A self-dual theory of real determinants

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One of the salient features of the theory of determinants is the metamathematical fact that every theorem about rows is also valid for columns, and vice versa. The traditional foundations of the theory, however, lean towards one side. WEIERSTRASS[1]), CARATHÉODORY[2]), and SCHREIER[3]) introduced determinants as matrix functions postulated to be linear (i. e., additive and homogeneous) in the *rows* and to assume the value 0 for every matrix with two equal *rows*. They then proved, among other theorems, the corresponding facts for *columns*. ARTIN[4]) more recently based a variant of that theory on postulates concerning *columns*, and GÁSPÁR[5]) developed determinants from new, natural postulates concerning either rows or columns.

In the present paper we show that, for real matrix functions, sublinearity in the columns and superlinearity in the rows, combined with nonpositivity for matrixes with equal columns and nonnegativity for matrices with equal rows constitute a self-dual foundation of the theory. [6]) The following proof of this result of one of us[7]) makes essential use of a theorem on functions of three vectors by the other one[8]).

If A is a square matrix of n^2 elements A^i_k, we denote[9]) the i-th row of A by A_i, and the k-th column by A_k. Thus A_i and A^k are the sequences

$$A^1_i, \ldots, A^n_i \quad \text{and} \quad A^k_1, \ldots, A^k_n,$$

[1]) K. WEIERSTRASS, Werke, vol. 3, p. 271. It is interesting that in MUIR's voluminous compendium on determinants WEIERSTRASS' fundamental paper is not even mentioned.

[2]) C. CARATHÉODORY, Reelle Functionen, 1 st. ed., *Leipzig*, 1918, p. 318.

[3]) O. SCHREIER—E. SPERNER, Einführung in die Analytische Geometrie und Algebra I, *Leipzig*, 1931, p. 69.

[4]) E. ARTIN—N. MILGRAM, Galois Theory, 2 nd. ed., *Notre Dame, Indiana* 1944, p. 11.

[5]) G. GÁSPÁR, Eine neue Definition der Determinanten, *Publ. Math. Debrecen* **3** (1953—54) 257—260. Cf. also the interesting related paper by M. STOJAKOVIÇ, Une théorie générale des déterminants, *Bull. Soc. Math. Phys. Serbie* **6** (1954) 41—55.

[6]) Other self-dual theories can be found in A. CLIMESCU, Une définition axiomatiquedes des déterminants, *Bul. Inst. Politehn Iasi* **2** (6) (1956) nr. 3—4, 1—7, and A. BERGMANN, Ein Axiomensystem für Determinanten, *Arch. Math.* **10** (1959) 243—256.

[7]) K. MENGER, Une théorie aximatique générale des déterminants *C. R. Acad. Sci. Paris*, **234** (1952), pp. 1941—1943.

[8]) F. KOZIN, On Functions of Three Vectors, *preceding paper*.

[9]) We here adopt MENGER's typographical convention according to which references to numbers (such as i, j, n, ...) are printed in roman type while functions are designated in *italic* type. A matrix A is a function, A^k_i is its value for i and k. A row A_i is a 1-place function assuming for k the value A^k_i. It may also be considered as a vector. In the proof of Theorem II., vectors will also be denoted by letters **x**, **y**, **z** in bold type.

respectively. Both rows and columns are referred to as *arrays* of the matrix. Arrays will be treated like vectors. For instance,

$$A^{k} = B^{k} + C^{k} \quad \text{means} \quad A_i^k = B_i^k + C_i^k \quad \text{for} \quad 1 \leqq i \leqq n.$$

In this notation, the afore-mentioned result reads:

Theorem I. *A real-valued function f of the square matrices with* n^2 *real elements is the determinant if and only if f has the following properties:*

($I^k \leqq$) ($1 \leqq k \leqq n$). *f is subadditive in the* k*-th column*, that is to say, for any three matrices A, B, C,

$$\text{if } A^k = B^k + C^k \text{ and } A^{k'} = B^{k'} = C^{k'} \text{ for each } k' \neq k, \text{ then } f(A) \leqq f(B) + f(C).$$

($I_i \geqq$) ($1 \leqq i \leqq n$). *f is superadditive in the* i*-th row*, that is,

$$A_i = B_i + C_i \text{ and } A_{i'} = B_{i'} = C_{i'} \text{ for } i' \neq i \text{ implies } f(A) \geqq f(B) + f(C).$$

($II^k \leqq$) ($1 \leqq k \leqq n$). *f is positively subhomogeneous in the* k*-th column*, that is, if $c \geqq 0$, then for any two matrices A and B,

$$A^k = cB^k \text{ and } A^{k'} = B^{k'} \text{ for } k \neq k' \text{ imply } f(A) \leqq cf(B).$$

($II_i \geqq$) ($1 \leqq i \leqq n$). *f is positively superhomogeneous in the* i*-th row*.

(III^{hj}) ($1 \leqq h, j \leqq n, h \neq j$). *f is nonpositive if the* h*-th and* j*-th column are equal*, that is, $A^h = A^j$ implies $f(A) \leqq 0$;

(III_{hj}). *f is nonnegative if the* h*-th and* i*-th row are equal.*

(IV). *$f(D) = 1$ if D is the unit matrix*, for which $D_i^k = 1$ or 0 according as $i = k$ or $\neq k$.

We begin by drawing conclusions from Postulates ($I^k \leqq$) and ($I_k \geqq$) alone.

Theorem II. *If a real-valued function f of square matrices is subadditive in the columns and superadditive in the rows, then f is additive in all arrays.*

For any square matrix A of n^2 numbers and any j ($1 \leqq j \leqq n$), let $A[j]$ be the matrix obtained by deleting from A the j-th row and the j-th column. Conversely, let B be a square matrix of $(n-1)^2$ numbers

$$A_i^k (i, k = 1, \ldots, j-1, j+1, \ldots, n);$$

and let **x**, **y**, and **z** be a triple of vectors, belonging to vector spaces of the dimensions 1, n − 1, and n − 1, respctively. If their components are

$$\mathbf{x} = \{A_j^j\}, \ \mathbf{y} = \{A_1^j, \ldots, A_{j-1}^j, A_{j+1}^j, \ldots, A_n^j\}, \ \mathbf{z} = \{A_j^1, \ldots, A_j^{j-1}, A_j^{j+1}, \ldots, A_j^n\},$$

then the matrix of the n^2 numbers A_i^k will be denoted by

$$\langle B, \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle$$

Clearly, $\langle B, \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle [j] = B$.

For each square matrix B of $(n-1)^2$ numbers, we introduce a function g_B on the triples of vectors of dimensions 1, n − 1, and n − 1 by defining its value for the triple **x**, **y**, **z** as follows:

$$g_B(\mathbf{x}, \mathbf{y}, \mathbf{z}) = f(\langle B, \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle).$$

For each B, by virtue of $(\mathbf{I}^j \leqq)$, the function g_B is subadditive in $\mathbf{x}$ and $\mathbf{y}$ for each $\mathbf{z}$; by virtue of $(\mathbf{I}_j \geqq)$, it is superadditive in $\mathbf{x}$ and $\mathbf{z}$ for each $\mathbf{y}$. According to the theorem on functions of three vectors, loc. cit[8]), the function g_B, therefore, is additive 1) in $\mathbf{x}$ and $\mathbf{y}$ for each $\mathbf{z}$, and 2) in $\mathbf{x}$ and $\mathbf{z}$ for each $\mathbf{y}$. Because of 1), the function f is additive in the j-th column; because of 2) it is additive in the j-th row. This completes the proof of Theorem II.

Clearly, a function that is additive in a certain array assumes the value 0 for each matrix in which that array is O. (The italic letter O will denote an array each element of which is the number 0.) Hence Theorem II has the following

Corollary 1. *A function f satsfying Postulates* $(\mathbf{I}^k \leqq)$ *and* $(\mathbf{I}_i \geqq)$ $(1 \leqq i, k \leqq n)$ *assumes the value* 0 *for each matrix including an array* O.

It further follows that, since f is additive in the first column,

$$f(A) = f(B_1) + f(B_2) + \ldots + f(B_n),$$

where B_i is obtained from A by replacing with 0 all elements in the first column except A_i^1. Since f is additive in the second column,

$$f(B_i) = f(B_{i,1}) + f(B_{i,2}) + \ldots + f(B_{i,n}) \quad \text{for} \quad 1 \leqq i \leqq n,$$

where $B_{i,j}$ is obtained from B_i by replacing with 0 all elements in the second column except A_j^2.

Continuing this producere one can represent $f(A)$ as the sum of n^n terms of the form $f(B_{i_1, i_2, \ldots, i_n})$, where $B_{i_1, i_2, \ldots, i_n}$ is the matrix whose k-th column, besides $A_{i_k}^k$ includes only 0 (for $1 \leqq k \leqq n$). With at most n! exceptions, each of the n^n terms equals 0 since every nonexceptional matrix includes at least one row O. The n! exceptional matrices are those for which $i_1, \ldots, i_n$ is a permutation π of $1, \ldots, n$. Such a matrix, obtained by replacing with 0 all numbers in A except exactly one in each array, will be denoted by $A\pi$ and called *quasi-diagonal*.

Corollary 2. *If f satisfies Postulates* $(\mathbf{I}^k \leqq)$ *and* $(\mathbf{I}_i \geqq)$, *then*

$$f(A) = \sum_{\pi} f(A\pi), \tag{1}$$

where the summation is extended over the n! *permutations of* 1, 2, ..., n.

We shall call A_h^j a *cross-element* of the matrix A if all the $2n-1$ other elements in the row A_h and the column A^j are 0. (The cross-element itself may or may not be 0.) If the cross-element A_h^j is positive, then from $(\mathbf{II}^j \leqq)$ and $(\mathbf{II}_h \geqq)$ it follows that

$$f(A) = A_h^j \cdot f(A^*), \tag{2}$$

where A^* denotes the matrix obtained from A by replacing A_h^j with 1. If the cross-element A_h^j is negative, then let B denote the matrix obtained from A by replacing A_h^j with $-A_h^j$. Since $A_h + B_h = 0$ and f is additive in the rows, it follows that

$f(A)+f(B) = 0$. Using ($\mathbf{II}^j \leqq$) and ($\mathbf{II}_h \geqq$) we find

$$f(A) = -f(B) = -(-A^j_h)\cdot f(A^*) = A^j_h \cdot f(A^*).$$

If $A^j_h = 0$, then $f(A)=0$. Thus (2) holds for each cross-element.

Since in the quasi-diagonal matrix $A\pi = B_{i_1 \ldots i_n}$ each element $A^k_{i_k}$ is a cross-element, it follows that

$$f(A\pi) = A^1_{i_1} \cdot A^2_{i_2} \cdot \ldots \cdot A^n_{i_n} \cdot f(D\pi),$$

where $D\pi$ is the quasi-diagonal unit matrix obtained by replacing each cross-element of A with 1. From (1) it thus follows

$$f(A) = \sum_\pi A^1_{i_1} \cdot A^2_{i_2} \cdot \ldots \cdot A^n_{i_n} \cdot f(D\pi). \tag{3}$$

The equality of $f(A)$ with the determinant $|A|$ will be established by proving that in (3)

(4) $f(D\pi) = 1$ or -1 according as π is an even or odd permutation.

Since by virtue of **IV**, (4) holds if π is the identity we merely have to show that f assumes opposite values (thus either 1 or -1) for any two quasi-diagonal unit matrices obtained from one another by interchanging two rows.

Consider a quasi-diagonal matrix A with cross-elements $A^j_h = 1$ and $A^k_i = 1$. We shall say that A contains

$$\begin{matrix} 1^j_h & 0^k_h \\ 0^j_i & 1^k_h. \end{matrix}$$

The matrix A' obtained from A by interchanging the columns A^j and A^k (or, which is tantamount, the rows A_h and A_i) contains

$$\begin{matrix} 0^j_h & 1^k_h \\ 1^j_i & 0^k_i \end{matrix}$$

and is otherwise identical with A. In order to prove $f(A) = -f(A')$, let C'' be the matrix that contains

$$\begin{matrix} 0^j_h & 1^k_h \\ 0^j_i & 1^k_i \end{matrix}$$

but otherwise is identical with A and A'. Since the j-th column of C'' is O, Corollary 1 implies that $f(C'')=0$. Now A' and C'' differ only in the i-th row. The matrix C' obtained by adding the i-th rows of A' and C'' contains

$$\begin{matrix} 0^j_h & 1^k_h \\ 1^j_i & 1^k_i \end{matrix}.$$

Since f is additive in the rows it follows that

$$f(C') = f(A') + f(C'') = f(A').$$

Similarly,

$$f(C) = f(A),$$

where C is the matrix containing

$$\begin{matrix} 1_h^j & 1_h^k \\ 0_i^j & 1_i^k \end{matrix} \qquad (h \neq i,\ j \neq k)$$

and otherwise identical with A and A'. Now C and C' differ only in the j-th column. The matrix A'' obtained by adding the j-th columns of C and C' contains

$$\begin{matrix} 1_h^j & 1_h^k \\ 1_i^j & 1_i^k \end{matrix}.$$

From Postulates ($\mathbf{III}^{jk} \leqq$) and ($\mathbf{III}_{hi} \geqq$) it follows that $f(A'') = 0$. By virtue of the additivity of f, we have

$$f(C) + f(C') = f(A'') = 0 \text{ and thus } f(A) + f(A') = 0.$$

This completes the proof of Theorem I.

(Received December 5, 1962.)