Considerations concerning the complex roots of Riemann's Zeta-function*

By E. GROSSWALD (Philadelphia, Pa.)

1. Introduction

For $s = \sigma + it$, let $\zeta(s)$ be the RIEMANN zeta-function and denote, as usual by θ the least upper bound of those values of σ, for which $\zeta(\sigma+it) = 0$. Then it is well-known that $\frac{1}{2} \leqq \theta \leqq 1$. The first equality, $\theta = \frac{1}{2}$, would assert the truth of the famous Riemann hypothesis, while the second, $\theta = 1$, would assert that inside the „critical strip" $0 < \sigma < 1$, there is no zero-free vertical strip adjacent to $\sigma = 1$.

There are many known equivalent formulations, i. e. necessary and/or sufficient conditions for the truth, resp. the negation, of the Riemann hypothesis. And in what follows, the already lengthy list of such criteria receives some further additions. At this time such an action requires some justification; hence, a few words concerning the purpose of this paper are in order. This is to indicate (besides some rather routine generalizations of results due to M. RIESZ [2] and G. H. HARDY & J. E LITTLEWOOD [1]; see Theorem 1) and one apparently new, but probably hopeless, function theoretic approach (Theorem 4), two less hopelessly looking types of criteria. The first appears in the form of Tauberian theorems. While statements, equivalent to the Prime Number Theorem and the Riemann hypothesis, in the form of summability and Tauberian theorems are not new, the author is not aware of any results resembling the present Theorems 2 (A and B). Their interest seems to lie in that it is sufficient to prove or disprove the validity of a certain equation for any one of the functions of a certain class, in order to determine the value of θ.

The criteria of the other type mentioned (Theorem 3) may be considered as analogues of the classical formula $\psi(x) = x + R(x)$, where $R(x) = O(x^{\theta})$ or $o(x)$, respectively, accordingly as $\theta < 1$ or $\theta = 1$.

2. Notations and definitions

In order to state the results succinctly, we need a few notations and definitions. We shall say that a function $R(x)$ is of exact order α, if $R(x) = O(x^{\alpha+\varepsilon})$ and $R(x) = \Omega(x^{\alpha-\varepsilon})$ (i. e. $R(x) \neq o(x^{\alpha-\varepsilon})$) for any $\varepsilon > 0$, as $x \to \infty$. In general, ε stands for a positive quantity, that may be taken arbitrarily small; $\int\limits_{(\sigma)} \ldots ds$ stands for an integral

* Part of this work was done under contract with the Office of Naval Research Contract No. NONR-551(43).

along a parallel to the imaginary axis, of abscissa σ, usually taken between well-defined, indicated bounds; γ stands for the Euler—Mascheroni constant.

Definition 1. $\Phi = \{\varphi(s)\}$ is the set of functions of the complex variable $s = \sigma + it$, analytic for $\sigma > \frac{\theta}{2}$, monotonically increasing for real, positive, increasing argument and satisfying the conditions:

(I) $$|\zeta(2s)\cdot\varphi(s)| > Ce^{-\pi|t|}\cdot|t|^{1+\varepsilon}, \text{ as } |t| \to \infty, \text{ for fixed } \sigma > \frac{\theta}{2}$$

$\left(\text{or } \sigma \geqq \frac{1}{2} \text{ if } \theta = 1\right)$, with $C = C(\sigma) > 0, \varepsilon = \varepsilon(\sigma) > 0$; and

(II) $$|\varphi(s)|^{\frac{1}{\sigma}} \to \infty \text{ as } \sigma \to +\infty \text{ for fixed } t.$$

REMARK. Φ consists of functions that behave essentially like $\Gamma(s)$; $\Gamma(s)$, s^s, and also $\log^s(s+1)$ are elements of Φ.

On account of (II), the series $\sum_{m=1}^{\infty}(-1)^m \frac{y^m}{\varphi(m)}$ converges absolutely for every complex y; hence, it represents an entire function that we denote by $E_\varphi(-y)$ or $E(-y)$, whenever there is no danger of confusion. If $E(-y) = O(y^{1-\varepsilon})$ for $y \to +\infty$ and some $\varepsilon > 0$, then $\int_0^\infty y^{-s-\frac{3}{2}}E(-y)\,dy$ converges for $\frac{1}{2} - \varepsilon < \sigma < \frac{1}{2}$. The convergence is uniform in every closed subinterval and the integral represents there an analytic function of s, which we denote by $H_\varphi(\frac{1}{2} - s)$ or $H(\frac{1}{2} - s)$, if there is no danger of confusion. The same symbol denotes the function obtained by analytic continuation of $H(\frac{1}{2} - s)$ outside the strip of convergence $\frac{1}{2} - \varepsilon < \sigma < \frac{1}{2}$.

Definition 2. $\Phi_1 \subset \Phi$ is the subset of functions $\varphi(s) \in \Phi$ satisfying also

(III) $E_\varphi(-y) = O(y^{1-\varepsilon})$, for some $\varepsilon > 0$ and

(IV) $H_\varphi(\frac{1}{2} - s) \neq 0$ for $-\frac{1}{4} < \sigma < 0$.

REMARKS. a) $\Gamma(s) \in \Phi_1$; hence, Φ_1 is not empty. Proof: It is trivial to check $\Gamma \in \Phi$; also, $E_\Gamma(-y) = \sum_{m=1}^{\infty}(-1)^m \frac{y^m}{\Gamma(m)} = -ye^{-y}$, so that (III) holds. Finally, $H_\Gamma(\frac{1}{2} - s) = -\int_0^\infty y^{-s-\frac{3}{2}}\, ye^{-y}\,dy = -\Gamma(\frac{1}{2} - s) \neq 0$; hence (IV) holds and $\Gamma(s) \in \Phi_1$.

b) The conditions defining Φ and Φ_1 are sufficient for our purpose; but they are far from necessary. They have been selected as stated for convenience, being satisfied in many cases. Occasionally some $\varphi \notin \Phi_1$ will be considered and the desired properties will be proven directly, without appeal to the general theorems.

For every $\varphi \in \Phi$ we define also the following functions:

Definition 3.

(a) $$F_\varphi(x) = F(x) = \frac{-1}{2i}\int_{(\sigma)} \frac{x^s}{\zeta(2s)\varphi(s)\sin(\pi s)}\,ds \quad \text{for} \quad \frac{1}{2} < \sigma < 1;$$

(*b*) $G_\varphi(s) = G(s) = \int\limits_0^\infty F(x) x^{-s-\frac{3}{2}}\, dx$ for $0 < \sigma < \frac{1}{2}$;

(*c*) $S_\varphi(s) = S(s) = \sum\limits_{m \geqq \frac{s}{2}} \frac{(-1)^m}{\zeta(2m)\varphi(m)}$;

(*d*) $S_1(s) = S(s)$; $S_n(s) = \int\limits_s^\infty S_{n-1}(t)\, dt$ for $n > 1$;

(*e*) $V(v) = S(e^{-v}) = \sum\limits_{m \geqq \frac{1}{2} e^{-v}} \frac{(-1)^m}{\zeta(2m) \cdot \varphi(m)}$;

(*f*) for u, v real, $K_u(v) = K(v) = \begin{cases} \exp(e^v + v) & \text{for } v > u, \\ 0 & \text{for } v < u. \end{cases}$

Obverse that $F(1) = S(1) = V(0)$; also $S(r) = S(1)$ for $0 < r < 2$.

3. Main results

With preceding notations, the main results can now be stated as follows:

Theorem 1. a) *If* $\varphi \in \Phi$, *then* $F(x) = \sum\limits_{m=1}^{\infty} (-1)^m \frac{x^m}{\zeta(2m) \cdot \varphi(m)}$ *and* $F(x) = O(x^\alpha)$ *with* $\alpha \leqq \frac{\theta}{2}$; *in particular,* $F(x) = o(x^{\frac{1}{2}})$. *Also,* $F(x) = \Omega(x^{\frac{1}{2}-\varepsilon})$ *implies* $\theta = 1$.

b) *If* $\varphi \in \Phi_1$, *then* $F(x)$ *is of exact order* $\frac{\theta}{2}$; *in particular,* $F(x) = \Omega(x^{\frac{1}{2}-\varepsilon})$. *And* $F(x) = O(x^{\frac{1}{4}+\varepsilon})$ *is the necessary and sufficient condition for the validity of the Riemann hypothesis.*

Theorem 2 (*A*). *If* $\varphi \in \Phi_1$, *then*:

a) $\lim\limits_{x\to\infty} \int\limits_1^\infty x^{s-1} \log x\, S(s)\, ds = -S(1)$; *more generally,*

b) $\lim\limits_{x\to\infty} \int\limits_r^\infty x^{s-r} \log x\, S(s)\, ds = -S(1)$ *holds for* $\theta < r < 2$ *and is false for* $r < \theta$.

c) *If* $r_0 = \underline{\lim}\, r$, *where* $R = \{r\}$ *is the set of values for which* b) *holds, then* $r_0 = \theta$.

d) *If* b) *holds for any* $\varphi_0 \in \Phi_1$ *and some* r, *then it holds, with the same* r, *for all* $\varphi \in \Phi_1$, *and* $\theta \leqq r$.

e) *If for some* $\varphi_0 \in \Phi_1$ *and some* $r < 1$, $\lim\limits_{x\to\infty} \int\limits_r^\infty x^{s-r} \log x\, S(s)\, ds \neq -S(1)$, *or if the limit does not exist, then the same is true with the same r for all other* $\varphi \in \Phi_1$, *and* $\theta \geqq r$. *In particular, if that is the case for any one function* $\varphi_0 \in \Phi_1$ *and all* $r < 1$, *then* $\theta = 1$.

f) *A necessary and sufficient condition for the validity of the Riemann hypothesis is that there should exist some function* $\varphi_0 \in \Phi_1$ *such that for every* $\varepsilon > 0$,

$$\lim_{x \to \infty} \int_{\frac{1}{2}+\varepsilon}^{\infty} x^{s-\frac{1}{2}-\varepsilon} \log x \, S(s)\, ds = -S(1).$$

g) *If* $\varphi \in \Phi$ *and* $0 < r < 2$, *then*

$$F(x^2) = \int_r^{\infty} x^s \cdot \log^n x \, S_n(s)\, ds + x^r\big(S_1(r) + S_2(r)\log x + \ldots + S_n(r)\log^{n-1} x\big).$$

h) *If* $\varphi \in \Phi$, $0 < r < \theta$, *then, as* $x \to \infty$, *for any fixed* n,

$$F(x^2) = \big(1 + o(1)\big) \int_r^{\infty} x^s \cdot \log^n x \cdot S_n(s)\, ds;$$

hence if $\varphi \in \Phi_1$, *then also* $\int_r^{\infty} x^s \cdot \log^n x \cdot S_n(s)\, ds$ *is of exact order* θ, *for every* n.

Theorem 2 (*B*). *If* $\varphi \in \Phi_1$, *then, as* $u \to +\infty$ *through real values*:

a) $\int_{-\infty}^{\infty} K_u(v)\, V(u-v)\, dv = -e^{e^u} V(0)\big(1 + o(1)\big)$; *more generally, for* $-\log 2 \leqq \varrho \leqq 0$ *and with* $\tau = \log \theta (\leqq 0)$,

b) $\int_{-\infty}^{\infty} K_{u+\varrho}(v)\, V(u-v)\, dv = -e^{e^{u+\varrho}} V(0)\big(1 + O(\exp\{e^u(e^\tau - e^\varrho)\})\big)$.

c) *If* $\varrho_1 = \liminf_{\varrho \in P} \varrho$, *where* $P = \{\varrho\}$ *is the set of those* ϱ, *for which* $\lim_{u \to \infty} e^{-e^{u+\varrho}} \cdot \int_{-\infty}^{\infty} K_{u+\varrho}(v)\, V(u-v)\, dv = -V(0)$, *then* $\varrho_1 = \tau = \log \theta$.

d) *If the equality of* c) *holds for any* $\varphi_0 \in \Phi_1$, *and some* ϱ *then it holds for all* $\varphi \in \Phi_1$ *with the same* ϱ *and* $\tau \leqq \varrho$.

e) *If for some* $\varphi_0 \in \Phi_1$ *and some* $\varrho(-\log 2 \leqq \varrho < 0)$, *the equality fails to hold in* c), *then, for the same* ϱ, *it fails to hold for all* $\varphi \in \Phi_1$ *and* $\tau \geqq \varrho$. *In particular, if for any one* $\varphi_0 \in \Phi_1$ *the equality in* c) *fails to hold for all* $\varrho < 0$, *then* $\tau = 0$ *and* $\theta = 1$.

f) *A necessary and sufficient condition for the validity of the Riemann hypothesis is that there should exist some function* $\varphi_0 \in \Phi_1$, *such that the equality in* c) *should hold for every* $\varrho > -\log 2$. $(\varrho \leqq 0)$

Theorem 3. *If* $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_b^{\alpha_b}$ $(\alpha_i > 0)$, $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_b \geqq b$, *denote as usual by* $\mu(n)$ *the Mobius function* $\mu(n) = \begin{cases} (-1)^b & \text{if } b = \alpha \\ 0 & \text{if } b < \alpha \end{cases}$; *set* $\lambda(n) = (-1)^\alpha$ *and* $\Lambda(n) = \begin{cases} \log p_1 & \text{if } b = 1 \\ 0 & \text{otherwise} \end{cases}$. *Then the following relations hold;*

(a) $\sum_{n=1}^{\infty} \mu(n) \frac{x}{n^2} e^{-\frac{x}{n^2}} = R_1(x)$ (see [2]);

(b) $\sum_{n=1}^{\infty} \lambda(n) \frac{x}{n^2} e^{-\frac{x}{n^2}} = R_2(x)$;

(c) $\sum_{n=2}^{\infty} \Lambda(n) \frac{x}{n^2} e^{-\frac{x}{n^2}} = \frac{1}{2}(\pi x)^{\frac{1}{2}} + R_3(x)$;

(d) $\sum_{n=2}^{\infty} \Lambda(n) \frac{x^2}{n^4} e^{-\frac{x}{n^2}} = \frac{1}{4}\cdot(\pi x)^{\frac{1}{2}} + R_4(x)$;

(e) $\sum_{n=1}^{\infty} |\mu(n)| \frac{x^2}{n^2} e^{-\frac{x}{n}} = \frac{x}{\zeta(2)} + R_5(x)$;

(f) $\sum_{n=1}^{\infty} |\mu(n)| \frac{x}{n} (1-e^{-\frac{x}{n}}) = \frac{x}{\zeta(2)}\cdot\left(\log x + 2\gamma - 2\frac{\zeta'}{\zeta}(2)\right) + R_6(x)$.

The functions $R_i(x)$ $(i=1, 2, 3, 5, 6)$ are of exact order $\frac{\theta}{2}$; in particular, all are $\Omega(x^{\frac{1}{4}-\varepsilon})$ and $o(x^{\frac{1}{2}})$. $(R_4 = O(x^{\frac{\theta}{2}+\varepsilon})$, not necessarily $\Omega(x^{\frac{\theta}{2}-\varepsilon}))$. Furthermore if $\theta=1$ and $M(x) = \sum_{n \leqq x} \mu(n) = O(xg(x))$ $(g(x) \to 0,\ xg(x) \to \infty$ monotonically, as $x \to \infty)$, then the estimate $R_1(x) = o(x^{\frac{1}{2}})$ can be improved to $R_1(x) = x^{\frac{1}{2}}\cdot$ $\cdot O\{g(x)^c + g(x)^{-2c} g(x^{\frac{1}{2}} g(x)^{-c})\}$. Taking, in particular, $g(x) = \exp\{-\alpha \log^{\frac{4}{7}} x\}$ (with $c = 3^{-1}\cdot 2^{-\frac{4}{7}}$), it follows that $R_1(x) = o(x^{\frac{1}{2}} e^{-\beta(\log x)^{4/7}})$ for every $\beta < 2^{\frac{3}{7}}\cdot 6^{-1}\alpha$.

REMARK. Similar improvements are possible for the other $R_i(x)$.

Theorem 4. *Denote by $\Phi_2 \subset \Phi$ the set of those functions $\varphi \in \Phi$, for which the entire function* $\Psi(u) = \sum_{n=1}^{\infty} \frac{u^n}{\zeta\left(1+\frac{n}{2}\right)\varphi\left(1+\frac{n}{2}\right)}$ *is of order $< \frac{1}{2}$ (or of order $\frac{1}{2}$, minimal type). Unless Φ_2 is empty, $\theta = 1$.*

4. Some comments

a) Theorem 2(B) follows from the more natural Theorem 2(A) simply by the change of variables $u = \log\log x$, $v = -\log s$, $\varrho = \log r$, $\tau = \log\theta$. It is quoted here mainly because it appears in a form familiar from other Tauberian theorems.

b) In case $\theta = 1$, the estimates $F(x) = o(x^{\frac{1}{2}})$ can be sharpened, using $M(x) = O(xg(x))$, as in Theorem 3.

c) One may extend ad lib. the list of formulae in Theorem 3.

d) The relevance of the results seems to lie in that it is sufficient to find even a single function $\varphi \in \Phi_1$ for which it is possible to show that the (unlikely looking) statements of Theorem 2(B) are actually false for $r<1$ (or $\varrho<o$, resp.). It would follow that $\theta=1$. Or else, it might be possible to estimate by methods of elementary number theory at least one of the sums occurring in Theorem 3 (or a similar one) with sufficient accuracy, in order to obtain at least some non-trivial bounds for θ. Finally, there might be a faint hope to construct a function $\varphi\in\Phi_2$ (although, most likely, Φ_2 is actually empty), which also would be sufficient to settle the problem.

5. Proofs

Proof of Theorem 1. (see [1] and [2]). If $\varphi\in\Phi$, consider the integral $F(x)=-\frac{1}{2i}\int\limits_{(\sigma)}\frac{x^s}{\zeta(2s)\,\varphi(s)\sin(\pi s)}\,ds$. On account of (I), the integral converges for $\frac{1}{2}\leqq\sigma<1$. Integration along the rectangle $\frac{1}{2}\pm iT$, $N+\frac{1}{2}\pm iT$, Cauchy's theorem on residues, followed by the successive passages to the limit $T\to\infty$ and $N\to\infty$ (using (II)) leads to $-\frac{1}{2i}\int\limits_{(\sigma)}\frac{x^s}{\zeta(2s)\,\varphi(s)\sin(\pi s)}\,ds=\sum_{m=1}^{\infty}(-1)^m\frac{x^m}{\zeta(2m)\,\varphi(m)}$. If $\theta<1$, then, using the properties of $\varphi(s)$ and Littlewood's theorem[1]) (see [3], pp. 282—283) that for $\sigma>\frac{\theta}{2}$, $|\zeta(2s)|^{-1}=o(|t|^{\varepsilon_1})$, the line of integration may be moved to $\sigma=\frac{1}{2}+\varepsilon$. It follows that $F(x)=O(x^{\frac{\theta}{2}+\varepsilon})$. If $\theta=1$, one may take $\sigma=\frac{1}{2}$ and, by the convergence of the integral and the Riemann—Lebesgue theorem one obtains $F(x)=o(x^{\frac{1}{2}})$; this proves part a). For part b), we assume that $\varphi\in\Phi_1$. As $F(x)=o(x^{\frac{1}{2}})$ for $x\to\infty$ and $F(x)=-\frac{x}{\zeta(2)\,\varphi(1)}(1+o(1))$ for $x\to 0$, it is clear that $G_\varphi(s)=\int\limits_0^\infty F(x)x^{-s-\frac{3}{2}}dx$ converges for $0<\sigma<\frac{1}{2}$, and uniformly in any closed subinterval. Hence

$$G(s)=\int\limits_0^\infty x^{-s-\frac{3}{2}}\left\{\sum_{m=1}^{\infty}\frac{(-1)^m x^m}{\varphi(m)\zeta(2m)}\right\}dx=\int\limits_0^\infty x^{-s-\frac{3}{2}}\sum_{m=1}^{\infty}\frac{(-1)^m x^m}{\varphi(m)}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{2m}}\,dx.$$

As both series converge absolutely,

$$G(s)=\int\limits_0^\infty x^{-s-\frac{3}{2}}\sum_{n=1}^{\infty}\mu(n)\sum_{m=1}^{\infty}\frac{(-1)^m\left(\frac{x}{n^2}\right)^m}{\varphi(m)}\,dx=\int\limits_0^\infty x^{-s-\frac{3}{2}}\left(\sum_{n=1}^{\infty}\mu(n)E\left(\frac{-x}{n^2}\right)\right)dx.$$

As $\varphi\in\Phi_1$, by (III) $\left|\mu(n)x^{-s-\frac{3}{2}}E\left(-\frac{x}{n^2}\right)\right|<C\left(\frac{x}{n^2}\right)^{1-\varepsilon}\cdot\left(x^{-\sigma-\frac{3}{2}}\right)=\frac{C}{n^{2(1-\varepsilon)}}x^{-\frac{1}{2}-\sigma-\varepsilon}$

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[1]) Littlewood's method permits us to prove the present statement, although the original formulation asserts less.

and the series converges uniformly in x, justifying termwise integration, which yields

$$G(s) = \sum_{n=1}^{\infty} \mu(n) \cdot \int_0^{\infty} x^{-s-\frac{3}{2}} E\left(-\frac{x}{n^2}\right) dx =$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2s+1}} \cdot \int_0^{\infty} \left(\frac{x}{n^2}\right)^{-s-\frac{3}{2}} E\left(-\frac{x}{n^2}\right) d\left(\frac{x}{n^2}\right) = \frac{H\left(\frac{1}{2}-s\right)}{\zeta(2s+1)}$$

valid at least for $\frac{1}{2}-\varepsilon<\sigma<\frac{1}{2}$. If $F(x)=O(x^{\alpha})$ $(\alpha<\frac{1}{2})$, then $\int_0^{\infty} F(x) x^{-s-\frac{3}{2}} dx$ converges for $\alpha-\frac{1}{2}<\sigma<\frac{1}{2}$; consequently, $\frac{H(\frac{1}{2}-s)}{\zeta(2s+1)}$ is regular within that range. By (IV), $H(\frac{1}{2}-s)\neq 0$ for $-\frac{1}{4}<\sigma<0$ so that also $\zeta(2s+1)\neq 0$ within the overlapping part of these ranges, or, for $\sigma_1 = 2\sigma+1$ in $2\alpha<\sigma_1<1$. Hence, $\theta\leqq 2\alpha$, finishing the proof of Theorem 1.

Proof of Theorem 2(A). Using the uniform convergence of $S(s)$, one obtains, for every $0<r<2$:

$$\int_r^{\infty} x^s \log x\, S(s)\, ds = \int_r^{\infty} x^s \log x \left(\sum_{m\geq \frac{s}{2}} \frac{(-1)^m}{\zeta(2m)\varphi(m)} \right) ds =$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m}{\zeta(2m)\varphi(m)} \cdot \int_r^{2m} x^s \log x\, dx = \sum_{m=1}^{\infty} \frac{(-1)^m}{\zeta(2m)\varphi(m)} (x^{2m}-x^r) = F(x^2) - x^r S(1),$$

or, dividing by x^r,

$$(*) \qquad \int_r^{\infty} x^{s-r} \log x\, S(s)\, ds = -S(1) + x^{-r} F(x^2).$$

Assertions a)—f) now follow immediately from $(*)$ on account of Theorem 1. For g), observe that for every finite a, $S(s)=o(\log^{-1} a \cdot a^{-\sigma})$ as $\sigma\to\infty$; hence, $S_2(s) = o(\log^{-2} a\cdot a^{-\sigma})$ and, by induction on n, $S_n(s)=o(\log^{-n} a\cdot a^{-\sigma})$. Therefore, $\int_r^{\infty} x^s (\log^i x)\, S_n(s)\, ds$ converges, $\lim_{\sigma\to\infty} x^s \log^{i-1} x\, S_n(s) = 0$ and, integrating by parts.

$$\int_r^{\infty} x^s \log x\, S_1(s)\, ds = - S_2(s) x^s \log x \Big|_r^{\infty} +$$

$$+ \int_r^{\infty} x^s (\log x) S_2(s)\, ds = S_2(r) x^r \log x + \int_r^{\infty} x^s \log^2 x\, S_2(s) = \ldots =$$

$$= x^r \left(S_2(r)\log x + S_3(r)\log^2 x + \ldots + S_n(r)\log^{n-1} x\right) + \int_r^{\infty} x^s \log^n x\, S_n(s)\, ds.$$

By ($\ast$) the first member equals $F(x^2)-x^r S_1(r)$ and this finishes the proof of g). Finally, for fixed n, $x^r \sum_{j=1}^{n} S_j(r)\log^{j-1}x = x^r S_n(r)\log^{n-1}x\bigl(1+o(1)\bigr)$ as $x\to\infty$ and this proves h) and finishes the proof of Theorem 2(A).

REMARK. $S_1(1)=S_1(r)$ for $0<r<2$, while $S_n(1)\neq S_n(r)$, in general.

PROOF OF THEOREM 2(B). With the changes of variables indicated under **4.**, ($\ast$) becomes $\int_{-\infty}^{\infty} K_{u+\varrho}(u-v)V(v)\,dv = -\exp\{e^{u+\varrho}\}V(0)+F(e^{2e^u})$. Using Theorem 1 and the convolution theorem one obtains b) of Theorem 2(B). The particular case $\varrho=0$ yields a) and the other assertions follow as in Theorem 2(A).

Most statements of Theorem 3 could be proven in a unified way, by an appeal to the theory of MELLIN transforms. However, in some cases, it seemed preferable to avoid said theory and to rely, instead, upon the following

Lemma. *Let* $\varphi(s)=\dfrac{\Gamma(s)}{f(s)}\in\Phi$, *with* $\dfrac{f(s)}{\zeta(2s)}=q(s)+\sum_{n=1}^{\infty}\dfrac{a_n}{n^{2s}}$, *where the Dirichlet series converges at least for* $\sigma\geqq\dfrac{1}{2}$,

$$\sum_{m=1}^{\infty}(-1)^m\frac{q(m)}{\Gamma(m)}x^m=L(x)$$

converges for every $|x|<\infty$, *and* $G_1(s)=\int_0^{\infty}x^{-s-\frac{3}{2}}L(x)\,dx$ *converges at least in some strip* $\frac{1}{2}-\varepsilon<\sigma<\frac{1}{2}$. *Then with* $F(x)$ *and* $G(s)$ *as in Definition* 3,

$$F(x)=L(x)-\sum_{m=1}^{\infty}a_m\frac{x}{m^2}e^{-\frac{x}{m^2}} \text{ is of exact order } \frac{\theta}{2} \text{ if } \theta<1,$$

$$F(x)=o\bigl(x^{\frac{1}{2}}\bigr) \text{ if } \theta=1 \text{ and } G(s)=G_1(s)-\Gamma\left(\frac{1}{2}-s\right)\left\{\frac{f\left(s+\frac{1}{2}\right)}{\zeta(2s+1)}-q\left(s+\frac{1}{2}\right)\right\}.$$

PROOF. As in the proof of Theorem 1,

$$F(x)=-\frac{1}{2i}\int_{(\sigma)}\frac{f(s)x^s}{\zeta(2s)\Gamma(s)\sin(\pi s)}\,ds$$

converges at least for $\frac{1}{2}\leqq\sigma<1$, more generally, for $\sigma>\dfrac{\theta}{2}$, so that it is $O(x^{\frac{\theta}{2}+\varepsilon})$, or $o\bigl(x^{\frac{1}{2}}\bigr)$, if $\theta=1$.

Also, as in Theorem 1, the integral equals $\sum_{m=1}^{\infty}(-1)^m\dfrac{x^m}{\Gamma(m)}\dfrac{f(m)}{\zeta(2m)}$.

Hence,

$$F(x) = \sum_{m=1}^{\infty} (-1)^m \frac{x^m}{\Gamma(m)} \left(q(m) + \sum_{n=1}^{\infty} \frac{a_n}{n^{2m}} \right) =$$

$$= L(x) + \sum_{n=1}^{\infty} a_n \sum_{m=1}^{\infty} (-1)^m \frac{\left(\frac{x}{n^2}\right)^m}{(m-1)!} = L(x) - \sum_{n=1}^{\infty} a_n \frac{x}{n^2} e^{-\frac{x}{n^2}}.$$

Finally,

$$G(s) = \int_0^\infty x^{-s-\frac{3}{2}} F(x)\, dx = \int_0^\infty x^{-s-\frac{3}{2}} \left\{ L(x) - \sum_{n=1}^{\infty} a_n \frac{x}{n^2} e^{-\frac{x}{n^2}} \right\} dx =$$

$$= G_1(s) + G_2(s), \text{ with } G_2(s) = - \sum_{n=1}^{\infty} a_n \int_0^\infty x^{-s-\frac{1}{2}} e^{-\frac{x}{n^2}} d\left(\frac{x}{n^2}\right) =$$

$$= - \sum_{n=1}^{\infty} \frac{a_n}{n^{2s+1}} \int_0^\infty \left(\frac{x}{n^2}\right)^{-s-\frac{1}{2}} e^{-\frac{x}{n^2}} d\left(\frac{x}{n^2}\right) = - \sum_{n=1}^{\infty} \frac{a_n}{n^{2s+1}} \Gamma\left(\frac{1}{2} - s\right) =$$

$$= \left\{ q\left(s+\frac{1}{2}\right) - \frac{f\left(s+\frac{1}{2}\right)}{\zeta(2s+1)} \right\} \cdot \Gamma\left(\frac{1}{2} - s\right).$$

The interchange of integration and summation can be justified at least in the strip $\frac{1}{2} - \varepsilon < \sigma < \frac{1}{2}$ and the Lemma follows.

PROOF OF THEOREM 3. a) Let $f(s) = 1$, i. e. $\varphi(s) = \Gamma(s)$, and $\frac{f(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2s}}$. The result now follows directly from the Lemma with $F(x) = R_1(x)$.

(*b*) Let $f(s) = \zeta(4s)$, $\varphi(s) = \frac{\Gamma(s)}{\zeta(4s)}$ and $\frac{f(s)}{\zeta(2s)} = \frac{\zeta(4s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^{2s}}$. The result follows from the Lemma, with $F(x) = R_2(x)$.

(*c*) Let $f(s) = \zeta'(2s) + \frac{\zeta(2s)}{2s-1}$, $\frac{f(s)}{\zeta(2s)} = \frac{\zeta'}{\zeta}(2s) + \frac{1}{2s-1} = \frac{1}{2s-1} - \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{2s}}$, $q(s) = \frac{1}{2s-1}$. By the Lemma, $F(x) = \frac{-1}{2i} \int_{(\sigma)} \left(\frac{\zeta'}{\zeta}(2s) + \frac{1}{2s-1} \right) \cdot \frac{x^s}{\Gamma(s) \sin \pi s}\, ds =$
$= \sum_{m=1}^{\infty} (-1)^m \frac{x^m}{\Gamma(m)} \left\{ \frac{\zeta'}{\zeta}(2m) + \frac{1}{2m-1} \right\}$ is of exact order $\frac{\theta}{2}$ (if $\theta < 1$) or $o(x^{\frac{1}{2}})$ (if $\theta = 1$). Also,

$$L(x) = \sum_{m=1}^{\infty} (-1)^m \frac{x^m}{(2m-1)\Gamma(m)} =$$

$$= -x^{\frac{1}{2}} \int_0^{x^{\frac{1}{2}}} e^{-u^2}\, du = -x^{\frac{1}{2}} \left\{ \frac{1}{2} \pi^{\frac{1}{2}} - \int_{x^{\frac{1}{2}}}^{\infty} e^{-u^2}\, du \right\} = -\frac{1}{2} (\pi x)^{\frac{1}{2}} + O(e^{-x}).$$

Hence, by the Lemma, $-\frac{(\pi x)^{\frac{1}{2}}}{2}+\sum_{n=2}^{\infty}\Lambda(n)\frac{x}{n^2}e^{-\frac{x}{n^2}}=F(x)+O(e^{-x})=R_3(x)$, so that the order α of $R_3(x)$ satisfies $\alpha\leqq\frac{\theta}{2}$.

For the opposite inequality, consider $G_1(s)=\int_0^\infty x^{-s-\frac{3}{2}}L(x)\,dx=$ $=-\int_0^\infty x^{-s-1}\left\{\int_0^{x^{\frac{1}{2}}}e^{-u^2}du\right\}dx=-\int_0^\infty e^{-u^2}\left\{\int_{u^2}^\infty x^{-s-1}\,dx\right\}du$. For $\sigma>0$, this equals $-\frac{1}{s}\int_0^\infty u^{-2s}e^{-u^2}\,du$ and, if also $\sigma<\frac{1}{2}$, $G_1(s)=-\frac{1}{s}\int_0^\infty u^{-2s}e^{-u^2}\,du=$ $=-\frac{1}{2s}\int_0^\infty v^{-\frac{1}{2}-s}e^{-v}\,dv=-\frac{1}{2s}\Gamma\left(\frac{1}{2}-s\right)$, so that, by the Lemma,

$G(s)=\frac{1}{2s}\Gamma\left(\frac{1}{2}-s\right)+\Gamma\left(\frac{1}{2}-s\right)\left\{-\frac{\zeta'}{\zeta}(2s+1)\right\}=-\Gamma\left(\frac{1}{2}-s\right)\left\{\frac{\zeta'}{\zeta}(2s+1)+\frac{1}{2s}\right\}$, whence follows, as in the proof of Theorem 1, that $\alpha\geqq\frac{\theta}{2}$.

(*d*) Let $f(s)=(2s-1)\cdot\zeta'(2s)$, $\varphi(s)=\frac{\Gamma(s)}{(2s-1)\zeta'(2s)}\in\Phi$. Although the Lemma is not directly applicable, one may proceed as in its proof and obtains that for $\frac{1}{2}\leqq\sigma<1$, $F(x)=\frac{1}{2i}\int_{(\sigma)}\frac{\zeta'(2s)}{\zeta(2s)}(2s-1)\frac{x^s}{\Gamma(s)\sin\pi s}\,ds$ *converges* and *equals* $\sum_{m=1}^{\infty}(-1)^m\frac{x^m}{\Gamma(m)}(2m-1)\frac{\zeta'}{\zeta}(2m)$. Hence, $F(x)=O(x^{\frac{\theta}{2}+\varepsilon})$ if $\theta<1$, $F(x)=o(x^{\frac{1}{2}})$ if $\theta=1$. Also $F(x)=-\sum_{m=1}^{\infty}(-1)^m\frac{x^m}{\Gamma(m)}(2m-1)\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{2m}}=\sum_{n=1}^{\infty}\Lambda(n)g\left(\frac{x}{n^2}\right)$ where $g(u)=\sum_{m=1}^{\infty}(-1)^m\frac{2m-1}{(m-1)!}u^m=e^{-u}(2u^2-u)$; hence, $F(x)=-2\sum_{n=2}^{\infty}\Lambda(n)\cdot$ $\cdot\frac{x^2}{n^4}e^{-\frac{x}{n^2}}+F_1(x)$, where $F_1(x)=\sum_{n=2}^{\infty}\Lambda(n)\frac{x}{n^2}e^{-\frac{x}{n^2}}$ is the function considered under c). It easily follows now that α, the order of $R_4(x)=R_3(x)-F(x)$ satisfies $\alpha\leqq\frac{\theta}{2}$. The proof is completed by computing $G(s)=\int_0^\infty F(x)x^{-s-\frac{3}{2}}dx=-2s\Gamma(\frac{1}{2}-s)\cdot$ $\cdot\frac{\zeta'}{\zeta}(2s+1)$, which proves that α_1, the order of $F(x)$, (but not necessarily α, the order of $R_4(x)$), satisfies $\alpha_1\geqq\frac{\theta}{2}$.

e) Let $f(s) = (s-1)\zeta(s)$. The proof proceeds as for d), except that now the residue of $-\frac{1}{2i}\int\limits_{(\sigma)} \frac{(s-1)x^s ds}{\zeta(2s)\Gamma(s)\sin \pi s}$ at $s=1$ is $\left(\frac{1}{2\pi i}\right)\frac{x}{\zeta(2)}$, so that, denoting the integral by $F(x)$, it follows that[2])

$$F(x) = -\frac{x}{\zeta(2)} + \sum_{m=2}^{\infty} (-1)^m \frac{x^m}{(m-2)!}\,\frac{\zeta(m)}{\zeta(2m)} = O(x^{\frac{\theta}{2}+\varepsilon})\,(\text{or } o(x^{\frac{1}{2}}) \text{ if } \theta=1).$$

Furthermore,

$$F(x) = -\frac{x}{\zeta(2)} + \sum_{m=2}^{\infty} (-1)^m \frac{x^m}{(m-2)!} \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^m} =$$

$$= -\frac{x}{\zeta(2)} + \sum_{n=1}^{\infty} |\mu(n)| \cdot \frac{x^2}{n^2} e^{-\frac{x}{n}} = O(x^\alpha) \text{ with } \alpha \leqq \frac{\theta}{2}.$$

For the inverse inequality we use MELLIN's reciprocal formulae written as (see [5], p. 246, with $t=x^{-1}$)

$$G(s) = \int\limits_0^\infty x^{-1-s}\varphi(x)\,dx,$$

$$\varphi(x) = \frac{1}{2\pi i}\int\limits_{(\sigma)} G(s)x^s\,ds.$$

The function $G(s) = \frac{\zeta\left(s+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-s\right)}{\zeta(2s+1)}$ is regular for $0<\sigma<\frac{1}{2}$. Setting it equal to $\int\limits_0^\infty x^{-1-s}x^{-\frac{1}{2}}F(x)dx$ leads to the inversion formula

$$x^{-\frac{1}{2}}F(x) = \frac{1}{2\pi i}\int\limits_{(\sigma)} \frac{\zeta\left(s+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-s\right)}{\zeta(2s+1)} x^s\,dx = \frac{1}{2\pi i}\int\limits_{(\sigma)} \frac{\zeta(s)\Gamma(2-s)}{\zeta(2s)} x^{s-\frac{1}{2}}\,ds =$$

$$\left(0<\sigma<\frac{1}{2}\right) \qquad\qquad \left(\frac{1}{2}<\sigma<1\right)$$

$$= \frac{x^{-\frac{1}{2}}}{2\pi i}\int\limits_{(\sigma)} \frac{x^s\zeta(s)(1-s)\Gamma(1-s)}{\zeta(2s)}\,ds = -\frac{x^{-\frac{1}{2}}}{2i}\int\limits_{(\sigma)} \frac{x^s(s-1)\zeta(s)}{\zeta(2s)\Gamma(s)\sin(\pi s)}\,ds;$$

[2]) The term obtained formally in the sum for $m=1$, namely $-\frac{x}{(-1)!}\frac{\zeta(1)}{\zeta(2)}$ is meaningless. If we write it, however, as $-\frac{x}{\zeta(2)}\left\{\frac{(m-1)\zeta(m)}{(m-1)!}\right\}_{m=1}$ and agree to replace $\{(m-1)\cdot\zeta(m)\}_{m=1}$ by unity, we obtain precisely $-\frac{x}{\zeta(2)}$.

hence, if $F(x)=\frac{1}{2i}\int\limits_{(\sigma)}\frac{(s-1)\zeta(s)}{\zeta(2s)}\cdot\frac{x^s}{\Gamma(s)\sin(\pi s)}\,ds$, then $G(s)=\int\limits_0^\infty x^{-s-\frac{3}{2}}F(x)\,dx=$
$=\frac{\zeta\left(s+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-s\right)}{\zeta(2s+1)}=-\Gamma\left(\frac{1}{2}-s\right)\frac{\left(s-\frac{1}{2}\right)\zeta\left(s+\frac{1}{2}\right)}{\zeta(2s+1)}$ and it follows, as in Theorem 1, that $\alpha\geqq\frac{\theta}{2}$, completing the proof.

f) If $f(s)=\zeta(s)$, $\varphi(s)=\Gamma(s)/\zeta(s)$ still belongs to Φ. But now $\varphi\notin\Phi_1$ and the pole at $s=1$ is double and has the residue $\frac{x}{2\pi i\,\zeta(2)}\left(\log x+2\gamma-2\frac{\zeta'}{\zeta}(2)\right)$. Hence,

$$F(x)=\frac{-1}{2i}\int\limits_{(\sigma)}\frac{\zeta(s)}{\zeta(2s)}\cdot\frac{x^s}{\Gamma(s)\sin\pi s}\,ds=-\frac{x}{\zeta(2)}\left(\log x+2\gamma-2\frac{\zeta'}{\zeta}(2)\right)+$$

$$+\sum_{m=2}^{\infty}(-1)^m\frac{x^m}{\Gamma(m)}\frac{\zeta(m)}{\zeta(2m)}\quad\text{and}\quad F(x)=O(x^\alpha)\ \text{with}\ \alpha\leqq\frac{\theta}{2}.$$

From

$$\sum_{m=2}^{\infty}(-1)^m\frac{x^m}{\Gamma(m)}\frac{\zeta(m)}{\zeta(2m)}=\sum_{m=2}^{m}(-1)^m\frac{x^m}{\Gamma(m)}\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^m}=\sum_{n=1}^{\infty}|\mu(n)|\frac{x}{n}\left(1-e^{-\frac{x}{n}}\right),$$

now follows the statement of the Theorem, except for $\alpha\geqq\frac{\theta}{2}$. This last inequality is obtained as under e) by using MELLIN's inversion formula to prove that

$$G(s)=\int\limits_0^\infty F(x)x^{-s-\frac{3}{2}}\,dx=-\Gamma\left(\frac{1}{2}-s\right)\frac{\zeta\left(s+\frac{1}{2}\right)}{\zeta(2s+1)}.$$

Finally, it remains to show that the estimates $R_i=o\left(x^{\frac{1}{2}}\right)$ can be sharpened even if $\theta=1$. The proof will be given in detail for $R_1(x)=\sum_{n=1}^{\infty}\mu(n)\frac{x}{n^2}e^{-\frac{x}{n^2}}$ and proceeds along similar lines in the other cases. Let $g(x)$ be a function such that $M(x)=$ $=\sum_{n\leqq x}\mu(n)=O\left(xg(x)\right)$, $g(x)\to 0$, $xg(x)\to\infty$ monotonically, as $x\to\infty$. Such functions are, of course well-known. Denote by n_0 the integer closest to $x^{\frac{1}{2}}\{g(x)\}^{-c}$ where c is a positive parameter, to be selected later. $R_1(x)=\sum_{n\leqq n_0}^{1}+\sum_{n>n_0}^{2}$. Also, $|\sum^2|\leqq x\sum_{n>n_0}\frac{1}{n^2}<\frac{x}{n_0-1}=(1+\varepsilon)x^{\frac{1}{2}}\{g(x)\}^c=x^{\frac{1}{2}}O\left(\{g(x)\}^c\right)$. By partial summation $|\sum^1|=\left|\sum_{n\leqq n_0}\left(M(n)-M(n-1)\right)\left(\frac{x}{n^2}\right)e^{-\frac{x}{n^2}}\right|\leqq\left|\sum_{n\leqq n_0}M(n)\left(\frac{x}{n^2}e^{-\frac{x}{n^2}}-\frac{x}{(n+1)^2}e^{-\frac{x}{(n+1)^2}}\right)\right|+$

$+\left|M(n_0)\frac{x}{(n_0+1)^2}e^{-\frac{x}{(n_0+1)^2}}\right|$. One observes that $D(n)=\left(\frac{x}{n^2}\right)e^{-\frac{x}{n^2}}-\frac{x}{(n+1)^2}e^{-\frac{x}{(n+1)^2}}$ tends to zero as n (provisionally treated as a continuous variable) tends to either zero or infinity; also, $D(n_1)=0$ for a value $n_1\cong x^{\frac{1}{2}}-\frac{1}{2}$. The extrema of $D(n)$ are taken for $n^2=2x$, when $D=\frac{\sqrt{2}}{4}e^{-\frac{1}{2}}x^{-\frac{1}{2}}(1+O(x^{-1}))$ and for $n^2=\frac{x}{3}$, when $D=-12\sqrt{3}e^{-3}x^{-\frac{1}{2}}\left(1+\frac{9\sqrt{3}}{8}x^{-\frac{1}{2}}+O(x^{-1})\right)$; hence, $|D(n)|=0(x^{-\frac{1}{2}})$. Consequently, $|\sum_{n\leqq n_0}M(n)D(n)|=x^{-\frac{1}{2}}O(\sum_{n\leqq n_0}|M(n)|)=x^{-\frac{1}{2}}O\{n_0\cdot n_0g(n_0)\}=x^{-\frac{1}{2}}O\{x(g(x))^{-2c}\cdot$ $\cdot g(x^{\frac{1}{2}}g(x)^{-c})\}=x^{\frac{1}{2}}O\cdot\{g(x)^{-2c}g(x^{\frac{1}{2}}g(x)^{-c})\}$. Finally, the last term in $|\sum^1|$ is of lower order, namely $O(n_0g(n_0)\cdot g(x)^{2c}e^{-g(x)^{2c}})=O\{x^{\frac{1}{2}}g(x)^c g(x^{\frac{1}{2}}g(x)^{-c})\}=$ $=\{g(x)\}^{3c}x^{\frac{1}{2}}O\{(g(x))^{-2c}\cdot g(x^{\frac{1}{2}}g(x)^{-c})\}$ and is absorbed into the previous term, on account of the factor $\{g(x)\}^{3c}$ which tends to zero for $x\to\infty$, $c>0$. Hence, $R_1(x)=(x^{\frac{1}{2}})\cdot O\{g(x)^c+g(x)^{-2c}g(x^{\frac{1}{2}}g(x)^{-c})\}$, as stated. The proof of the last assertion of Theorem 3 reduces to a simple computation.

PROOF OF THEOREM 4. Let us assume that $\theta<1$ and that Φ_2 is not empty; we shall arrive at a contradiction. Consider the function of a complex variable $\Psi(z)=z^{1-2m}\sum_{n=2}^{\infty}\frac{z^{nm}}{\zeta\left(\frac{n}{2}\right)\varphi\left(\frac{n}{2}\right)}$, for some $\varphi\in\Phi$ and an arbitrary natural integer m. If $z\to\infty$ along the ray $z=xe^{\frac{\pi i}{2m}}$ $(x>0)$, one obtains $\Psi(xe^{\frac{\pi i}{2m}})=ie^{\frac{\pi i}{2m}}x^{1-2m}\cdot$ $\cdot\{F_1(x^{2m})+iF_2(x^{2m})\}$, with

$$F_1(y)=-\frac{1}{2i}\int_{(\sigma)}\frac{y^s\,ds}{\zeta(s)\varphi(s)\sin(\pi s)}=\sum_{n=1}^{\infty}(-1)^n\frac{y^n}{\zeta(n)\varphi(n)}\quad\text{for}\quad\theta<\sigma<1;$$

and

$$F_2(y)=-\frac{1}{2i}\int_{(\sigma)}\frac{y^s\,ds}{\zeta(s)\varphi(s)\sin\left\{\pi\left(s-\frac{1}{2}\right)\right\}}=\sum_{n=1}^{\infty}(-1)^n\frac{y^{n+\frac{1}{2}}}{\zeta\left(n+\frac{1}{2}\right)\varphi\left(n+\frac{1}{2}\right)}$$

$$\text{for}\quad\theta<\sigma<1.$$

The first equalities define $F_i(x)$ $(i=1, 2)$ while the second ones are proven as in the proof of Theorem 1. It follows that $F_i(x)=o(y^{\theta+\varepsilon})$; hence, $\Psi(xe^{\frac{\pi i}{2m}})=$ $=o(x^{1-2m(1-\theta)+\varepsilon})$. Letting $m_0=$ smallest integer in excess of $\frac{1}{2}(1-\theta)^{-1}$, it follows that, for $m\geqq m_0$, the exponent of x is negative and $\Psi(z)\to 0$ as $z\to\infty$ along the selected ray. We make the change of variables $z^m = u$. Then

$$\Psi(z) = u^{\frac{1}{m}-2} \sum_{n=2}^{\infty} \frac{u^n}{\zeta\left(\frac{n}{2}\right)\varphi\left(\frac{n}{2}\right)}.$$

If along some ray going to infinity $\Psi(z) \to 0$, then it follows a fortiori that

$$\Psi_1(u) = u^{-2} \sum_{n=2}^{\infty} \frac{u^n}{\zeta\left(\frac{n}{2}\right)\varphi\left(\frac{n}{2}\right)} = \sum_{n=0}^{\infty} \frac{u^n}{\zeta\left(1+\frac{n}{2}\right)\varphi\left(\frac{n}{2}+1\right)} =$$

$$= \sum_{n=1}^{\infty} \frac{u^n}{\zeta\left(1+\frac{n}{2}\right)\varphi\left(1+\frac{n}{2}\right)} \to 0$$

as $u \to \infty$ along the corresponding ray, $u = iy (y > 0)$. However, by assumption Φ_2 is not empty; hence there exists a function $\varphi \in \Phi$ for which previous conclusions hold and such that $\Psi_1(u)$ is of order less than 1/2, (or of order 1/2 and minimal type). But then, by WIMAN's theorem (see [4], p. 275), (or by its extension to functions of order 1/2 and minimal type[3])) $\Psi_1(u)$ cannot stay bounded along any path going to infinity, contradicting $\Psi_1(u) \to 0$. Hence, unless Φ_2 is empty, $\theta = 1$.

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(Received January 11, 1963.)

[3]) A short proof of this extension is due to L. A. RUBEL, unpublished.