On independent systems of axioms for lattices

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1. Introduction

Let $\mathfrak{L}=(L;\cap,\cup)$ be an algebra[1]) with two binary operations denoted by $\cap$ and $\cup$. Further, let us consider the equations

$$(1)\qquad (x\cap y)\cap z = x\cap(y\cap z),$$

$$(2)\qquad (x\cup y)\cup z = x\cup(y\cup z),$$

$$(3)\qquad x\cap y = y\cap x,$$

$$(4)\qquad x\cup y = y\cup x,$$

$$(5)\qquad x\cap(x\cup y) = x,$$

$$(6)\qquad x\cup(x\cap y) = x,$$

$$(7)\qquad x\cap x = x,$$

$$(8)\qquad x\cup x = x,$$

$$(9)\qquad x\cap(y\cap z) = (x\cap y)\cap(x\cap z),$$

$$(10)\qquad x\cup(y\cup z) = (x\cup y)\cup(x\cup z),$$

$$(11)\qquad x\cap(y\cap z) = (y\cap x)\cap(z\cap x),$$

$$(12)\qquad x\cup(y\cup z) = (y\cup x)\cup(z\cup x)$$

and the relation

$$(13)\qquad x\cap y = y \quad \textit{if and only if} \quad y\cup x = x.$$

It is well-known that each of the systems

$$S_{\rm I}=\{(1)-(6)\},$$
$$S_{\rm II}=\{(1)-(4),(7),(13)\},$$
$$S_{\rm III}=\{(3)-(6),(9),(10)\}$$

[1]) In the sense of [1], p. VII. (It may be found also in [4], p. 37.)

is an independent system of axioms for lattices[2]) and it is not hard to see that also

$$S_{IV} = \{(5), (6), (11), (12)\}$$

is a such one[3]).

The purpose of our discussion is to determine independence examples consisting of the fewest possible number of elements for these axiom systems.

2. The System S_I

2. 1 For this system, R. Croisot gave (in [2], pp. 27—29) independence examples with a finite number of elements. But, there is a certain disproportionality which appears in the Croisot's examples. Namely, in order to prove the independence of (1) or (2) in S_I, he used algebras consisting of five elements, in contrast with the other axioms in S_I, whose independence was proved by algebras consisting only of two elements.

We show that this disproportionality follows as a matter of course. Indeed, we prove below

Theorem 1. *Let $\mathfrak{L} = (L, \cap, \cup)$ be an algebra consisting at most of four elements. If the axioms* (3)—(6) *hold, then either both or none of* (1) *and* (2) *are satisfied in $\mathfrak{L}$.*

For proving this theorem, we shall use the well-known facts[4]) that

(14) (5) *and* (6) *imply* (7) *and* (8),

(15) (3)—(6) *imply* (13).

Furthermore, we need also the following

Lemma. *Let $\mathfrak{M} = (M; \cap, \cup)$ be an algebra satisfying the axioms* (3)—(6). *If the equation*

$$p \cap x = p$$

for a fixed p in M has no solution other than $x = p$, then

(i) *$p \cap u = u$ for each element u of M,*

(ii) *each equation $q \cap x = q$ ($q \in M$, $q \neq p$) has at least one solution different to q.*

Indeed, by (5)

$$p \cap (p \cup u) = p$$

for every element u of M. Therefore, if the assumption of the lemma is satisfied,

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[2]) For S_I and S_{II} see, for example, [4], pp. 40—41, 44—45 and 61—64 (where references to the original papers are given too). For S_{III} see [3].

[3]) S_{III} implies S_{IV} obviously. On the other hand, (5) and (6) imply (7) and (8) (see [1], p. 18 footnote 5 or [4], p. 44, Satz 4) whence, by using (11), $a \cap b = a \cap (b \cap b) = (b \cap a) \cap (b \cap a) = b \cap a$. This proves (3). Similarly, by using (12) and (8) we get (4). Thus, S_{IV} implies S_{III}. For the independence of S_{IV}, see section 5 of this paper.

[4]) See the references in footnote 3.

then[5]) $p \cup u = p$. But

$$p \cup u = p \overset{(4)}{\Rightarrow} u \cup p = p \overset{(13)}{\Rightarrow} p \cap u = u$$

which proves (i).

Let, especially, $q \neq p$. Then, by (3) and (i),

$$q \cap p = p \cap q = q.$$

Thus the equation $q \cap x = q$ has also the solution $x = p \neq q$.

For sake of brevity, we introduce some notations. If an equation $u \cap x = u$ (with fixed u) does not hold for $x = v_1, \ldots, x = v_n$, then we write $[u \cap x; v_1, \ldots, v_n]$. Further, $(x, y, z)^\frown$ will mean that $(x \cap y) \cap z \neq x \cap (y \cap z)$; the symbol $(x, y, z)^\smile$ will be used in similar sense.

Finally, we remark that the results implied by (3) or (4) are not always enumerated explicitely.

After these preliminairies we begin the proof of Theorem 1.

2. 2 For algebras consisting of two elements the statement of the theorem may be easily proved as follows.

Let $\mathfrak{L}^\frown = (L; \cap)$ and $\mathfrak{L}^\smile = (L; \cup)$ denote the algebras (with one operation) obtained from $\mathfrak{L} = (L; \cap, \cup)$ by considering only the operation $\cap$ or $\cup$, respectively. If $\mathfrak{L} = (L; \cap, \cup)$ satisfies the axioms (3)—(6), then, by (14), $\mathfrak{L}^\frown$ and $\mathfrak{L}^\smile$ both are commutative and idempotent. Now, one can verify by direct computation that there are only two two-element algebras with a commutative and idempotent operation, namely the semilattices

	a	b
a	a	a
b	a	b

and

	a	b
a	a	b
b	b	b

Consequently, in case of two-element algebras $\mathfrak{L} = (L; \cap, \cup)$, (3)—(6) imply both of (1) and (2).

2. 3 Now, we consider the set $L_3 = \{a, b, c\}$ and we try to define an algebra $\mathfrak{L}_3 = (L_3; \cap, \cup)$ satisfying (3)—(6). Then, by (14), we have to take

(16) $$a \cap a = a, \quad b \cap b = b, \quad c \cap c = c,$$

(17) $$a \cup a = a, \quad b \cup b = b, \quad c \cup c = c.$$

Furthermore in accordance to the statement (ii) of our Lemma we may assume:

(18) *The equations $a \cap x = a$ and $b \cap y = b$ both have at least two different solutions,*

and we may use (13) by (15).

[5]) If A, B, C are arbitrary propositions, then the symbol A $\Rightarrow$ B resp. A $\overset{C}{\Rightarrow}$ B will mean „A implies B" resp. „A implies B by C".

In what follows we distinguish three cases.

2. 3. 1 *Case* $a \cap b = a$. Then:

$$a \cap b = a \overset{(3)}{\Rightarrow} b \cap a = a \overset{(13)}{\Rightarrow} a \cup b = b$$
$$\Downarrow (18)$$
(19) $$b \cap c = b \overset{(3),(13)}{\Longrightarrow} b \cup c = c.$$

Now, if we take $a \cap c = c$, then we get, by (13) and (4), $a \cup c = a$ and, consequently, $(a, b, c)^{\cap}$ and $(a, b, c)^{\cup}$; thus neither (1) nor (2) are satisfied. But if we take, on the contrary, $a \cap c \neq c$, then

$$a \cap c \neq c \overset{(3)}{\Rightarrow} c \cap a \neq c \overset{(19)(3)}{\Longrightarrow} [c \cap x;\ b, a] \Rightarrow$$
$$\overset{\text{Lemma } (i)}{\Longrightarrow} c \cap a = a \overset{(13)}{\Rightarrow} a \cup c = c$$

and thus we get the lattice in which $a < b < c$, that is (1) and (2) both are satisfied.

2. 3. 2 *Case* $a \cap b = b$. This case may be treated similarly to 2. 3. 1

2. 3. 3 *Case* $a \cap b = c$. Then

$$a \cap b = c \overset{(6)}{\Rightarrow} a \cup c = a \overset{(4)}{\Rightarrow} c \cup a = a \overset{(13)}{\Rightarrow} a \cap c = c.$$

Thus the equation $a \cap x = a$ would have only the solution $x = a$, in contradiction to our assumption (18).

By 2.3.1—2.3.3, the statement of Theorem 1 for $\mathfrak{L}_3$ is proved.

2. 4 Finally, we consider the set $L_4 = \{a, b, c, d\}$ and we try to define an algebra $\mathfrak{L}_4 = (L_4;\ \cap, \cup)$ satisfying (3)—(6). Then (16) and (17) hold again, completed also by

$$d \cap d = d \quad \text{and} \quad d \cap d = d.$$

Furthermore, similarly as before, we may assume (13) and:

(20) *The equations* $a \cap x = a, b \cap x = b$ *and* $c \cap x = c$ *have at least two different solutions.*

We distinguish four cases according to the value of $a \cap b$.

2. 4. 1 *Case* $a \cap b = d$. Then we have

(21) $$a \cap b = d \overset{(3)(6)}{\Longrightarrow} u \cup d = u \cup (a \cap b) = u \overset{(4)(13)}{\Longrightarrow} u \cap d = d \Rightarrow$$
$$\Rightarrow [u \cap x;\ b, d] \overset{(20)}{\Rightarrow} u \cap c = u \overset{(3)}{\Rightarrow} [c \cap x;\ u] \qquad (\text{for } u = a, b).$$
$$\Downarrow (3)(13)$$
$$u \cup c = c$$

The last statement implies, by our assumption (20), $c \cap d = c$. Consequently, $c \cup d = d$ and thus $(b, c, d)^{\cap}$, $(b, c, d)^{\cup}$.

2. 4. 2 *Case* $a \cap b = c$. If we take, in (21), c in place of d and conversely, then we get relations valid in the present case. $[d \cap x; a, b]$ implies (only) that either

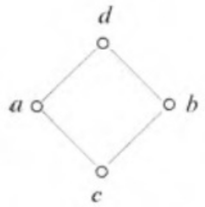

Diagramm 1

$d \cap c = d$ or $d \cap c = c$. But, as in 2. 4. 1, the relation $d \cap c = d$ would imply $(b, d, c)^{\frown}$ and $(b, d, c)^{\smile}$.

Therefore we suppose $d \cap c = c$, whence by (13) $c \cup d = d$. As for $a \cup b$, our assumption $a \cap b = c$ implies (by (13) and (4)) $a \cup b \neq a$ and $a \cup b \neq b$; further, $a \cup b = c$ would imply $a \cap c = a \cap (a \cup b) = a$, in contradiction to the result $a \cap c = c$ obtained above. Thus $a \cup b = d$, so that we get the lattice given by the diagramm 1. This means: If $a \cap b = c$ and $c \cap d = c$, then both of (1) and (2) hold.

2. 4. 3 *Case* $a \cap b = b$. During the discussion of this case we shall often use the following obvious consequences of (5) resp. (6):

(22) $$x \cap z \neq x \textit{ implies } x \cup y \neq z,$$

(23) $$x \cup z \neq x \textit{ implies } x \cap y \neq z.$$

Thus we have

(24) $$\begin{cases} a \cap b = b \overset{(22)}{\Rightarrow} a \cup y \neq b \text{ (for all } y \in L_4) \\ \Downarrow (13) \\ b \cup a = a \overset{(23)}{\Rightarrow} b \cap c \neq a. \end{cases}$$

We distinguish three sub-cases according to the value of $a \cup c$. (The sub-case $a \cup c = b$ is impossible by (24).)

2. 4. 3. 1 *Sub-case* $a \cup c = a$. Now

$$a \cup c = a \overset{(4)(13)}{\Longrightarrow} a \cap c = c$$
$$\Downarrow$$
$$[a \cap x;\, b, c] \overset{(20)}{\Rightarrow} a \cap d = a \overset{(3)(13)}{\Longrightarrow} a \cup d = d$$
$$\Downarrow (3)(22)$$

(25) $$d \cup c \neq a.$$

Thus, if we take $d \cup b \neq d$ ($\overset{(4)(13)}{\Longrightarrow} d \cap b \neq b$), then it is easy to verify $(d, a, b)^{\smile}$ and $(d, a, b)^{\frown}$, so that we suppose $d \cup b = d$. Hence we obtain

(26) $$d \cup b = d \overset{(4)(23)}{\Longrightarrow} b \cap c \neq d$$
$$\Downarrow (4)(13)$$

(27) $$d \cap b = b \overset{(22)}{\Rightarrow} d \cup c \neq b.$$

It follows by (25) and (27) that $d \cup c$ must be equal either to c or to d.

In case of $d \cup c = c$ ($\overset{(13)}{\Rightarrow} c \cap d = d$) we have $(a, c, d)^{\smile}$ and $(a, c, d)^{\frown}$.

In case of $d \cup c = d$ ($\overset{(4)(13)}{\Longrightarrow} d \cap c = c$) we take into account (24) and (26) which imply: the value of $b \cap c$ is either b or c (and, consequently, the value of $b \cup c$ is c or b, respectively). Thus we get the lattices $b < c < a < d$ and $c < b < a < d$, respectively.

2. 4. 3. 2 *Sub-case* $a\cup c=c(\overset{(13)}{\Rightarrow} c\cap a=a)$. Now, if we take $b\cap c=d$, we get

$$\left.\begin{array}{l} b\cap c=d \overset{(4)(13)(3)}{\Longrightarrow} b\cup c\neq b,c \\ \quad\Downarrow (6) \\ b\cup d=b \overset{(4)(13)}{\Longrightarrow} b\cap d=d \overset{(22)}{\Rightarrow} b\cup c\neq d \end{array}\right\} \begin{array}{c} \overset{(3)}{\Rightarrow} c\cup b=a \\ \Downarrow (5) \\ c\cap a=c, \end{array}$$

in contradiction to our assumption. Further, if we take $b\cap c=c\ (\overset{(13)}{\Rightarrow} c\cup b=b)$, then $(a,b,c)^{\cap}$ and $(a,b,c)^{\cup}$. Thus, by (24), we may suppose $b\cap c=b$. Then

$$\begin{array}{l} b\cap c=b \overset{(3)(13)}{\Longrightarrow} b\cup c=c \\ \quad\Downarrow (3) \\ [c\cap x;\ a,b] \overset{(20)}{\Rightarrow} c\cap d=c \overset{(3)(13)}{\Longrightarrow} c\cup d=d \\ \qquad\qquad\qquad\qquad\qquad \Downarrow (3)(22) \end{array}$$

(28) $$\ldots\ldots\ldots\ldots\ldots\ d\cup y\neq c \quad (\text{for all } y\in L)$$
$$\Downarrow (4)$$
(29) $$\ldots\ldots\ldots\ldots\ldots\ a\cup d\neq c.$$

Since $a\cup d=a$ implies $(a,d,c)^{\cup}$ and (by $a\cup d=a \overset{(4)(13)}{\Longrightarrow} a\cap d=d$) also $(a,d,c)^{\cap}$, by (24) and (29) we have only to treat the case of $a\cup d=d$.

But, under the assumption $a\cup d=d\ (\overset{(13)}{\Rightarrow} d\cap a=a)$, by taking $b\cap d\neq b$, we have

$$\begin{array}{l} b\cap d\neq b\Rightarrow (b,d,c)^{\cap} \\ \quad\Downarrow (3)(13) \\ b\cup d\neq d\Rightarrow (c,b,d)^{\cap} \end{array}$$

and, by taking $b\cap d=b\ (\overset{(3)(13)}{\Longrightarrow} b\cup d=d)$ we get the lattice $b<a<c<d$.

2. 4. 3. 3 *Sub-case* $a\cup c=d$. Now we have

(30) $$\begin{array}{l} a\cup c=d \overset{(5)}{\Rightarrow} a\cap d=a \overset{(3)(22)}{\Longrightarrow} d\cup b\neq a \\ \quad\Big\| (3) \qquad\quad \Downarrow (3)(13) \\ \quad\Big\| (13) \quad a\cup d=d \\ \quad\Downarrow (4) \qquad\quad \Downarrow (23) \\ \underbrace{a\cap c\neq a,c \quad a\cap c\neq d} \\ \qquad\qquad \Downarrow \\ \qquad\quad a\cap c=b \end{array}$$

$$\left.\begin{array}{l} \quad a\cap c=b \\ (3)(6)\Downarrow \\ c\cup b=c \overset{(4)(13)}{\Longrightarrow} c\cap b=b \end{array}\right\} \overset{(3)}{\Rightarrow} [c\cap x;\ a,b] \overset{(20)}{\Rightarrow} \begin{array}{c} c\cap d=c \\ \Downarrow (3)(13) \\ c\cup d=d \end{array}$$

$$\Downarrow (4)(23)$$
(31) $$b\cap d\neq c.$$

If we take $b\cap d\neq b$, then we get

$$\begin{array}{l} b\cap d\neq b \overset{(31)}{\Rightarrow} [b\cap d=a \text{ or } b\cap d=d]\Rightarrow (c,b,d)^{\cup} \\ \quad\Downarrow (13)(3) \\ b\cup d\neq d \overset{(30)(4)}{\Longrightarrow} [b\cup d=b \text{ or } b\cup d=c]\Rightarrow (c,b,d)^{\cup}. \end{array}$$

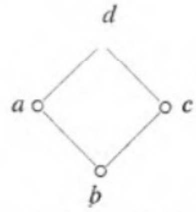

Diagramm 2

If we take, on the contrary, $b \cap d = b \ (\overset{(3)\,(13)}{\Longrightarrow} b \cup d = d)$, then we obtain the lattice given by the diagramm 2.

By 2. 4. 1—2. 4. 3, the statement of Theorem 1 is proved for $\mathfrak{L}_4$.

3. The System S_{II}

For this system I gave in [4] independence examples with two resp. three elements[6]). Especially, the independence of the axioms (1)—(4) was proved by algebras with three elements and that of (7), (9) by algebras with two elements. Now I prove that these examples are the best possible in the following sense:

Theorem 2. *Let $\mathfrak{L}=(L;\ \cap, \cup)$ be an algebra with two elements. If the axioms* (3), (4), (7) *and* (13) *hold, then either both or none of* (1) *and* (2) *are satisfied in $\mathfrak{L}$.*

Theorem 3. *Let $\mathfrak{M}=(M;\ \cap, \cup)$ be an algebra with two elements. If the axioms* (1), (2), (7) *and* (13) *hold, then either both or none of* (3) *and* (4) *are satisfied in $\mathfrak{M}$.*

PROOF. Since (7) and (13) imply (8), the discussion in section 2. 2 may be verbatim applied for S_{II} too. Thus Theorem 2 is already proved.

As for Theorem 3, let us consider the set $M=\{a, b\}$ and try to define an algebra $\mathfrak{M}=(M;\ \cap, \cup)$ satisfying the axioms (1), (2), (7) and (13). Then

$$a \cap b = a \overset{(13)}{\Rightarrow} b \cup a \neq a \Rightarrow b \cup a = b,$$

$$a \cap b = b \overset{(13)}{\Rightarrow} b \cup a = a,$$

$$b \cap a = a \overset{(13)}{\Rightarrow} a \cup b = b,$$

$$b \cap a = b \overset{(13)}{\Rightarrow} a \cup b \neq b \Rightarrow a \cup b = a.$$

Consequently

$$a \cap b = b \cap a \Rightarrow a \cup b = b \cup a,$$

and, similarly,

$$a \cup b = b \cup a \Rightarrow a \cap b = b \cap a.$$

Thus (3) and (4) are equivalent in $\mathfrak{M}$.

[6]) The independence example given in [4] for (3) (with the notation of [4], for L_3) is erroneous. A correct example is the following:

$\cap$	a	b	c
a	a	a	a
b	b	b	b
c	a	b	c

$\cup$	a	b	c
a	a	c	c
b	c	b	c
c	c	c	c

4. The System S_{III}

Felscher proved in [3]: If we assume that in the algebra $\mathfrak{L}=(L;\ \cap,\ \cup)$ the axioms (3)—(6) hold, then (1) and (9), resp. (2) and (10) are equivalent. Thus the statement of Theorem 1 remains true if we replace (1) by (9) and (2) by (10). Consequently, in order to prove the independence of (9) or (10) in S_{III}, we need use algebras with five elements.

5. The System S_{IV}

For this system we may give very simple independence examples. In fact, we show that the algebras

$$(32)\qquad \mathfrak{L}:\ \begin{array}{c|cc} \cap & a & b \\ \hline a & b & b \\ b & b & b \end{array} \qquad \begin{array}{c|cc} \cup & a & b \\ \hline a & a & a \\ b & a & b \end{array}$$

and

$$(33)\qquad \mathfrak{M}:\ \begin{array}{c|cc} \cap & a & b \\ \hline a & a & a \\ b & b & b \end{array} \qquad \begin{array}{c|cc} \cup & a & b \\ \hline a & a & a \\ b & a & b \end{array}$$

prove the independence of (5) resp. of (11) in S_{IV}; in order to prove the independence of (6) or (12) in S_{IV}, we need only ot change the signs $\cap$ and $\cup$ in (32) and (33), respectively.

For the algebra $\mathfrak{L}$:

Firstly, (5) does not hold and (11) does obviously. Next, (6) holds, because $x\cup(x\cap y) = x\cup b = x$ for each $x\in\mathfrak{L}$. Finally, $\mathfrak{L}^{\cup}$ being a (well-known) semilattice, (12) is also satisfied.[7])

For the algebra $\mathfrak{M}$:

The equations $x\cap y = x$ resp. $x\cup(x\cap y) = x\cup x = x$ being true for each $x\in\mathfrak{M}$, the axioms (5) and (6) are satisfied in $\mathfrak{M}$. (12) is also satisfied (by the same argument as for $\mathfrak{L}$). Finally, (11) does not hold: $a\cap(b\cap b) = a$ and $(b\cap a)\cap(b\cap a) = b$.

We call the attention of the reader that, in consideration of the independence examples, the axiom system S_{IV} is as simple as possible.

References

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[7]) Indeed, $x\cup(y\cup z)=(x\cup y)\cup(x\cup z)$ may be obtained by the argument used in [3], p. 172; hence, by use of the commutativity, we get (12).