A note on endomorphism groups

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Suppose G is an abelian p-group with endomorphism group End G and endomorphism ring $\mathcal{E}(G)$. It is well known that G is determined by $\mathcal{E}(G)$ [1]. It is also known that most abelian p-groups are determined by their multiplicative groups of automorphisms [3]. Each of these theorems depends on the multiplicative structure of $\mathcal{E}(G)$ in an essential way. It is then unlikely that G should be determined by End G, and this is indeed the case. One may, however, ask for a characterization of those abelian p-groups which are determined by their endomorphism groups (Cf., Problem 41 of [1]). While this problem appears difficult for the class of all abelian p-groups, it is not difficult for the class of p-groups which are direct sums of cyclic groups and it is the purpose of this note characterize those groups in this restricted class which are determined by their endomorphism group. (It should be noted that we shall assume the generalized continuum hypothesis for this result.)

Lemma 1. *Let*

$$G \cong \sum_{i=1}^{\infty} \sum_{n_i} C(p^i), \qquad 0 \leqq n_i.$$

Put

$$s_i = \sum_{k \geqq i} n_k.$$

Then

$$\text{End}\, G \cong \prod_{i=1}^{\infty} \sum_{u_i} C(p^i),$$

where

$$u_i = \begin{cases} 0, & \text{if } n_i = 0, \\ n_i^2 + 2n_i s_{i+1}, & \text{if } n_i \neq 0 \text{ and } s_i \text{ is finite}, \\ 2^{s_i} & \text{if } n_i \neq 0 \text{ and } s_i \text{ is infinite}. \end{cases} \tag{1}$$

PROOF. $$\text{End}\, G \cong \prod_{i=1}^{\infty} \prod_{n_i} \text{Hom}\,(C(p^i), G) \cong \prod_{i=1}^{\infty} \prod_{n_i} G[p^i] \cong$$

$$\cong \prod_{i=1}^{\infty} \prod_{n_i} \left(\sum_{j=1}^{i} \sum_{n_j} C(p^j) + \sum_{j=i+1}^{\infty} \sum_{n_j} C(p^i) \right) \cong \prod_{i=1}^{\infty} \prod_{n_i} \prod_{j=1}^{i} \sum_{n_j} C(p^j) + \prod_{i=1}^{\infty} \prod_{n_i} \sum_{s_{i+1}} C(p^i).$$

Now

$$\prod_{i=1}^{\infty}\prod_{n_i}\prod_{j=1}^{i}\sum_{n_j} C(p^j) \cong \prod_{i=1}^{\infty}\prod_{j=1}^{i}(\prod_{n_i}\sum_{n_j} C(p^j)) \cong$$

$$\cong \prod_{j=1}^{\infty}\prod_{i=j}^{\infty}(\prod_{n_i}\sum_{n_j} C(p^j)) \cong \prod_{j=1}^{\infty}\prod_{s_j}\sum_{n_j} C(p^j).$$

Therefore

$$\operatorname{End} G \cong \prod_{i=1}^{\infty}(\prod_{s_i}\sum_{n_i} C(p^i) + \prod_{n_i}\sum_{s_{i+1}} C(p^i)) \cong \prod_{i=1}^{\infty}\sum_{u_i} C(p^i),$$

where u_i is given by (1).

Lemma 2. *Let G and H be abelian p-groups,*

$$G \cong \sum_{i=1}^{\infty}\sum_{n_i} C(p^i), \qquad 0 \leqq n_i,$$

and

$$H \cong \sum_{i=1}^{\infty}\sum_{m_i} C(p^i), \qquad 0 \leqq m_i.$$

Let $s_i = \sum_{k \geqq i} n_k$ and $t_i = \sum_{k \geqq i} m_k$. Then End $G \cong$ End H *if and only if*

(i) *for any i, $n_i = 0$ if and only if $m_i = 0$,*

and

(ii) $2^{s_i} = 2^{t_i}$ *for every i.*

Proof. By Lemma 1,

$$\operatorname{End} G \cong \prod_{i=1}^{\infty}\sum_{u_i} C(p^i)$$

and

$$\operatorname{End} H \cong \prod_{i=1}^{\infty}\sum_{v_i} C(p^i),$$

where u_i given by (1) and

$$v_i = \begin{cases} 0, & \text{if } m_i = 0, \\ m_i^2 + 2m_i t_{i+1}, & \text{if } m_i \neq 0 \text{ and } t_i \text{ is finite}, \\ 2^{t_i}, & \text{if } m_i \neq 0 \text{ and } t_i \text{ is infinite}. \end{cases}$$

It follows readily that (i) and (ii) imply End $G \cong$ End H. The converse follows from the fact that a basic subgroup of the torsion subgroup of

$$\prod_{i=1}^{\infty}\sum_{w_i} C(p^i), \qquad 0 \leqq w_i,$$

is isomorphic with

$$\sum_{i=1}^{\infty}\sum_{w_i} C(p^i).$$

For the following proposition we assume the generalized continuum hypothesis. In particular, we assume that for any cardinal numbers α and β, $2^\alpha = 2^\beta$ implies $\alpha = \beta$.

Proposition 1. *Let G be a p-group which is a direct sum of cyclic groups,*

$$G \cong \sum_{i=1}^{\infty} \sum_{n_i} C(p^i), \qquad 0 \leqq n_i.$$

Then End *G determines G if and only if G is a bounded group and the following condition holds for G: If for some index i, n_i is infinite, then $n_j > n_i$ whenever $j < i$ and $n_j \neq 0$.*

PROOF. Let $s_i = \sum_{k \geqq i} n_k$. If G is unbounded there is an index j with $n_j \leqq s_j$ (otherwise the n_i's would form an infinite strictly decreasing sequence of cardinal numbers). Set $m_i = n_i$ for each $i \neq j$ and let m_j be any cardinal number different from n_j and not exceeding s_j. Put

$$H \cong \sum_{i=1}^{\infty} \sum_{m_i} C(p^i).$$

Then End $H \cong$ End G but $H \not\cong G$. Hence if G is determined by End G, G must be bounded. Let k be an index such that n_k is infinite. Suppose there were $n_j, j < k$, such that $0 \neq n_j \leqq n_k$. Define H,

$$H \cong \sum_{i=1}^{\infty} \sum_{m_i} C(p^i)$$

by cardinal numbers m_i such that $m_i = n_i$ if $i \neq j$ and m_j is any cardinal number different from n_j and not exceeding n_k. Then $s_i = t_i$ whenever $i > j$. If $i < j$, then $s_{j+1} \geqq n_k \geqq n_j$ and $n_k \geqq m_j$ imply $n_j + s_{j+1} = m_j + s_{j+1} = m_j + t_{j+1}$ and so $s_i = n_i + \ldots + n_{j-1} + (n_j + s_{j+1}) = m_i + \ldots + m_{j-1} + (m_j + t_{j+1}) = t_i$. That is, $s_i = t_i$ for every i and End $G \cong$ End H by Lemma 2. But clearly $G \not\cong H$. This proves the necessity.

To prove sufficiency, let

$$G \cong \sum_{i=1}^{r} \sum_{n_i} C(p^i), \qquad 0 \leqq n_i.$$

If each n_i is finite then G is finite and therefore is determined by End G. So suppose some n_i is infinite. Let H

$$H \cong \sum_{i=1}^{r} \sum_{m_i} C(p^i), \qquad 0 \leqq m_i,$$

be such that End $G \cong$ End H. Let k be the largest index for which n_k is infinite. Then $s_i = t_i$ for all $i > k$ and hence $n_i = m_i$ for all $i > k$. Let $j \leqq k$. End $G \cong$ End H implies $2^{s_j} = 2^{t_j}$ which implies (assuming the generalized continuum hypothesis) that $s_j = t_j$. Since the condition holds for G, if $n_j \neq 0$ then $n_j = s_j = t_j = m_j$. Thus $n_i = m_i$ for every i and $G \cong H$. Hence End G determines G.

Clearly any p-group G which is not determined by its endomorphism group has the property that there exists another p-group H with End $H \cong$ End G but $\mathscr{E}(H) \not\cong \mathscr{E}(G)$ (Cf. Problem 42 of [1]). Proposition 1 then provides simple examples of such p-groups. For example, any unbounded direct sum of cyclic p-groups is one of these. (See [2] for a more complicated example.)

Problem 44 of [1] asks for a characterization of groups which are endomorphism groups. We specialize this problem and conclude with the following

Proposition 2. *A p-group T is the group of endomorphisms of some group G if and only if T has the form*

$$T \cong \sum_{i=1}^{r} \sum_{n_i} C(p^i),$$

and there exist cardinal numbers $m_1, \ldots, m_r$ such that if $s_i = \sum_{k=i}^{r} m_k$, then $n_i = 2^{s_i}$ if s_i infinite and $n_i = m_i^2 + 2m_i s_{i+1}$ if s_i is finite.

PROOF. If T has the indicated form and such cardinal numbers exist, then T is isomorphic to the endomorphism group of

$$G \cong \sum_{i=1}^{r} \sum_{m_i} C(p^i).$$

If $T \cong \text{End}\, G$ for some G, then G is a p-group bounded by the order of the identity endomorphism. Write G as above for some cardinal numbers $m_1, \ldots, m_r$. Then T has the form indicated in the proposition.

REMARK. Since this note was prepared, R. S. PIERCE has kindly provided this author with a pre-print of his forth-coming paper *The Homomorphism Groups of Primary Abelian Groups* which contains results analagous to Lemmas 1 and 2 for arbitrary reduced abelian p-groups.

Bibliography

[1] L. FUCHS, Abelian Groups, *Budapest*, 1958.
[2] D. K. HARRISON, Two of the problems of L. Fuchs, *Publ. Math. Debrecen* **7** (1960), 316–319.
[3] H. LEPTIN, Abelsche p-Gruppen und ihre Automorphismen-gruppen, *Math. Z.* **73** (1960), 235–253.

(Received November 25, 1962)