On functions of three vectors*

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Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be three vector spaces (not necessarily of the same dimension). We say that a real-valued function f is defined on $\mathbf{X}\times\mathbf{Y}\times\mathbf{Z}$ if, with each ordered triple of vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ belonging to $\mathbf{X}, \mathbf{Y}$, and $\mathbf{Z}$, respectively, a real number $f(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is associated.

Theorem. *If a real-valued function on* $\mathbf{X}\times\mathbf{Y}\times\mathbf{Z}$ *is superadditive in* $\mathbf{x}$ *and* $\mathbf{y}$ *for every* $\mathbf{z}$, *and subbadditive in* $\mathbf{x}$ *and* $\mathbf{z}$ *for every* $\mathbf{y}$, *then* f *is additive in* $\mathbf{x}$ *and* $\mathbf{y}$ *for every* $\mathbf{z}$, *as well as in* $\mathbf{x}$ *and* $\mathbf{z}$ *for every* $\mathbf{y}$. *In other words, the inequalities*

(I) $f(\mathbf{x}+\mathbf{x}_1, \mathbf{y}+\mathbf{y}_1, \mathbf{z}) \geqq f(\mathbf{x}, \mathbf{y}, \mathbf{z})+f(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z})$

(I′) $f(\mathbf{x}+\mathbf{x}_1, \mathbf{y}, \mathbf{z}+\mathbf{z}_1) \leqq f(\mathbf{x}, \mathbf{y}, \mathbf{z})+f(\mathbf{x}_1, \mathbf{y}, \mathbf{z}_1)$

for all vectors $\mathbf{x}, \mathbf{x}_1, \mathbf{y}, \mathbf{y}_1, \mathbf{z}, \mathbf{z}_1$ *jointly imply the two opposite inequalities.*

It goes without saying that f is not necessarily additive in the three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

We begin by proving some auxiliary formulae in which $\mathbf{o}$ denotes the vector all components of which are 0. From (I) it follows that $f(\mathbf{o}, \mathbf{o}, \mathbf{z}) \geqq 2f(\mathbf{o}, \mathbf{o}, \mathbf{z})$ for any $\mathbf{z}$. Hence

(1) $$f(\mathbf{o}, \mathbf{o}, \mathbf{z}) \leqq 0 \text{ for any } \mathbf{z}.$$

Similarly, (I′) implies

(1′) $$f(\mathbf{o}, \mathbf{y}, \mathbf{o}) \geqq 0 \text{ for any } \mathbf{y}.$$

Hence, in particular,

(1*) $$f(\mathbf{o}, \mathbf{o}, \mathbf{o}) = 0.$$

Clearly, by virtue of (I) and (I′),

(2*) $$f(\mathbf{x}+\mathbf{x}_1, \mathbf{o}, \mathbf{o}) = f(\mathbf{x}, \mathbf{o}, \mathbf{o})+f(\mathbf{x}_1, \mathbf{o}, \mathbf{o}),$$

whence

(3*) $$f(-\mathbf{x}, \mathbf{o}, \mathbf{o}) = -f(\mathbf{x}, \mathbf{o}, \mathbf{o}) \text{ for any } \mathbf{x}.$$

(1*) and (I′) imply that $0=f(\mathbf{o}, \mathbf{o}, \mathbf{o}) \geqq f(\mathbf{x}, \mathbf{y}, \mathbf{o})+f(-\mathbf{x}, -\mathbf{y}, \mathbf{o})$. Thus

(4) $$f(\mathbf{x}, \mathbf{y}, \mathbf{o})+f(-\mathbf{x}, -\mathbf{y}, \mathbf{o}) \leqq 0 \text{ for any } \mathbf{x} \text{ and } \mathbf{y},$$

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and similarly,

(4′) $$f(\mathbf{x}, \mathbf{o}, \mathbf{z}) + f(-\mathbf{x}, \mathbf{o}, -\mathbf{z}) \geqq 0 \text{ for any } \mathbf{x} \text{ and } \mathbf{z}.$$

From (4) and (4′), for $\mathbf{x} = \mathbf{o}$, in view of (1) and (1′), we obtain

(5*) $$f(\mathbf{o}, \mathbf{y}, \mathbf{o}) = 0 = f(\mathbf{o}, \mathbf{o}, \mathbf{z}) \text{ for any } \mathbf{y} \text{ and } \mathbf{z}.$$

By (**I**), (5*), and (**I′**) we find

$$f(\mathbf{o}, -\mathbf{y}, \mathbf{z}) + f(\mathbf{o}, \mathbf{y}, \mathbf{z}) \leqq f(\mathbf{o}, \mathbf{o}, \mathbf{z}) = 0 = f(\mathbf{o}, -\mathbf{y}, \mathbf{o}) \leqq f(\mathbf{o}, -\mathbf{y}, \mathbf{z}) + f(\mathbf{o}, -\mathbf{y}, -\mathbf{z})$$

and, consequently,

$$f(\mathbf{o}, \mathbf{y}, \mathbf{z}) \leqq f(\mathbf{o}, -\mathbf{y}, -\mathbf{z}) \text{ for any } \mathbf{y} \text{ and } \mathbf{z}.$$

Here, $\mathbf{y}, \mathbf{z}$ and $-\mathbf{y}, -\mathbf{z}$ may change roles, whence

(6*) $$f(\mathbf{o}, \mathbf{y}, \mathbf{z}) = f(\mathbf{o}, -\mathbf{y}, -\mathbf{z}) \text{ for any } \mathbf{y} \text{ and } \mathbf{z}.$$

By (5*), (**I′**), (6*), and (**I**) we have

$$0 = f(\mathbf{o}, \mathbf{y}, \mathbf{o}) \leqq f(\mathbf{o}, \mathbf{y}, \mathbf{z}) + f(\mathbf{o}, \mathbf{y}, -\mathbf{z}) = f(\mathbf{o}, \mathbf{y}, \mathbf{z}) + f(\mathbf{o}, -\mathbf{y}, \mathbf{z}) \leqq f(\mathbf{o}, \mathbf{o}, \mathbf{z}) = 0.$$

Therefore,

(7*) $$f(\mathbf{o}, \mathbf{y}, -\mathbf{z}) = -f(\mathbf{o}, \mathbf{y}, \mathbf{z}) = f(\mathbf{o}, -\mathbf{y}, \mathbf{z}) \text{ for any } \mathbf{y} \text{ and } \mathbf{z}.$$

By (**I**), (4), (5*), and (3*) we find

$$f(\mathbf{x}, \mathbf{y}, \mathbf{o}) \leqq -f(-\mathbf{x}, -\mathbf{y}, \mathbf{o}) \leqq -f(-\mathbf{x}, \mathbf{o}, \mathbf{o}) - f(\mathbf{o}, -\mathbf{y}, \mathbf{o}) = -f(-\mathbf{x}, \mathbf{o}, \mathbf{o}) =$$
$$= f(\mathbf{x}, \mathbf{o}, \mathbf{o}) = f(\mathbf{x}, \mathbf{o}, \mathbf{o}) + f(\mathbf{o}, \mathbf{y}, \mathbf{o}) \leqq f(\mathbf{x}, \mathbf{y}, \mathbf{o}).$$

Consequently,

(8) $$f(\mathbf{x}, \mathbf{o}, \mathbf{o}) = f(\mathbf{x}, \mathbf{y}, \mathbf{o}) \text{ for any } \mathbf{x} \text{ and } \mathbf{y}.$$

Similarly,

(8′) $$f(\mathbf{x}, \mathbf{o}, \mathbf{o}) = f(\mathbf{x}, \mathbf{o}, \mathbf{z}) \text{ for any } \mathbf{x} \text{ and } \mathbf{z}.$$

By (8′), (**I**), and (**I′**), we find

$$f(\mathbf{x}, \mathbf{o}, \mathbf{o}) + f(\mathbf{o}, \mathbf{y}, \mathbf{z}) = f(\mathbf{x}, \mathbf{o}, \mathbf{z}) + f(\mathbf{o}, \mathbf{y}, \mathbf{z}) \leqq f(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leqq$$
$$\leqq f(\mathbf{x}, \mathbf{y}, \mathbf{o}) + f(\mathbf{o}, \mathbf{y}, \mathbf{z}) = f(\mathbf{x}, \mathbf{o}, \mathbf{o}) + f(\mathbf{o}, \mathbf{y}, \mathbf{z}).$$

Therefore

(9*) $$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = f(\mathbf{x}, \mathbf{o}, \mathbf{o}) + f(\mathbf{o}, \mathbf{y}, \mathbf{z}) \text{ for any } \mathbf{x}, \mathbf{y}, \mathbf{z}.$$

From (7*) and (**I**) it follows that

$$f(\mathbf{x}, \mathbf{y} + \mathbf{y}_1, \mathbf{z}) - f(\mathbf{o}, \mathbf{y}_1, \mathbf{z}) = f(\mathbf{x}, \mathbf{y} + \mathbf{y}_1, \mathbf{z}) + f(\mathbf{o}, -\mathbf{y}_1, \mathbf{z}) \leqq f(\mathbf{x}, \mathbf{y}, \mathbf{z}).$$

Hence by (9*)

$$f(\mathbf{x}, \mathbf{y} + \mathbf{y}_1, \mathbf{z}) \leqq f(\mathbf{x}, \mathbf{y}, \mathbf{z}) + f(\mathbf{o}, \mathbf{y}_1, \mathbf{z}) = f(\mathbf{x}, \mathbf{y}, \mathbf{z}) + f(\mathbf{x}, \mathbf{y}_1, \mathbf{z}) - f(\mathbf{x}, \mathbf{o}, \mathbf{o}).$$

Thus,

$$f(\mathbf{x}, \mathbf{y}+\mathbf{y}_1, \mathbf{z}) \leqq f(\mathbf{x}, \mathbf{y}, \mathbf{z})+f(\mathbf{x}, \mathbf{y}_1, \mathbf{z})-f(\mathbf{x}, \mathbf{o}, \mathbf{o}) \tag{10}$$

and, similarly,

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}+\mathbf{z}_1) \geqq f(\mathbf{x}, \mathbf{y}, \mathbf{z})+f(\mathbf{x}, \mathbf{y}, \mathbf{z}_1)-f(\mathbf{x}, \mathbf{o}, \mathbf{o}). \tag{10'}$$

From (10), (I′), (8), and (2*) it follows that

$$\begin{aligned} f(\mathbf{x}+\mathbf{x}_1, \mathbf{y}+\mathbf{y}_1, \mathbf{z}) &\leqq f(\mathbf{x}+\mathbf{x}_1, \mathbf{y}, \mathbf{z})+f(\mathbf{x}+\mathbf{x}_1, \mathbf{y}_1, \mathbf{z})-f(\mathbf{x}+\mathbf{x}_1, \mathbf{o}, \mathbf{o}) \leqq \\ &\leqq f(\mathbf{x}, \mathbf{y}, \mathbf{z})+f(\mathbf{x}_1, \mathbf{y}, \mathbf{o})+f(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z})+f(\mathbf{x}, \mathbf{y}_1, \mathbf{o})-f(\mathbf{x}+\mathbf{x}_1, \mathbf{o}, \mathbf{o})= \\ &= f(\mathbf{x}, \mathbf{y}, \mathbf{z})+f(\mathbf{x}_1, \mathbf{o}, \mathbf{o})+f(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z})+f(\mathbf{x}, \mathbf{o}, \mathbf{o})-f(\mathbf{x}+\mathbf{x}_1, \mathbf{o}, \mathbf{o})= \\ &= f(\mathbf{x}, \mathbf{y}, \mathbf{z})+f(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}). \end{aligned}$$

Therefore

$$f(\mathbf{x}+\mathbf{x}_1, \mathbf{y}+\mathbf{y}_1, \mathbf{z}) \leqq f(\mathbf{x}, \mathbf{y}, \mathbf{z})+f(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}) \tag{11}$$

and, similarly,

$$f(\mathbf{x}+\mathbf{x}_1, \mathbf{y}, \mathbf{z}+\mathbf{z}_1) \geqq f(\mathbf{x}, \mathbf{y}, \mathbf{z})+f(\mathbf{x}_1, \mathbf{y}, \mathbf{z}_1). \tag{11'}$$

This completes the proof of our theorem.

In conclusion, we mention a corollary of (8) and (8′). If the vector space $\mathbf{Z}$ has the dimension 0, then f is independent of $\mathbf{y}$. If the dimension of $\mathbf{Y}$ is 0, then f is independent of $\mathbf{z}$.

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