A note on order relations

By VLADIMIR DEVIDÉ (Zagreb)

1. Let S be a set and $\mathfrak{U}S$ the set of all subsets of S.

DEFINITIONS. a) Any subset $\mathfrak{P}$ of $\mathfrak{U}S$, having the power of S ($\overline{\overline{\mathfrak{P}}} = \overline{\overline{S}}$), we call a *P-partial ordering* of S.

b) A P-partial ordering $\mathfrak{P}$ of S we call a *P-(total) ordering* of S if for every finite subset $\mathfrak{W}$ of $\mathfrak{P}$ for the subsets Z of S which are elements of $\mathfrak{W}$ it holds

$$(1) \qquad (\bigcup_{Z \in \mathfrak{W}} Z) \in \mathfrak{W}.$$

c) A P-partial ordering $\mathfrak{P}$ of S we call a *P-well-ordering* of S if for every (finite or infinite) subset $\mathfrak{W}$ of $\mathfrak{P}$ (1) holds.

d) Two P-partial orderings $\mathfrak{P}', \mathfrak{P}''$ of S we call *isomorphic* in $\mathfrak{U}S$ if there exists a $(1-1)$ maping φ of $\mathfrak{P}'$ onto $\mathfrak{P}''$ such that

$$(\forall X, Y \in \mathfrak{P}) \quad X \subset Y \Leftrightarrow \varphi X \subset \varphi Y.$$

We shall show that to the just introduced notions of P-orderings there can be corresponded in a natural way the respective notions of orderings in the ordinary sense.

2. a) Let $\mathfrak{P}$ be a P-partial ordering of S, and f the assumed $(1-1)$ maping of S onto $\mathfrak{P}$. For the elements s of S we define a binary operation $\leqq$, associated with $\mathfrak{P}$, by

$$(2) \qquad s_1 \leqq s_2 \Leftrightarrow fs_1 \supset fs_2.$$

If X, Y, Z are any elements of $\mathfrak{P}$ and $X = fx$, $Y = fy$, $Z = fz$, it will be:

$(\forall X)\ X \supset X$, hence $(\forall x)\ x \leqq x$;

$(\forall x, y)\ x \leqq y \,\&\, y \leqq x \Rightarrow X \supset Y \,\&\, Y \supset X \Rightarrow X = Y \Rightarrow x = y$;

$(\forall x, y, z)\ x \leqq y \,\&\, y \leqq z \Rightarrow X \supset Y \,\&\, Y \supset Z \Rightarrow X \supset Z \Rightarrow x \leqq z$.

b) Moreover, if $\mathfrak{P}$ is a P-ordering, it is also

$(\forall x, y)\ X \cup Y = X \vee X \cup Y = Y \Rightarrow x \leqq y \vee y \leqq x$.

c) Finally, if $\mathfrak{P}$ is a P-well-ordering, it is also

$(\forall Z \subset S) \exists z_0 \in Z,\ \bigcup_{z \in Z} fz = fz_0$, hence $(\forall Z \subset S) \exists z_0 \in Z,\ \forall z \in Z,\ z_0 \leqq z$.

3. a) Conversely, if $\leqq$ is a partial ordering of S, we can define a subset $\mathfrak{P}$ of $\mathfrak{U}S$ by

(3) $$Z\in\mathfrak{P}\Leftrightarrow(\exists s\in S,\ \{z|s\leqq z\}=Z).$$

Then because of $s_1\leqq s_2\ \&\ s_2\leqq s_1\Rightarrow s_1=s_2$, the power of $\mathfrak{P}$ equals the power of S.

b) Moreover, if $\leqq$ is a (total) ordering, then for any finite subset $\{Z_i\}$, $i=1, 2, \ldots, n$ of $\mathfrak{P}$ it is $\bigcup\limits_{i=1}^{n} Z_i=Z_1\vee Z_2\vee\ldots\vee Z_n$.

c) Finally, if $\leqq$ is a well-ordering, it will also hold: For any (finite or infinite) subset $\{Z_\nu\}_{\nu\in\Gamma}$ of $\mathfrak{P}$ it is $(\exists\mu\in\Gamma,\ \bigcup\limits_{\nu\in\Gamma} Z_\nu=Z_\mu)$.

4. If to the (in 3.) just obtained orderings we correspond orderings (in the ordinary sense) by the procedure used in 2, taking as f the $(1-1)$ maping between $\mathfrak{P}$ and S determined by (3), then these orderings will coincide with those from which we started in 3. Similarly, if we start from P-orderings $\mathfrak{P}$ and associate with them first orderings according to 2, and then to these orderings new P-orderings according to 3, then the new P-orderings will coincide with $\mathfrak{P}$ up to isomorphism in $\mathfrak{U}S$.

5. Let us point out, that so far in this note we had no need to use the axiom of choice. Hence from the preceeding considerations (by the well-ordering theorem of ZERMELO) we can infer as a corollary the following result (which was proved in a different way in the author's note „An Equivalent of the Axiom of choice", *Nieuw Archief voor Wiskunde* (3), X, 53-54 (1962)):

The axiom of choice is equivalent with the following proposition: For any set S, there exists a subset $\mathfrak{P}$ of $\mathfrak{U}S$ having the power of S, such that for any subset $\mathfrak{W}$ of $\mathfrak{P}$ the union of all subsets of S which are elements of $\mathfrak{W}$ is again an element of $\mathfrak{W}$.

6. With the use of the axiom of choice the requirements of Def. c) can be weakened in that „every (finite or infinite)" is replaced by „every at most denumerable".

For, let us suppose that a P-ordering, meeting these weaker requirements, yields by (2) an ordering which is not a well-ordering. Then the for the so obtained relation $\leqq$ there would exist an (infinite) subset Z of S without least element. Using the axiom of choice, we may infer that also a denumerable subset of Z will have no least element, what is in contradiction with the assumption of the weakened Def. c).

Obviously, this gives us no right to strenghten the statement of the corollary in 5. so that „any subset" be replaced by „ any at most denumerable subset".

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