in the prime implicants according to the prescribed sets of variables. (We remark that f^* does not depend on each variable of $\Gamma^{(s)}$ in general.) One can see that

$$\mathfrak{B}_{\Pi^{(0)} \cup \Gamma^{(1)} \cup \dots \cup \Gamma^{(s-1)}} \,\&\, x_i \,\&\, \mathfrak{A}_{\Gamma^{(s)}} \,\&\, x_j \,\&\, \mathfrak{B}_{\Gamma^{(s+1)} \cup \Pi^{(s+1)} \cup \dots \cup \Pi^{(t+1)}}$$

is a prime implicant of f; it is a proper partial conjuction of $\mathfrak{B}$. This contradiction proves that two conjunctions of form $\mathfrak{A}_{\Pi^{(s-1)} \cup \Gamma^{(s)} \cup \Pi^{(s)}}$ cannot be in a proper inclusion relation, consequently any conjunction of this form is a prime implicant of f_s.

The necessity of the third statement of the proposition is evident, the sufficiency can be verified by an extension of the above argument.

The irreducibility of f_s is also proved indirectly. Assume that f_s admits a nontrivial superpositional representation; denote the set of variables of the inner function by Δ, the substituted variable by x'. Consider the representation of f as the disjunction of conjuctions of type (**). Replace the single variable x' for those variables in each $\mathfrak{B}^{(s-1)}, \mathfrak{B}^{(s)}, \mathfrak{B}^{(s+1)}$ which occur in Δ. Thus we obtain a function which represents f by substituting the inner function of the representation of f_s, considered above, for x'. So Δ is separable also for f.

The quasi-series indecomposability of the functions f_s is evident.

Theorem 2. *Let f be a monotonic increasing truth function which depends effectively on its variables and is irreducible for superposition. If the quasi-series decomposition is not defined for f, then f admits no repetition-free realization by a two-terminal graph.*

Theorem 3. *Let f be a function as in Theorem 2. Suppose that the quasi-series decomposition is defined for f. f is realizable by a two-terminal graph without repetition if and only if all the quasi-series components of f admit such a realization.*

[9]) This common variable belongs necessarily to $\Pi^{(s)}$.

PROOF of Theorems 2, 3. Let f be realized by the (necessarily irreducible) graph $\mathfrak{G}$. By Theorem 1, all the concepts occurring in the course of the partial definitions can be interpreted graph-theoretically. (There exist the following correspondences: The classes $\Pi^{(s)}$ — the set of beginning edges, the set of final edges, the separating pairs. The two classes of $\Theta - \Pi_{x_i}$ (mod σ_{x_i}) — the set of the edges preceding resp. following the set corresponding to Π_{x_i}. The classes Γ — the inner edges of the quasi-series components of $\mathfrak{G}$.) These correspondences show that f cannot be a function for which the quasi-series decomposition is not defined, moreover, the quasi-series components of $\mathfrak{G}$ realize the quasi-series components of f. — Conversely, let every quasi-series component f_s of the truth function f be realizable. If we form the quasi-series composition of the graphs $\mathfrak{G}_s$ which realize the functions f_s (the identification of the edges happens according to the common variables of the functions), then we get a graph realizing f.

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