Admissible direct decompositions of direct sums of abelian groups of rank one

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The starting point of the theory of ordinary representations is MASCHKE's Theorem, which states that every representation of a finite group over a field whose characteristic does not divide the order of the group is completely reducible (see e. g. VAN DER WAERDEN [6], p. 182). A partial generalization of this theorem has recently been given by O. GRÜN in [2], and the main step of the classical proof of the theorem has been generalized by M. F. NEWMAN and the author in [4]. Both of these results arose out of a shift in the point of view: they do not refer to representations, but to direct decompositions of abelian groups, admissible with respect to a finite group of operators (in the sense of KUROSH [5], § 15). The aim of this paper is to present an extension of GRÜN's result, exploiting the start made in [4]. The terminology follows, apart from minor deviations, that of FUCHS's book [1].

From [4], only a special case of Theorem 2.2 is needed here:

Lemma. *Let X be an abelian group, and G a finite group of operators on X; suppose that (every element of) X is divisible (in X) by the order of G, and that X has no element (other than 0) whose order is a divisor of the order of G. If Y is an admissible subgroup of X which is also a direct summand of X, then Y has an admissible (direct) complement in X.*

The result of this paper is the following.

Theorem. *Let A be a direct sum of abelian groups of rank one, and G a finite group of operators on A; suppose that A is divisible by the order of G, and that A has no element (other than 0) whose order is a divisor of the order of G. Then A can be written as a direct sum of admissible, G-indecomposable subgroups, each of which is a direct sum of finitely many isomorphic groups of rank one.*

The proof splits into several steps, and occupies the rest of the paper.

(A) *A is a direct sum of countable admissible subgroups each of which is a direct sum of groups of rank one.*

PROOF. Let $A=\Sigma(C_\lambda:\lambda<\sigma)$ where the C_λ are groups of rank one, σ is an ordinal, and λ runs through all the ordinals which precede σ. Denote the corresponding canonical projections $A\to C_\lambda$ by γ_λ. For each ordinal μ such that $\mu<\sigma$, one makes simultaneously the following definitions. Let $\Lambda_\mu^0=\text{set}(\mu)$. If i is a finite ordinal and Λ_μ^i a countable set of ordinals preceding σ, let $C_\mu^i=\Sigma(C_\lambda:\lambda\in\Lambda_\mu^i)$, and let C_μ^iG be the smallest admissible subgroup containing C_μ^i. Then C_μ^i is countable;

as $C_\mu^i G$ is generated by the countably many elements cg with $c \in C_\mu^i$, $g \in G$, $C_\mu^i G$ is also countable. Hence $C_\mu^i G \gamma_\lambda = 0$ for all but countably many values of λ; so that the set Λ_μ^{i+1} defined by $\Lambda_\mu^{i+1} = \mathrm{set}(\lambda : \lambda < \sigma, C_\mu^i G \gamma_\lambda > 0)$ is countable. This inductive definition provides an increasing chain $\Lambda_\mu^0 \subseteq \Lambda_\mu^1 \subseteq \ldots \subseteq \Lambda_\mu^i \subseteq \ldots$ of countable sets of ordinals. In turn, one constructs another increasing chain by defining its general term Λ^ν as $\Lambda^\nu = \bigcup (\Lambda_\mu^i : \mu < \nu, i < \omega)$, for every ordinal ν with $\nu \leqq \sigma$. This chain has the following properties:

(A1) Λ^0 is empty.

(A2) If ϱ is a limit ordinal, $\varrho \leqq \sigma$, then $\Lambda^\varrho = \bigcup (\Lambda_\mu^i : \mu < \sigma, i < \omega) = \bigcup \left[\bigcup (\Lambda_\mu^i : \mu < \nu, i < \omega) : \nu < \varrho \right] = \bigcup (\Lambda^\nu : \nu < \varrho)$.

(A3) If $\mu < \nu \leqq \sigma$, then $\mu \in \Lambda^\nu$; for $\mu \in \Lambda_\mu^0 \subseteq \Lambda^\nu$.

(A4) If $\lambda < \sigma$, then the difference set $\Lambda^{\lambda+1} - \Lambda^\lambda$ is countable; for it is a subset of $\bigcup (\Lambda_\lambda^i : i < \omega)$ and each Λ_λ^i is countable.

Correspondingly, $C^\nu = \Sigma (C_\lambda : \lambda \in \Lambda^\nu)$ defines an increasing chain of partial sums of $\Sigma (C_\lambda : \lambda < \sigma)$, with the properties:

(A1′) $C^0 = 0$;

(A2′) if ϱ is a limit ordinal, $\varrho \leqq \sigma$, then $C^\varrho = \bigcup (C^\nu : \nu < \varrho)$;

(A3′) $C^\sigma = A$;

(A4′) if $\lambda < \sigma$, then $C^{\lambda+1}/C^\lambda$ is a countable direct sum groups of rank one.

Moreover, each C^ν is admissible; for, C^ν is generated by the elements c with $c \in C_\lambda$, $\lambda \in \Lambda^\nu$, and, if $\lambda \in \Lambda_\mu^i$, $\mu < \nu$, $i < \omega$, while g is an arbitrary element of G, then $cg \in C_\mu^{i+1} \leqq C^\nu$. Thus each C^λ with $\lambda < \sigma$ is an admissible direct summand in the admissible subgroup $C^{\lambda+1}$, and so the Lemma, with $X = C^{\lambda+1}$ and $Y = C^\lambda$, gives that C^λ has an admissible complement, say D_λ, in $C^{\lambda+1}$. In view of $D_\lambda \cong C^{\lambda+1}/C^\lambda$ and (A4′), it suffices to prove that $A = \Sigma (D_\lambda : \lambda < \sigma)$. This, in turn, will follow from (A3′) and the general relation $C^\nu = \Sigma (D_\lambda : \lambda < \nu)$ which holds for every ν with $\nu \leqq \sigma$. The validity of this relation is proved by a simple induction: it is valid if $\nu = 0$, because of (A1′); if it is valid for the predecessor $\nu - 1$ of ν, then $C^\nu = C^{\nu-1} + D_{\nu-1} = \Sigma (D_\lambda : \lambda < \nu - 1) + D_{\nu-1} = \Sigma (D_\lambda : \lambda < \nu)$; if it is valid for every ν preceding a limit ordinal ϱ, then $C^\varrho = \bigcup (C^\nu : \nu < \varrho) = \bigcup \left[\Sigma (D_\lambda : \lambda < \nu) : \nu < \varrho \right] = \Sigma (D_\lambda : \lambda < \varrho)$, by (A2′).

(B) Being a direct sum of groups of rank one, A is the direct sum of its maximal p-subgroups A_p and a torsion free subgroup A_0. The A_p are characteristic and therefore admissible subgroups, and so, by the Lemma with $X = A$, $Y = \Sigma A_p$ (where p runs through all primes), A_0 can also be chosen admissible. Moreover, both A_0 and the A_p are direct sums of groups of rank one. This and (A) make it possible to *assume, without loss of generality, that A is countable and either a torsion free or a p-group.* The torsion free case will be discussed first.

(C) *If A is torsion free and B is a subgroup of finite rank in A, then A has a direct decomposition $A = A' + A''$ such that A' is of finite rank and contains B; moreover, both A' and A'' are admissible subgroups of A, and are direct sums of groups of rank one.*

Proof. In order to prove this assertion, one first notes that there is no loss of generality in assuming that B is admissible and pure in A. The justification of this can be outlined as follows. Let B be any subgroup of finite rank and S a maximal independent subset of B. Consider the set SG defined by $SG = \mathrm{set}(sg: s \in S, g \in G)$; this is finite, for both S and G are finite. Let B_G be the set of those elements of A which depend on SG; this is an admissible subgroup of A: for, if $a, b \in B_G$ and $g \in G$, then $ma = m_1 s_1 g_1 + \ldots + m_k s_k g_k$, $nb = n_1 s_1 g_1 + \ldots + n_k s_k g_k$ with suitable integers $m, m_1, \ldots, m_k, n, n_1, \ldots, n_k$, $m \neq 0 \neq n$, and elements $s_1 g_1, \ldots, s_k g_k$ of SG; so that $mn[(a-b)g] = \Sigma[(m_i n - mn_i) s_i g_i g: 1 \leqq i \leqq k]$, $mn \neq 0$ shows that $(a-b)g$ is dependent on SG and hence belongs to B_G. It is easy to see that B_G contains B and is pure in A; moreover, its rank cannot be greater than the cardinal of SG. Thus B can be replaced by B_G.

Let it be assumed therefore that B is admissible and pure in A. Consider an arbitrary decomposition of A into a direct sum of groups of rank one:

$$\text{(C1)} \qquad A = \Sigma(C_\lambda: \lambda \in \Lambda),$$

with the corresponding canonical projections $\gamma_\lambda: A \to C_\lambda$; and define a subset $\Lambda(B, \mathrm{C1})$ of the index set Λ by $\Lambda(B, \mathrm{C1}) = \mathrm{set}(\lambda: \lambda \in \Lambda, B\gamma_\lambda > 0)$. It is easily seen that this subset is finite: if S is a maximal independent subset of B, then B consists precisely of those elements of A which depend on S; so, if $0 \neq b \in B$, then $nb = n_1 s_1 + \ldots + n_k s_k$ for some integrs $n, n_1, \ldots, n_k$, $n \neq 0$, and elements $s_1, \ldots, s_k$ of S; if $S\gamma_\lambda = 0$, then $(nb)\gamma_\lambda = 0$ and, as C_λ is torsion free, $n(b\gamma_\lambda) = 0$ and $n \neq 0$ imply that $b\gamma_\lambda = 0$; so that one has $\Lambda(B, \mathrm{C1}) = \mathrm{set}(\lambda: \lambda \in \Lambda, S\gamma_\lambda \neq 0)$ which, since S is finite, proves the finiteness of $\Lambda(B, \mathrm{C1})$. This subset is used to define $\mathfrak{A}(B, \mathrm{C1})$, a set of types of torsion free groups of rank one: put $\mathfrak{A}(B, \mathrm{C1}) = \mathrm{set}(T(C_\lambda): \lambda \in \Lambda(B, \mathrm{C1}))$; this set of types is clearly also finite.

The statement (C) will be proved by induction on the cardinal $|\mathfrak{A}(B, \mathrm{C1})|$ of $\mathfrak{A}(B, \mathrm{C1})$. If $\mathfrak{A}(B, \mathrm{C1})$ is empty, then $B = 0$ and so (C) is trivially true. Hence one can procede to the inductive step: let $B > 0$, and let (C) be assumed to be true for every choice of A, G, and B to which there is a decomposition like (C1) which yields a cardinal smaller than $|\mathfrak{A}(B, \mathrm{C1})|$. Let $\mathfrak{a}$ be a maximal type in $\mathfrak{A}(B, \mathrm{C1})$, and put $\Lambda_1 = \mathrm{set}(\lambda: \lambda \in \Lambda, T(C_\lambda) > \mathfrak{a})$, $\Lambda_2 = \mathrm{set}(\lambda: \lambda \in \Lambda, T(C_\lambda) = \mathfrak{a})$, and $\Lambda_3 = \mathrm{set}(\lambda: \lambda \in \Lambda, T(C_\lambda) \ngeq \mathfrak{a})$. The three sets so defined are pairwise disjoint and their union is Λ. Let $A^1 = \Sigma(C_\lambda: \lambda \in \Lambda_1)$; then A^1 is the characteristic subgroup of A which is generated by the elements whose types (in A) are greater than $\mathfrak{a}$; so A^1 is admissible. Moreover, A^1 is a direct summand of A, for $A = A^1 + A_1$ with $A_1 = \Sigma(C_\lambda: \lambda \in \Lambda_2 \cup \Lambda_3)$, and B is contained in this complement A_1. Consider the torsion free factor group A/B; this has a direct decomposition $A/B = (A^1 + B)/B + A_1/B$, with $(A^1 + B)/B$ admissible. Hence the Lemma, with $X = A/B$ and $Y = (A^1 + B)/B$, implies that $(A^1 + B)/B$ has an admissible complement, say A^*/B, in A/B. As $A^1 \cap A^* \leqq (A^1 + B) \cap A^* = B$ and $A^1 \cap B = 0$, A^* is in fact an admissible complement of A^1 in A. Let α be the canonical projection of A onto A^* corresponding to the direct decomposition

$$\text{(C2)} \qquad A = A^1 + A^*.$$

If $a \in A$, and $a = a^1 + a_1$ with $a^1 \in A^1, a_1 \in A_1$, then $a\alpha = a^1\alpha + a_1\alpha = a_1\alpha$, so that $A^* = A\alpha = A_1\alpha$. As A_1 avoids the kernel A^1 of α, it is mapped isomorphically by α, so that in fact $A^* = A_1\alpha = \Sigma(C_\lambda\alpha: \lambda \in \Lambda_2 \cup \Lambda_3)$. It is convenient now to change from (C1) to the new decomposition

$$\text{(C3)} \qquad A = A^1 + A^* = \Sigma(C_\lambda: \lambda \in \Lambda_1) + \Sigma(C_\lambda\alpha: \lambda \in \Lambda_2 \cup \Lambda_3) = \Sigma(D_\lambda: \lambda \in \Lambda)$$

where $D_\lambda = C_\lambda$ if $\lambda \in \Lambda_1$ and $D_\lambda = C_\lambda\alpha$ if $\lambda \in \Lambda_2 \cup \Lambda_3$. Of the corresponding canonical projections $\delta_\lambda: A \to D_\lambda$, one has to observe the following. Since $B \leqq A^* = \Sigma(D_\lambda: \lambda \in \Lambda_2 \cup \Lambda_3)$, $B\delta_\lambda = 0$ whenever $\lambda \in \Lambda_1$. On the other hand, if $\lambda \in \Lambda_2 \cup \Lambda_3$, then $\delta_\lambda = \gamma_\lambda\alpha$: for, then $\alpha\delta_\lambda = \delta_\lambda$ by definition; if a is an arbitrary element of A, then $a = \Sigma(a\gamma_\mu: \mu \in \Lambda)$; also, $\gamma_\mu\alpha = 0$ if $\mu \in \Lambda_1$ and $A\gamma_\mu\alpha = D_\mu$ if $\mu \in \Lambda_2 \cup \Lambda_3$, so that in this second case $\gamma_\mu\alpha\delta_\lambda = 0$ if $\mu \neq \lambda$ and $\gamma_\mu\alpha\delta_\lambda = \gamma_\lambda\alpha$ if $\mu = \lambda$; and hence it follows that $a\delta_\lambda = a\alpha\delta_\lambda = \Sigma(a\gamma_\mu\alpha\delta_\lambda: \mu \in \Lambda) = a\gamma_\lambda\alpha$. Also, if $\lambda \in \Lambda_2 \cup \Lambda_3$, then the kernel of α avoids C_λ and so, in this case, $B\delta_\lambda = B\gamma_\lambda\alpha > 0$ is equivalent to $B\gamma_\lambda > 0$. These observations yield the conclusion that $\Lambda(B, \mathrm{C3}) = \Lambda(B, \mathrm{C1})$, and so $\mathfrak{A}(B, \mathrm{C3}) = \mathfrak{A}(B, \mathrm{C1})$ as well.

Next, consider the subgroup A^2 defined by $A^2 = \Sigma(D_\lambda: \lambda \in \Lambda_2)$. This subgroup can be described as the set consisting of 0 and the elements of type $\mathfrak{a}$ in the admissible subgroup A^*; so that A^2 is characteristic in A^* and hence admissible. Also, A^2 is a direct summand in A^* and so the Lemma, with $X = A^*$ and $Y = A^2$, provides that A^2 has an admissible complement, say A^3, in A^*. Thus A has the admissible direct decomposition

$$A = A^1 + A^2 + A^3; \tag{C4}$$

let the corresponding cannonical projections $A \to A^i$ be denoted by α_i, for $i = 1, 2, 3$. Clearly $A^3 = A\alpha_3 = \Sigma(D_\lambda: \lambda \in \Lambda_3)\alpha_3$; as $\Sigma(D_\lambda: \lambda \in \Lambda_3)$ avoids the kernel $A^1 + A^2$ of α_3, this subgroup is mapped isomorphically by α_3, so that $A^3 = \Sigma(D_\lambda\alpha_3: \lambda \in \Lambda_3)$. Put $E_\lambda = D_\lambda$ if $\lambda \in \Lambda_1 \cup \Lambda_2$ and $E_\lambda = D_\lambda\alpha_3$ if $\lambda \in \Lambda_3$; then (C4) can be refined to the decomposition

$$A = \Sigma(E_\lambda: \lambda \in \Lambda). \tag{C5}$$

Like in a similar situation above, one checks that, for the canonical projections $\varepsilon_\lambda: A \to E_\lambda$ corresponding to (C5), $B\varepsilon_\lambda = 0$ if $\varepsilon \in \Lambda_1$ and $\alpha_3\varepsilon_\lambda = \varepsilon_\lambda = \delta_\lambda\alpha_3$ if $\lambda \in \Lambda_3$.

Put $B^2 = B\alpha_2$ and $B^3 = B\alpha_3$; both B^2 and B^3 are of finite rank, and $B \leqq B^2 + B^3$. If $B^3\varepsilon_\lambda > 0$, then $\lambda \in \Lambda_3$ and so $B^3\varepsilon_\lambda = B\alpha_3\varepsilon_\lambda = B\varepsilon_\lambda = B\delta_\lambda\alpha_3$ shows that also $B\delta_\lambda > 0$. Hence $\Lambda(B^3, \mathrm{C5}) \subseteqq \Lambda(B, \mathrm{C3}) = \Lambda(B, \mathrm{C1})$, so that $\mathfrak{A}(B^3, \mathrm{C5}) \subseteqq \mathfrak{A}(B, \mathrm{C1})$; moreover, as $T(E_\lambda) = T(D_\lambda) = T(C_\lambda)$ for every λ in Λ, and as $\Lambda(B^3, \mathrm{C5}) \subseteqq \Lambda_3$, the type $\mathfrak{a}$ does not belong to $\mathfrak{A}(B^3, \mathrm{C5})$. Hence $\mathfrak{A}(B^3, \mathrm{C5})$ is a proper subset of $\mathfrak{A}(B, \mathrm{C1})$, and therefore the induction hypothesis applies to A^3, G, B^3, with the conclusion that A^3 has an admissible direct decomposition $A^3 = V + W$ such that V is of finite rank and contains B^3, while both V and W are direct sums of groups of rank one.

Finally, consider B^2. By the initial step of this proof, A^2 has an admissible pure subgroup U of finite rank which contains B^2. The set $\Lambda(U, \mathrm{C5})$ is a finite subset of Λ_2; put $U' = \Sigma(E_\lambda: \lambda \in \Lambda(U, \mathrm{C5}))$ and $U'' = \Sigma(E_\lambda: \lambda \in \Lambda_2 - \Lambda(U, \mathrm{C5}))$; then $A^2 = U' + U''$ and $U \leqq U'$. Now U is a pure subgroup of the direct sum U' of finitely many groups of rank one which are all of the same type $\mathfrak{a}$; so that a theorem of ČERNIKOV, FUCHS, KERTÉSZ, and SZELE (Theorem 46. 8 in FUCHS [1]) implies that U is a direct summand of U'; hence U is a direct summand of A^2 as well. As U is admissible, the Lemma (with $X = A^2$, $Y = U$) provides that U has an admissible complement, say U^*, in A^2. It follows from a theorem of BAER (Theorem 46. 6 in FUCHS [1]) that both U and U^* are direct sums of groups of rank one.

It remains to put these results together: $A = A^1 + A^2 + A^3 = A^1 + (U + U^*) + (V + W) = (U + V) + (A^1 + U^* + W)$; all these summands are admissible subgroups and direct sums of groups of rank one; $U + V$ is of finite rank; and $B \leqq B^2 +$

$B^3 \leqq U+V$; so that A' and A'' given by $A' = U+V$ and $A'' = A^1 + U^* + W$ satisfy the claims made in (C).

(D) *If A is torsion free, then A can be written as a direct sum of admissible subgroups of finite rank such that each of the summands is a direct sum of groups of rank one.*

PROOF. In view of (A), A can be assumed to be countable; moreover, only the case when A is of infinite rank needs investigation. Let $A = \Sigma(C_i : 1 \leqq i < \omega)$ be a direct decomposition of A in which all the C_i are groups of rank one. According to (C), A can be written as $C^1 + D^1$ in such a way that $C_1 \leqq C^1$, both C^1 and D^1 are admissible subgroups and direct sums of groups of rank one, and the rank of C^1 is finite. Suppose that, for some positive integer n,

$$A = C^1 + C^2 + \dots + C^n + D^n \tag{D1}$$

is an admissible direct decomposition of A in which all the summands are direct sums of groups of rank one, all but the last are of finite rank, and $C_1 + \dots + C_n \leqq \leqq C^1 + \dots + C^n$. Let δ be the canonical projection of A onto D^n, corresponding to (D1). Then $C_{n+1}\delta$ is a subgroup of finite rank in D^n, and so (C) provides that D^n has a direct decomposition $D^n = C^{n+1} + D^{n+1}$ such that C^{n+1} and D^{n+1} are admissible subgroups which are again direct sums of groups of rank one, $C_{n+1}\delta \leqq C^{n+1}$, and C^{n+1} is of finite rank. Thus $A = C^1 + \dots + C^n + C^{n+1} + D^{n+1}$, and now $C_1 + \dots + C_n + C_{n+1} \leqq C^1 + \dots + C^n + C^{n+1}$, so that a decomposition like (D1) has been obtained for $n+1$ in place of n. This inductive process defines a subgroup C^i for each positive integer i. It is easily seen that the subgroup generated by the C^i is their direct sum, and it contains all the C_i. Therefore $A = \Sigma(C^i : 1 \leqq i < \omega)$, and this is a direct decomposition satisfyng the claims made in (D).

(E) *If A is torsion free, then the Theorem is true.*

PROOF. According to (D), it can be assumed that A is of finite rank. In this case A is trivially a direct sum of G-indecomposable subgroups; it remains to prove the assertion about the structure of its G-indecomposable summands. Let B be an arbitrary G-indecomposable summand of A, and let $B \neq 0$. First, a theorem of BAER (Theorem 46. 7 in FUCHS [1]) gives that B is a direct sum of groups of rank one. Let $B = C_1 + \dots + C_n$ with all the C_i of rank one, and let $\mathfrak{a}$ be a maximal element of the set of types $T(C_i)$, $i = 1, \dots, n$. Put $B_1 = \Sigma(C_i : T(C_i) = \mathfrak{a})$ and $B_2 = \Sigma(C_i : T(C_i) \neq \mathfrak{a})$; then $B = B_1 + B_2$. The subgroup B_1 consists precisely of 0 and the elements of type $\mathfrak{a}$ in B, so that B_1 is characteristic in B and hence admissible. Thus the Lemma, with $X = B$ and $Y = B_1$, gives that B_1 has an admissible complement in B; as B is G-indecomposable and $B_1 \neq 0$, this complement can only be 0. Hence $B_1 = B$, so that all the C_i are of the same type $\mathfrak{a}$.

In view of (B), it is possible to *assume for the rest of the proof that A is a countable p-group.* In (F) a special case will be discussed, and (G) will provide the key to the general case.

(F) *If $pA = 0$, then the Theorem is true.*

PROOF. If A is finite as well, this statement is trivially true. Let A be countably infinite; then $A = \Sigma(C_i : 1 \leqq i < \omega)$ where all the C_i are of order p. For each positive integer j, let $C^j = \Sigma(C_i : 1 \leqq i \leqq j)$, and let C^jG be the subgroup generated by all the elements of the form cg with $c \in C^j$, $g \in G$. Then all the C^j and the C^jG are finite; $C^j \leqq C^{j+1}$ and $C^j \leqq C^jG \leqq C^{j+1}G$ hold for every j; and $\bigcup (C^j: 1 \leqq j < \omega) = A$, so that also $\bigcup (C^jG: 1 \leqq j < \omega) = A$. Since now every subgroup of A is a direct summand of A, and since the C^jG are admissible, the Lemma (with $X = C^{j+1}G$, $Y = C^jG$) gives that each C^jG has an admissible complement, say D_{j+1}, in the corresponding $C^{j+1}G$. In addition, let $D_1 = C^1G$. Then it is easy to see that $C^jG = \Sigma(D_i: 1 \leqq i \leqq j)$ holds for every j, so that $A = \bigcup (C^jG: 1 \leqq j < \omega) = \bigcup [\Sigma(D_i: 1 \leqq i \leqq j): 1 \leqq j < \omega] = \Sigma(D_i: 1 \leqq i < \omega)$. Each D_i is admissible and, being contained in the finite C^iG, finite. Therefore each D_i is a direct sum of finitely many finite G-indecomposable subgroups, so that the direct decomposition of A obtained above can be refined to one in which all the summands are finite, admissible, and G-indecomposable. This refinement satisfies the Theorem.

(G) *Let T be an admissible, G-indecomposable subgroup in the socle S of A. Then T is finite, and A has a direct summand B which is admissible, G-indecomposable, and whose socle is precisely T; moreover, B is G-indecomposable.*

PROOF. It follows from (F) that T must be finite. If $T = 0$, then $B = 0$ will do; hence suppose that $T > 0$. Let k be one of the ordinals $0, 1, \ldots, \omega$; then p^kA is a characteristic and hence admissible subgroup of A. As every subgroup of T is a direct summand of T, the Lemma can be applied to $X = T$, $Y = T \cap p^kA$, with the conclusion that $T \cap p^kA$ has an admissible complement in T. Since T is G-indecomposable, it follows that either $T \cap p^kA = 0$ or $T \cap p^kA = T$. If $T \leqq p^\omega A$, let $m = \omega$. If $T \cap p^\omega A = 0$, then $T \cap p^kA = 0$ for some finite ordinals k; but not for all, for $T \leqq A = p^0A$. Hence the first of the ordinals k for which $T \cap p^kA = 0$, can be written in the form $m+1$, and then $T \leqq p^mA$, $T \cap p^{m+1}A = 0$.

Since every subgroup of S is a direct summand of S, the Lemma can be applied to $X = S$, $Y = T$ with the conclusion that $S = T + U$ for some admissible subgroup U. Let $U_k = U \cap p^kA$, for $k = 0, 1, \ldots, \omega$; then $S \cap p^kA = T + U_k$ for every k with $k \leqq m$.

Let it be agreed that $\omega - i = \omega$ for every finite ordinal i.

Put $B_0 = T$. If $m > 0$, suppose that, for some ordinal k with $k < m$, and increasing chain $B_0, \ldots, B_k$ of admissible subgroups has been defined in such a way that $B_i \leqq p^{m-i}A$, $T = p^iB_i$, and B_i has T as its socle, for $i = 0, \ldots, k$. Let V/B_k be the socle of $p^{m-k-1}A/B_k$. As the socle T of B_k intersects U_{m-k-1} in 0, the subgroup W generated by B_k and U_{m-k-1} is their direct sum: $W = B_k + U_{m-k-1}$. The factor group W/B_k is an admissible subgroup in V/B_k and, as every subgroup of V/B_k is a direct summand of V/B_k, the Lemma (with $X = V/B_k$, $Y = W/B_k$) provides that W/B_k has an admissible complement, say B_{k+1}/B_k, in V/B_k. The subgroup B_{k+1} so chosen is admissible, contains B_k, and is contained in $p^{m-k-1}A$. The socle T' of B_{k+1} contains T and is contained in $S \cap p^{m-k-1}A$; hence, as $S \cap p^{m-k-1}A = T + U_{m-k-1}$, $T' = T + (T' \cap U_{m-k-1})$. On the other hand, one knows that $T' \cap U_{m-k-1} \leqq B_{k+1} \cap W = B_k$, so that $T' \cap U_{m-k-1} \leqq B_k \cap U_{m-k-1} = 0$; hence it follows that $T' = T + 0 = T$. Next, note that $pB_{k+1} \leqq B_k$ is an immediate consequence of the choice of B_{k+1}. On the other hand, if $b \in B_k$, then $B_k \leqq p^{m-k}A = p(p^{m-k-1}A)$ implies that $b = pa$ for some a in $p^{m-k-1}A$; for this a, $a + B_k \in V/B_k = (B_k + U_{m-k-1})/B_k + B_{k+1}/B_k$, so that $a = u + b'$ with $u \in U_{m-k-1}$, $b' \in B_{k+1}$, and this shows that $b = pa = pu + pb' = pb' \in pB_{k+1}$; hence $B_k \leqq pB_{k+1}$. Thus in fact

$B_k = pB_{k+1}$, and therefore $T = p^k B_k = p^{k+1} B_{k+1}$. To sum up: $B_0, \dots, B_k, B_{k+1}$ has all the relevant properties of $B_0, \dots, B_k$, with $k+1$ in place of k.

If m is finite, this inductive process provides in a finite number of steps a subgroup B_m; in this case, put $B = B_m$. If $m = \omega$, then the process provides a subgroup B_k for every finite ordinal k; in this case, let $B = \cup (B_k : 0 \leqq k < \omega)$. In each case, the socle of B is precisely T. In the first case, every non-zero element of T has height m in A (for $T \leqq p^m A$ but $T \cap p^{m+1} A = 0$), and its height in B is also m (for $T = p^m B_m = p^m B$); hence [e. g. by J) on p. 78 of FUCHS [1]] B is a pure subgroup in A; moreover, B is bounded, so that a theorem of KULIKOV (Theorem 24. 5 in FUCHS [1]) implies that B is a direct summand of A. In the second case, every non-zero element of T is of infinite height in B (as $T = p^k B_k \leqq p^k B$ for every finite k), so that B is divisible [see e. g. (f) on p. 59 of FUCHS [1]], and hence, acording to a theorem of BAER (Theorem 18. 1 in FUCHS [1]), B is a direct summand of A. By construction, B is admissible; and the G-indecomposability of T implies that B is also G-indecomposable. This completes the proof of (G).

(H) If A is a p-group, then $A = \Sigma(C_\lambda : \lambda \in \Lambda)$ where each C_λ is either cyclic or of the type $C(p^\infty)$, Let $C = \Sigma(C_\lambda : \lambda \in \Lambda, C_\lambda$ cyclic) and $D = \Sigma(C_\lambda : \lambda \in \Lambda, C_\lambda \cong C(p^\infty))$; then D is precisely the maximal divisible subgroup of A, so that D is characteristic in A and is therefore also admissible. Now the Lemma (with $X = A$, $Y = D$) provides that D has an admissible complement C' in A. Of necessity, $C' \cong C$, so that C' is a direct sum of cyclic groups. Hence *it suffices to prove the Theorem under the further assumption that A is either divisible or a direct sum of cyclic groups.*

(I) *If A is a divisible p-group, the Theorem is true.*

PROOF. In view of (F), the socle S of A can be written as a direct sum of finite, admissible, G-indecomposable subgroups T_λ, with λ running through some index set Λ. According to (G), each T_λ is the socle of some admissible, G-indecomposable direct summand B_λ of A. Each B_λ is of finite rank, for its socle T_λ is finite, and each B_λ is divisible; hence each B_λ is a direct sum of finitely many (isomorphic) divisible groups of rank one (that is, of groups of the type $C(p^\infty)$; by another theorem of BAER, Theorem 19. 1 in FUCHS [1]). The subgroup generated by the B_λ is their direct sum, and it is divisible; moreover, it contains the whole socle of A; hence $A = \Sigma(B_\lambda : \lambda \in \Lambda)$.

(J) *If A is a direct sum of cyclic p-groups, then A is a direct sum of bounded admissible subgroups.*

PROOF. Now all the direct summands of A which are of rank one are cyclic groups. If A is of finite rank, then A itself is bounded, so there is nothing to prove. In view of (A), it can be assumed that A is countable, so that in the remaining case $A = \Sigma(C_i : 1 \leqq i < \omega)$ where all the C_i are cyclic. Let S be the socle of A and S_i the socle of C_i, for each i; then $S = \Sigma(S_i : 1 \leqq i < \omega)$. As before, VG will denote, for each subgroup V of A, the subgroup generated by all the elements of the form vg with $v \in V$, $g \in G$; VG is always admissible; and, if V is finite, then so is VG. Since A is a direct sum of cyclic groups, $p^\omega A = 0$, and so each finite subgroup of A must have zero intersection with $p^k A$ for some positive integer k.

Let $k(1)$ be the first positive integer for which $S_1 G \cap p^{k(1)} A = 0$. Since $S \cap p^{k(1)} A$ is characteristic in A, it is also admissible. Let $S^1 = S_1 G + (S \cap p^{k(1)} A)$; then $S^1 / S_1 G$

is an admissible subgroup and a direct summand in the elementary group S/S_1G. On applying the Lemma to $X=S/S_1G$, $Y=S^1/S_1G$, one obtains an admissible complement, say T_1/S_1G, for S^1/S_1G in S/S_1G. As T_1 and S^1 generate S, and as $S_1G\leqq T_1$, T_1 and $S\cap p^{k(1)}A$ also generate S; moreover, $T_1\cap(S\cap p^{k(1)}A)= = T_1\cap S^1\cap(S\cap p^{k(1)}A) = S_1G\cap(S\cap p^{k(1)}A)=0$; so that in fact $S=T_1+ +(S\cap p^{k(1)}A)$.

For an inductive construction, suppose that $T_1, \ldots, T_j, k(1), \ldots, k(j)$ have already been defined, in such a way that $T_1, \ldots, T_j$ are admissible subgroups, $S_1+\ldots+S_j\leqq T_1+\ldots+T_j$, $k(1)\leqq\ldots\leqq k(j)$, and $S=T_1+\ldots+T_i+(S\cap p^{k(i)}A)$ for every i with $1\leqq i\leqq j$. Let π denote the canonical projection of S onto $S\cap p^{k(j)}A$ corresponding to the direct decomposition $S=T_1+\ldots+T_j+(S\cap p^{k(j)}A)$, and let $k(j+1)$ be either $k(j)$ or the first positive integer for which $S_{j+1}G\pi\cap p^{k(j+1)}A=0$, whichever is the larger. Check that $S_{j+1}G\pi$ is admissible. Similarly to the application in the preceding paragraph, the Lemma can be used to prove the existence of an admissible complement T_{j+1} of $S\cap p^{k(j+1)}A$ in $S\cap p^{k(j)}A$ such that $S_{j+1}G\pi\leqq \leqq T_{j+1}$. It can easily be seen that the hypothesis carries over to $T_1, \ldots, T_j, T_{j+1}$, $k(1), \ldots, k(j), k(j+1)$.

This process defines, for each positive integer i, and admissible subgroup T_i and a positive integer $k(i)$. The subgroup generated by the T_i is their direct sum, and it contains all the S_i, so that it is equal to S. Thus, if $T_1+\ldots+T_j$ is denoted by T^j, one has that $S=\cup(T^j\colon 1\leqq j<\omega)$. Moreover, $S=T^j+(S\cap p^{k(j)}A)$ for every positive integer j.

Next, let B_1 be a subgroup of A maximal with respect to being admissible and having T^1 for its socle. For another induction, suppose that $B_1, \ldots, B_j$ are already defined in such a way that they form an increasing chain of admissible subgroups and the socle of B_i is T^i whenever $1\leqq i\leqq j$. Then B_j intersects T_{j+1} in 0, for its socle does; so the subgroup generated by B_j and T_{j+1} is their direct sum B_j+T_{j+1}, and its socle is T^{j+1}. Thus it is possible to choose B_{j+1} as a subgroup which contains B_j+T_{j+1} and is maximal with respect to being admissible and having T^{j+1} for its socle. This process provides, for each positive integer j, an admissible subgroup B_j, such that these subgroups form an increasing chain, and the socle of each B_j is the corresponding T^j.

Observe that, for each j, B_j and $p^{k(j)}A$ intersect in 0, for their socles do so. Therefore one can speak of the direct sum C of B_j and $S\cap p^{k(j)}A$ in A. Let U be the socle of A/B_j; clearly, C/B_j is an admissible subgroup and a direct summand in U. Hence, according to the Lemma, C/B_j has an admissible complement, say B/B_j, in U. Now B is admissible, and $B\cap C=B_j$. The socle of B contains T^j, and so it is $T^j+(B\cap S\cap p^{k(j)}A)$; but $B\cap(S\cap p^{k(j)}A)=B\cap C\cap(S\cap p^{k(j)}A)\leqq \leqq B_j\cap p^{k(j)}A=0$, so that in fact the socle of B is just T^j. Hence, by the maximality of B_j, it follows that $B=B_j$; therefore $U=C/B_j$. Now if E is any subgroup of A such that $B_j<E$, then $E/B_j\cap U=E/B_j\cap C/B_j>0$, and so $E\cap C>B_j$: it follows that $E\cap p^{k(j)}A>0$. Thus B_j is maximal among all the subgroups of A which intersect $p^{k(j)}A$ in 0, so that a Lemma of M. ERDÉLYI (Lemma 1 in [3]; or, the main step in the proof of Theorem 24. 8 in FUCHS [1]) proves that B_j is a direct summand of A.

Hence the union of the B_j is pure in A [e. g. by F) on p. 77 in FUCHS [1]]; and it contains the union of the T^j, which is the whole socle of A; so that in fact $\cup(B_j\colon 1\leqq j<\omega)=A$ [e. g. by K) on p. 78 in FUCHS [1]]. Put $B^1=B_1$, and apply the Lemma (with $X=B_{j+1}$, $Y=B_j$) to obtain an admissible complement B^{j+1} for each B_j in the corresponding B_{j+1}. Then it follows readily that $B_j = \Sigma(B^i\colon 1\leqq i\leqq j)$ for every j, and so $A = \bigcup(B_j\colon 1\leqq j<\omega) = \bigcup[\Sigma(B^i\colon 1\leqq i\leqq j)\colon 1\leqq j<\omega] = \Sigma(B^i\colon 1\leqq i<\omega)$. Here all the B^i are admissible subgroups, and $p^{k(i)}B^i\leqq B^i\cap p^{k(i)}A\leqq B_i\cap p^{k(i)}A=0$ shows that they are all bounded subgroups as well.

(K) *If A is a direct sum of cyclic p-groups, the Theorem is true.*

PROOF. In view of (J), it may be assumed that A is bounded; say, $p^nA=0$. Let the socle of A be S. For $i=1, \dots, n$, let T_i be an admissible complement of $S\cap p^iA$ in $S\cap p^{i-1}A$; such complements exist, for each $S\cap p^iA$ and $S\cap p^{i-1}A$ is characteristic in A and is therefore admissible, and each subgroup of S is a direct summand in every subgroup of S which contains it, so that the Lemma can be applied to $X = S\cap p^iA$, $Y = S\cap p^{i-1}A$. Then $S = T_1+\dots+T_n$, and $S = T_1+\dots+T_i+ +(S\cap p^iA)$ for every i. As in the proof of (J), one constructs admissible subgroups $B^1, \dots, B^n$ such that

$$A = B^1+\dots+B^n, \tag{K1}$$

the socle of $B^1+\dots+B^i$ is precisely $T_1+\dots+T_i$, and

$$B^i\cap p^iA=0, \tag{K2}$$

whenever $1\leqq i\leqq n$.

One checks that the socle of B^i is precisely T_i, and that $T_i=p^{i-1}B^i$, as follows. The assertion is trivial for $i=1$; in fact, $B^1=T_1$. Let $1<i\leqq n$ and $t\in T_i$. Then $t\in p^{i-1}A$; say, $t=p^{i-1}a$, $a\in A$. Write a as $b_1+\dots+b_n$, according to (K1). By (K2), $p^{i-1}b_1=\dots=p^{i-1}b_{i-1}=0$, so that

$$t = p^{i-1}a = p^{i-1}b_i+\dots+p^{i-1}b_n. \tag{K3}$$

On the other hand, (K3) is a decomposition of t corresponding to (K1), and $t\in B^1+\dots+B^i$, so that one must have $t=p^{i-1}b_i$. This proves that $T_i\leqq p^{i-1}B^i\leqq B^i$. Since now the socle of B^i contains T_i and is contained in $T_1+\dots+T_i$, it is in fact $T_i+[B^i\cap(T_1+\dots+T_{i-1})]$; but $B^i\cap(T_1+\dots+T_{i-1})\leqq B^i\cap(B^1+\dots+B^{i-1})=0$, and so the socle of B^i is precisely T_i. Also, (K2) implies that $p(p^{i-1}B^i)=0$, so that $p^{i-1}B^i\leqq T_i$; the converse inclusion has already been seen, so that $T_i=p^{i-1}B^i$.

It follows that every non-zero element in the socle of B^i is of height $i-1$ in B^i, so that B^i is a direct sum of isomorphic cyclic groups of order p^i. This and (K1) imply that it can be assumed without loss of generality that A is a direct sum of isomorphic cyclic groups; each of order p^m, say. The proof will be completed under this additional hypothesis.

In view of (F), the socle S of A is a direct sum of finite, admissible, G-indecomposable subgroups T_λ. According to (G), each T_λ is the socle of some admissible, G-indecomposable direct summand B_λ of A. The B_λ are then also direct sums of cyclic groups of order p^m, as is every direct summand of A; the subgroup generated by the B_λ is their direct sum, and it contains the whole socle S of A; it is also a direct sum of cyclic groups of order p^m, so that it must be the whole of A. Finally, each B_λ is finite, for its socle T_λ is finite.

The steps (B), (E), (H), (I), (K) together prove the Theorem.

Remark. After the preparation of the paper had been completed, PROFESSOR REINHOLD BAER kindly called the attention of the author to the fact that a combination of results of KULIKOV and KAPLANSKY implies that every direct summand of A is a direct sum of groups of rank one; this would allow some minor cuts in the present proof.

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