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On the curvature tensor fields of a type of contact metric manifolds and of its certain submanifolds

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Abstract. It is well-known that on a Sasakian manifold the curvature tensor is determined by the pointwise constant ϕ-sectional curvature completely and this pointwise ϕ-sectional curvature is constant on a Sasakian manifold of dimension ≥ 5. In this paper, we consider contact metric manifolds with ξ belonging to the k-nullity distribution and with pointwise constant ϕ-sectional curvature. Then we shall see that the curvature tensor on its contact metric manifold is determined by the pointwise constant ϕ-sectional curvature perfectly and this sectional curvature is constant on its manifold of dimension ≥ 5, that is, the former result on Sasakian manifolds is a special case in our result. We shall also study the curvature tensor of invariant submanifolds in its contact metric manifold.

1. Introduction

It is well-known that on a Sasakian manifold M^{2n+1} the curvature tensor is completely determined by the pointwise constant ϕ-sectional curvatures H, that is

$$
\begin{aligned}
(1.1)\quad R(X,Y)Z = {} & \frac{(H+3)}{4}(g(Y,Z)X - g(X,Z)Y) \\
& + \frac{(H-1)}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
& + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi + g(\phi Y,Z)\phi X \\
& - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z)
\end{aligned}
$$

for any vector fields X, Y, Z on M^{2n+1}. In particular, H is constant on a Sasakian manifold of dimension ≥ 5 (e.g., see [15] p. 11 and [16] p. 280). A Sasakian manifold M^{2n+1} is called a Sasakian space form if M^{2n+1} has constant ϕ-sectional curvature H. Sasakian space forms have been studied by many authors (e.g., see the last part of Chapter VI of [15] and Chapter VI of [16]).

On the other hand, Kon ([9], [10] and [11]), Harada ([6]) and Kenmotsu ([7] and [8]) studied an invariant submanifold of a Sasakian space form.

In this paper we consider the pointwise constant ϕ-sectional curvature on a contact metric manifold with ξ belonging to the k-nullity distribution. We shall see that on its contact metric manifold the curvature tensor is perfectly determined by the pointwise constant ϕ-sectional curvature (Theorem 3.1 of the present paper). This is a generalization of (1.1) in this introduction. We shall also study the curvature tensor of an invariant submanifold in its contact metric manifold.

2. Preliminaries

Let M be a $(2n+1)$-dimensional contact metric manifold and (ϕ, ξ, η, g) be its contact metric structure. Then we have

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$
$$g(X, \xi) = \eta(X), \quad d\eta(X, Y) = g(\phi X, Y)$$

for any vector field X and Y on M. On such a manifold we define an operator h by $h = -\frac{1}{2}\mathcal{L}_\xi \phi$, where $\mathcal{L}$ denotes the Lie differentiation. Then h is symmetric, h anti-commutes with ϕ (i.e., $\phi h + h\phi = 0$), $h\xi = 0$, $\eta \circ h = 0$ and $\operatorname{Tr} h = 0$ ($\operatorname{Tr} h$ is the trace of h). It is well-known that the vector field ξ is a Killing vector field if and only if h vanishes, and

$$\nabla_X \xi = \phi X + \phi h X \quad (\text{and thus } \nabla_\xi \xi = 0), \tag{2.1}$$

where ∇ is the Riemannian connection of g (e.g., [5], cf. [1]). A contact metric manifold M for which ξ is Killing is called a K-contact manifold. We also recall that on a K-contact manifold $R(X, \xi)\xi = X - \eta(X)\xi$. A contact structure on M^{2n+1} gives rise to an almost complex structure on the product $M^{2n+1} \times R$, where R is the real line. If this almost complex structure is integrable, the contact metric manifold is said to be Sasakian. Equivalently, a contact metric manifold is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \quad ((\nabla_X \phi)Y = \eta(Y)X - g(X, Y)\xi).$$

The k-nullity distribution (e.g., see [13]) of a Riemannian manifold (M, g) for a real number k is a distribution

$$N(k) : p \to N_p(k) = \{Z \in T_p(M) \mid R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\}$$

for any $X, Y \in T_p(M)$. From now on (if we do not refer to something else) we suppose that M is a contact metric manifold with ξ belonging to the k-nullity distribution, i.e.,

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y). \tag{2.2}$$

In particular, if M is Sasakian, then $k = 1$.

The following two lemmas are needed later.

Lemma A ([12] and [13]). *Let M be a contact metric manifold with ξ belonging to the k-nullity distribution. Then we have*

$$h^2 = (k-1)\phi^2 \quad \textit{(and hence } k \leq 1\textit{) and} \tag{2.3}$$

$$\begin{aligned} &-(\nabla_X h)Y + (\nabla_Y h)X \\ &= (1-k)(2g(X,\phi Y)\xi + \eta(X)\phi Y - \eta(Y)\phi X) \\ &\quad + \eta(X)\phi h Y - \eta(Y)\phi h X \end{aligned} \tag{2.4}$$

for any vector fields X and Y on M (cf. [12]).

Lemma B ([12]). *Let M be a contact metric manifold with ξ belonging to the k-nullity distribution. Then*

$$\begin{aligned} R(X,Y)\phi Z - \phi R(X,Y)Z = &\Big\{(1-k)(\eta(X)g(\phi Y, Z) \\ &- \eta(Y)g(\phi X, Z)) + \eta(X)g(\phi h Y, Z) - \eta(Y)g(\phi h X, Z)\Big\}\xi \\ &- g(Y + hY, Z)(\phi X + \phi h X) + g(X + hX, Z)(\phi Y + \phi h Y) \\ &- g(\phi Y + \phi h Y, Z)(X + hX) + g(\phi X + \phi h X, Z)(Y + hY) \\ &- \eta(Z)\Big\{(1-k)(\eta(X)\phi Y - \eta(Y)\phi X) + \eta(X)\phi h Y \\ &- \eta(Y)\phi h X\Big\} \quad \textit{and} \end{aligned} \tag{2.5}$$

$$\begin{aligned} g(\phi R(\phi X, \phi Y)Z, \phi W) = &\, g(R(X,Y)Z, W) \\ &+ \eta(Y)\Big\{(1-k)(\eta(Z)g(W,X) - \eta(W)g(Z,X)) \end{aligned} \tag{2.6}$$

$$
\begin{aligned}
&+ \eta(Z)g(hW,X) - \eta(W)g(hZ,X)\Big\} \\
&- \eta(X)\Big\{(1-k)(\eta(Z)g(W,Y) - \eta(W)g(Z,Y)) \\
&+ \eta(Z)g(hW,Y) - \eta(W)g(hZ,Y)\Big\} + g(X,\phi Z + \phi hZ) \\
&\times g(W + hW, \phi Y) - g(X, \phi W + \phi hW)g(Z + hZ, \phi Y) \\
&- g(X, W + hW)g(Y, Z + hZ) \\
&+ g(X, Z + hZ)g(Y, W + hW)
\end{aligned}
$$

for any vector fields X, Y, Z and W on M.

3. Contact metric manifolds of constant ϕ-sectional curvature

If X is a unit vector which is orthogonal to ξ, we say that X and ϕX span a ϕ-section. If the sectional curvature $H(X)$ of all ϕ-sections is independent of X we say that M is of pointwise constant ϕ-sectional curvature.

Now we get (cf. (1.1)) the following.

Theorem 3.1. *Let M be a contact metric manifold with ξ belonging to the k-nullity distribution. If M is of pointwise constant ϕ-sectional curvature H, then the curvature tensor has the following form*

$$
\begin{aligned}
(3.1)\quad 4R(X,Y)Z &= (H+3)(g(Y,Z)X - g(X,Z)Y) \\
&+ (H-1)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi \\
&- \eta(X)g(Y,Z)\xi + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\
&- 2g(\phi X,Y)\phi Z) + 4(k-1)(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\
&+ \eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi) + 4(g(hY,Z)X \\
&- g(hX,Z)Y + g(Y,Z)hX - g(X,Z)hY + \eta(X)\eta(Z)hY \\
&- \eta(Y)\eta(Z)hX + \eta(Y)g(hX,Z)\xi - \eta(X)g(hY,Z)\xi) \\
&+ 2(g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY \\
&- g(\phi hY,Z)\phi hX),
\end{aligned}
$$

where H is constant on M if $n \neq 1$.

PROOF. By the assumption we have $g(R(X,\phi X)X,\phi X)+H_m\|X\|^4=0$ at any point $m\in M$ and for any $X\in T_m(M)$, $X\perp\xi$. It is clear that this condition implies

$$g(R(\phi X,\phi^2X)\phi X,\phi^2X)+H_m g(\phi X,\phi X)g(\phi X,\phi X)=0 \tag{3.2}$$

at any point $m\in M$, and for any $X\in T_m(M)$. Set

$$P(X,Y,Z,W)=g(R(\phi X,\phi^2Y)\phi Z,\phi^2W)+H_m g(\phi X,\phi Z)g(\phi Y,\phi W).$$

The tensor P satisfies $P(X,Y,Z,W)=P(Z,W,X,Y)$. Therefore (3.2) is equivalent to

$$\begin{aligned}&P(X,Y,Z,W)+P(X,Y,W,Z)+P(Y,X,Z,W)\\&\quad+P(Y,X,W,Z)+P(X,W,Y,Z)+P(X,W,Z,Y)\\&\quad+P(W,X,Y,Z)+P(W,X,Z,Y)+P(X,Z,Y,W)\\&\quad+P(X,Z,W,Y)+P(Z,X,Y,W)+P(Z,X,W,Y)=0.\end{aligned} \tag{3.3}$$

On the other hand, using (2.5) and the property of the Riemannian curvature tensor, we generally have

$$\begin{aligned}&g(R(X,Y)\phi Z,W)+g(R(X,Y)Z,\phi W)=g(R(\phi Z,W)X,Y)\\&\quad+g(R(Z,\phi W)X,Y)=((1-k)(\eta(X)g(\phi Y,Z)\\&\quad-\eta(Y)g(\phi X,Z))+\eta(X)g(\phi hY,Z)-\eta(Y)g(\phi hX,Z))\eta(W)\\&\quad-g(Y+hY,Z)g(\phi X+\phi hX,W)+g(X+hX,Z)\\&\quad\times g(\phi Y+\phi hY,W)-g(\phi Y+\phi hY,Z)g(X+hX,W)\\&\quad+g(\phi X+\phi hX,Z)g(Y+hY,W)-\eta(Z)((1-k)(\eta(X)g(\phi Y,W)\\&\quad-\eta(Y)g(\phi X,W))+\eta(X)g(\phi hY,W)-\eta(Y)g(\phi hX,W)).\end{aligned} \tag{3.4}$$

By using (3.4), the property of the Riemannian curvature tensor and (2.2), after some lengthy computation (3.3) leads us to

$$\begin{aligned}&2g(R(\phi X,Y)\phi Z,W)+2g(R(\phi X,W)\phi Y,Z)\\&\quad+2g(R(\phi X,Z)\phi W,Y)+2g(\phi W,Y)g(\phi X,Z)\\&\quad+2g(\phi X,Y)g(\phi Z,W)+2g(\phi X,W)g(\phi Y,Z)\\&\quad+k(\eta(Y)\eta(Z)g(X,W)+\eta(Y)\eta(W)g(X,Z)\\&\quad+\eta(Z)\eta(W)g(X,Y)-\eta(X)\eta(Y)g(Z,W)\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&- \eta(X)\eta(Z)g(Y,W) - \eta(X)\eta(W)g(Y,Z)) \\
&+ (k-2H)(\eta(X)\eta(Y)g(Z,W) + \eta(X)\eta(Z)g(Y,W) \\
&+ \eta(X)\eta(W)g(Y,Z) + \eta(Y)\eta(Z)g(X,W) \\
&+ \eta(Y)\eta(W)g(X,Z) + \eta(Z)\eta(W)g(X,Y)) \\
&+ 2H(g(X,Y)g(Z,W) + g(X,Z)g(Y,W) \\
&+ g(X,W)g(Y,Z)) + 6(H-k)\eta(X)\eta(Y)\eta(Z)\eta(W) \\
&+ 2(g(W,\phi hY)g(\phi X,Z) - g(\phi W,Y)g(\phi hX,Z) \\
&+ g(\phi X,Y)g(\phi hZ,W) - g(\phi hX,Y)g(\phi Z,W) \\
&+ g(\phi X,W)g(\phi hY,Z) - g(\phi hX,W)g(\phi Y,Z) \\
&+ g(hY,Z)g(X,W) - g(hX,W)g(Z,Y) + g(hY,W)g(X,Z) \\
&- g(Y,W)g(hX,Z) + g(hW,Z)g(X,Y) - g(W,Z)g(hX,Y) \\
&+ \eta(Y)\eta(W)g(hX,Z) - \eta(X)\eta(W)g(hY,Z) \\
&+ \eta(Y)\eta(Z)g(hX,W) - \eta(X)\eta(Z)g(hY,W) \\
&+ \eta(Z)\eta(W)g(hX,Y) - \eta(X)\eta(Y)g(hW,Z)) = 0.
\end{aligned}
$$

On the other hand, using (3.4) and the property of the curvature tensor again, we have

$$
\begin{aligned}
(3.6)\quad & g(R(\phi X,W)\phi Y,Z) + g(R(\phi X,Z)\phi W,Y) \\
&= g(R(\phi W,X)\phi Y,Z) + g(R(\phi Z,X)\phi Y,W) \\
&\quad - g(\phi W,Y)g(\phi X,Z) - g(\phi X,Y)g(\phi Z,W) \\
&\quad - k\eta(Y)\eta(Z)g(X,W) + k\eta(X)\eta(W)g(Y,Z) \\
&\quad - \eta(Y)\eta(W)g(X,Z) + \eta(X)\eta(Y)g(Z,W) \\
&\quad - (k-1)\eta(Z)\eta(W)g(X,Y) + (k-1)\eta(X)\eta(Z)g(Y,W) \\
&\quad - g(X,Y)g(Z,W) + g(X,Z)g(Y,W) - g(W,\phi hY)g(\phi X,Z) \\
&\quad + g(\phi W,Y)g(\phi hX,Z) - 3g(\phi X,Y)g(\phi hZ,W) \\
&\quad + g(\phi hX,Y)g(\phi Z,W) + g(\phi hW,Y)g(\phi hX,Z) \\
&\quad - g(\phi hX,Y)g(\phi hZ,W) - 2g(hY,Z)g(X,W) \\
&\quad + 2g(hX,W)g(Z,Y) - g(hY,W)g(X,Z) + g(Y,W)g(hX,Z)
\end{aligned}
$$

$$
\begin{aligned}
&- g(hW,Z)g(X,Y) + g(W,Z)g(hX,Y) - g(hY,W)g(hX,Z)\\
&+ g(hW,Z)g(hX,Y) - \eta(Y)\eta(W)g(hX,Z)\\
&+ 2\eta(X)\eta(W)g(hY,Z) - 2\eta(Y)\eta(Z)g(hX,W)\\
&+ \eta(X)\eta(Z)g(hY,W) - \eta(Z)\eta(W)g(hX,Y)\\
&+ \eta(X)\eta(Y)g(hW,Z).
\end{aligned}
$$

Substituting (3.6) in (3.5), we get

$$
\begin{aligned}
(3.7)\quad & 2g(R(\phi X,Y)\phi Z,W) + 2g(R(\phi Y,X)\phi Z,W)\\
&- 2g(R(\phi Y,X)\phi Z,W) + 2g(R(\phi Z,X)\phi Y,W)\\
&+ 2g(R(\phi Y,X)\phi W,Z) - 2g(R(\phi Y,X)\phi W,Z)\\
&+ 2g(R(\phi W,X)\phi Y,Z) + 2g(\phi X,W)g(\phi Y,Z)\\
&- 2H\eta(Y)\eta(Z)g(X,W) + 2(k-1-H)\eta(Y)\eta(W)g(X,Z)\\
&+ 2(1-H)\eta(Z)\eta(W)g(X,Y) + 2(1-H)\eta(X)\eta(Y)g(Z,W)\\
&+ 2(k-1-H)\eta(X)\eta(Z)g(Y,W) + 2(k-H)\eta(X)\eta(W)g(Y,Z)\\
&+ 2(H-1)g(X,Y)g(Z,W) + 2(H+1)g(X,Z)g(Y,W)\\
&+ 2Hg(X,W)g(Y,Z) + 6(H-k)\eta(X)\eta(Y)\eta(Z)\eta(W)\\
&- 4g(\phi X,Y)g(\phi hZ,W) + 2g(\phi X,W)g(\phi hY,Z)\\
&- 2g(\phi hX,W)g(\phi Y,Z) + 2g(\phi hW,Y)g(\phi hX,Z)\\
&- 2g(\phi hX,Y)g(\phi hZ,W) - 2g(hY,Z)g(X,W)\\
&+ 2g(hX,W)g(Z,Y) - 2g(hY,W)g(hX,Z)\\
&+ 2g(hW,Z)g(hX,Y) + 2\eta(X)\eta(W)g(hY,Z)\\
&- 2\eta(Y)\eta(Z)g(hX,W) = 0.
\end{aligned}
$$

Here, from $g(R(\phi X,Z)\phi Y,W) + g(R(Z,\phi Y)\phi X,W) + g(R(\phi Y,\phi X)Z,W) = 0$, we generally get

$$
(3.8)\quad g(R(\phi X,Z)\phi Y,W) - g(R(\phi Y,Z)\phi X,W) = g(R(\phi X,\phi Y)Z,W).
$$

However, by (2.6), (2.2) and the property of the curvature tensor, we find

$$
\begin{aligned}
(3.9)\quad & g(R(\phi X,\phi Y)Z,W) = g(\phi R(\phi X,\phi Y)Z,\phi W)\\
&= g(R(X,Y)Z,W) + (1-k)\eta(Y)\eta(Z)g(W,X)
\end{aligned}
$$

$$
\begin{aligned}
&- (1-k)\eta(Y)\eta(W)g(Z,X) + \eta(Y)\eta(Z)g(hW,X)\\
&- \eta(Y)\eta(W)g(hZ,X) - (1-k)\eta(X)\eta(Z)g(W,Y)\\
&+ (1-k)\eta(X)\eta(W)g(Z,Y) - \eta(X)\eta(Z)g(hW,Y)\\
&+ \eta(X)\eta(W)g(hZ,Y) + g(X,\phi Z)g(W,\phi Y)\\
&+ g(X,\phi hZ)g(W,\phi Y) + g(X,\phi Z)g(hW,\phi Y)\\
&+ g(X,\phi hZ)g(hW,\phi Y) - g(X,\phi W)g(Z,\phi Y)\\
&- g(X,\phi hW)g(Z,\phi Y) - g(X,\phi W)g(hZ,\phi Y)\\
&- g(X,\phi hW)g(hZ,\phi Y) - g(X,W)g(Y,Z)\\
&- g(X,hW)g(Y,Z) - g(X,W)g(Y,hZ)\\
&- g(X,hW)g(Y,hZ) + g(X,Z)g(Y,W)\\
&+ g(X,hZ)g(Y,W) + g(X,Z)g(Y,hW)\\
&+ g(X,hZ)g(Y,hW).
\end{aligned}
$$

Moreover, using (3.4) again, we get

$$
\begin{aligned}
(3.10)\quad & g(R(\phi Y,X)\phi Z,W) + g(R(\phi Y,X)\phi W,Z)\\
&= g(R(\phi X,Y)\phi Z,W) + g(R(\phi X,Y)\phi Z,W)\\
&\quad + g(\phi W,Y)g(\phi X,Z) + g(\phi X,W)g(\phi Y,Z)\\
&\quad + \eta(Y)\eta(Z)g(X,W) + (2k-1)\eta(Y)\eta(W)g(X,Z)\\
&\quad - (k-1)\eta(X)\eta(Z)g(Y,W) - (k-1)\eta(X)\eta(W)g(Y,Z)\\
&\quad + g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\\
&\quad + g(W,\phi hY)g(\phi X,Z) + g(\phi W,Y)g(\phi hX,Z)\\
&\quad + 3g(\phi X,W)g(\phi hY,Z) - 3g(\phi hX,W)g(\phi Y,Z)\\
&\quad + g(\phi hW,Y)g(\phi hX,Z) - g(\phi hX,W)g(\phi hY,Z)\\
&\quad + g(hY,Z)g(X,W) - g(hX,W)g(Z,Y)\\
&\quad + 3g(hY,W)g(X,Z) - 3g(Y,W)g(hX,Z)\\
&\quad + g(hY,Z)g(hX,W) - g(hY,W)g(hX,Z)\\
&\quad + 3\eta(Y)\eta(W)g(hX,Z) - 3\eta(X)\eta(Z)g(hY,W)\\
&\quad - \eta(X)\eta(W)g(hY,Z) + \eta(Y)\eta(Z)g(hX,W).
\end{aligned}
$$

Substituting (3.10) into the second term and the fifth term of (3.7), and furthermore using the result which we substitute (3.9) into (3.8) to two terms in the middle of (3.7) and to the last two terms in (3.7), we get, by the property of the curvature tensor,

$$\begin{aligned}
(3.11)\quad & 3g(R(\phi X,Y)\phi Z,W)-g(R(Y,Z)X,W)\\
&+g(R(W,Y)X,Z)+3g(\phi X,W)g(\phi Y,Z)\\
&+(k-H)\eta(Y)\eta(Z)g(X,W)\\
&+((k-H)+3(k-1))\eta(Y)\eta(W)g(X,Z)\\
&+((k-H)-3(k-1))\eta(Z)\eta(W)g(X,Y)\\
&+((k-H)-3(k-1))\eta(X)\eta(Y)g(Z,W)\\
&+((k-1)-(H+2))\eta(X)\eta(Z)g(Y,W)\\
&+(k-H)\eta(X)\eta(W)g(Y,Z)+(H-3)g(X,Y)g(Z,W)\\
&+(H+3)g(X,Z)g(Y,W)+Hg(X,W)g(Y,Z)\\
&+3(H-k)\eta(X)\eta(Y)\eta(Z)\eta(W)+3g(\phi X,W)g(\phi hY,Z)\\
&-3g(\phi hX,W)g(\phi Y,Z)+g(\phi hW,Y)g(\phi hX,Z)\\
&+g(\phi hX,Y)g(\phi hZ,W)-2g(\phi hX,W)g(\phi hY,Z)\\
&+g(hY,Z)g(X,W)+g(hX,W)g(Z,Y)\\
&+4g(hY,W)g(X,Z)-2g(Y,W)g(hX,Z)\\
&-2g(hW,Z)g(X,Y)-2g(W,Z)g(hX,Y)\\
&+2g(hY,Z)g(hX,W)-g(hY,W)g(hX,Z)\\
&-g(hW,Z)g(hX,Y)+2\eta(Y)\eta(W)g(hX,Z)\\
&-\eta(X)\eta(W)g(hY,Z)-\eta(Y)\eta(Z)g(hX,W)\\
&-4\eta(X)\eta(Z)g(hY,W)+2\eta(Z)\eta(W)g(hX,Y)\\
&+2\eta(X)\eta(Y)g(hW,Z)=0.
\end{aligned}$$

Writing ϕX and ϕZ instead of X and Z in (3.11) respectively, and using

$$\begin{aligned}
-3\eta(X)g(R(\xi,Y)Z,W)&=-3\eta(X)g(R(Z,W)\xi,Y)\\
&=-3k\eta(X)\eta(W)g(Z,Y)+3k\eta(X)\eta(Z)g(W,Y),\\
-3\eta(Z)g(R(X,Y)\xi,W)&=-3k\eta(Y)\eta(Z)g(X,W)
\end{aligned}$$

$$
\begin{aligned}
&+ 3k\eta(X)\eta(Z)g(Y,W)3\eta(X)\eta(Z)g(R(\xi,Y)\xi,W) \\
&= 3k\eta(X)\eta(Y)\eta(Z)\eta(W) - 3k\eta(X)\eta(Z)g(Y,W) \quad \text{and} \\
&g(R(\phi X,\phi Z)W,Y) = -g(R(Y,W)X,Z) \\
&+ (k-1)(\eta(Y)\eta(Z)g(X,W) - \eta(Z)\eta(W)g(X,Y) \\
&+ \eta(X)\eta(W)g(Y,Z) - \eta(X)\eta(Y)g(Z,W)) \\
&- \eta(Z)\eta(Y)g(hX,W) + \eta(Z)\eta(W)g(hX,Y) \\
&+ \eta(X)\eta(Y)g(hW,Z) - \eta(X)\eta(W)g(hY,Z) \\
&+ g(\phi X,Y)g(\phi Z,W) + g(\phi X,W)g(\phi Y,Z) \\
&- g(\phi X,Y)g(\phi hZ,W) - g(\phi hX,Y)g(\phi Z,W) \\
&+ g(\phi X,W)g(\phi hY,Z) - g(\phi hX,W)g(\phi Y,Z) \\
&+ g(\phi hX,Y)g(\phi hZ,W) - g(\phi hX,W)g(\phi hY,Z) \\
&+ g(X,W)g(Y,Z) - g(X,Y)g(Z,W) + g(hY,Z)g(X,W) \\
&+ g(hX,W)g(Z,Y) - g(hW,Z)g(X,Y) - g(hX,Y)g(W,Z) \\
&+ g(X,hW)g(Z,hY) - g(X,hY)g(Z,hW)
\end{aligned}
$$

(where we used (2.2) and (3.9)), we obtain

$$
\begin{aligned}
(3.12)\quad &g(R(Y,\phi Z)\phi X,W) = 3g(R(X,Y)Z,W) - g(R(Y,W)X,Z) \\
&+ (H-2)g(\phi X,Y)g(\phi Z,W) - (H-1)g(\phi X,W)g(\phi Y,Z) \\
&- 2(k-1)\eta(Y)\eta(Z)g(X,W) \\
&+ ((k-H) + 3(k-1))\eta(Y)\eta(W)g(X,Z) \\
&- (k-1)\eta(Z)\eta(W)g(X,Y) - (k-1)\eta(X)\eta(Y)g(Z,W) \\
&+ (-H + 3(k-1))\eta(X)\eta(Z)g(Y,W) \\
&- 2(k-1)\eta(X)\eta(W)g(Y,Z) - g(X,Y)g(Z,W) \\
&+ (H+3)g(X,Z)g(Y,W) - 2g(X,W)g(Y,Z) \\
&- (k-H)\eta(X)\eta(Y)\eta(Z)\eta(W) - 2g(hY,Z)g(X,W) \\
&- 2g(hX,W)g(Z,Y) + 4g(hY,W)g(X,Z) \\
&+ 2g(Y,W)g(hX,Z) - g(hW,Z)g(X,Y) \\
&- g(W,Z)g(hX,Y) - g(hY,Z)g(hX,W) \\
&+ g(hY,W)g(hX,Z) + g(\phi X,Y)g(\phi hZ,W)
\end{aligned}
$$

$$
\begin{aligned}
&+ g(\phi hX, Y)g(\phi Z, W) - g(\phi hW, Y)g(\phi hX, Z) \\
&+ g(\phi hX, W)g(\phi hY, Z) - 2\eta(Y)\eta(W)g(hX, Z) \\
&+ 2\eta(X)\eta(W)g(hY, Z) + 2\eta(Y)\eta(Z)g(hX, W) \\
&- 4\eta(X)\eta(Z)g(hY, W) + \eta(Z)\eta(W)g(hX, Y) \\
&+ \eta(X)\eta(Y)g(hW, Z).
\end{aligned}
$$

Exchanging Y for W in (3.12) and using the property of the curvature tensor, we find

$$
\begin{aligned}
(3.13)\quad & g(R(\phi X, Y)\phi Z, W) = 3g(R(X, W)Y, Z) \\
&- g(R(Y, W)X, Z) + g(X, W)g(Z, Y) \\
&+ 2g(X, Y)g(W, Z) - (H + 3)g(X, Z)g(W, Y) \\
&+ (H - 3(k - 1))\eta(X)\eta(Z)g(W, Y) \\
&+ ((H - k) - 3(k - 1))\eta(W)\eta(Y)g(X, Z) \\
&- (H - 2)g(\phi X, W)g(\phi Z, Y) + (H - 1)g(\phi X, Y)g(\phi W, Z) \\
&- (H - k)\eta(X)\eta(W)\eta(Z)\eta(Y) + 2(k - 1)\eta(W)\eta(Z)g(X, Y) \\
&+ (k - 1)\eta(Z)\eta(Y)g(X, W) + (k - 1)\eta(X)\eta(W)g(Z, Y) \\
&+ 2(k - 1)\eta(X)\eta(Y)g(W, Z) + 2g(hW, Z)g(X, Y) \\
&+ 2g(hX, Y)g(Z, W) - 4g(hW, Y)g(X, Z) \\
&- 2g(W, Y)g(hX, Z) + g(hY, Z)g(X, W) \\
&+ g(Y, Z)g(hX, W) + g(hW, Z)g(hX, Y) \\
&- g(hW, Y)g(hX, Z) - g(\phi X, W)g(\phi hZ, Y) \\
&- g(\phi hX, W)g(\phi Z, Y) + g(\phi hY, W)g(\phi hX, Z) \\
&- g(\phi hX, Y)g(\phi hW, Z) + 2\eta(W)\eta(Y)g(hX, Z) \\
&- 2\eta(X)\eta(Y)g(hW, Z) - 2\eta(W)\eta(Z)g(hX, Y) \\
&+ 4\eta(X)\eta(Z)g(hW, Y) - \eta(Z)\eta(Y)g(hX, W) \\
&- \eta(X)\eta(W)g(hY, Z).
\end{aligned}
$$

Moreover, exchanging Z for W in (3.13), we see that

$$
\begin{aligned}
(3.14)\quad & - g(R(\phi X, Y)\phi W, Z) = -3g(R(X, Z)Y, W) \\
&+ g(R(Y, Z)X, W) - g(X, Z)g(W, Y)
\end{aligned}
$$

$$
\begin{aligned}
&- 2g(X,Y)g(Z,W) + (H+3)g(X,W)g(Z,Y)\\
&- (H - 3(k-1))\eta(X)\eta(W)g(Z,Y)\\
&- ((H-k) - 3(k-1))\eta(Z)\eta(Y)g(X,W)\\
&+ (H-2)g(\phi X,Z)g(\phi W,Y) - (H-1)g(\phi X,Y)g(\phi Z,W)\\
&+ (H-k)\eta(X)\eta(Z)\eta(W)\eta(Y) - 2(k-1)\eta(Z)\eta(W)g(X,Y)\\
&- (k-1)\eta(W)\eta(Y)g(X,Z) - (k-1)\eta(X)\eta(Z)g(W,Y)\\
&- 2(k-1)\eta(X)\eta(Y)g(Z,W) - 2g(hZ,W)g(X,Y)\\
&- 2g(hX,Y)g(W,Z) + 4g(hZ,Y)g(X,W)\\
&+ 2g(Z,Y)g(hX,W) - g(hY,W)g(X,Z)\\
&- g(Y,W)g(hX,Z) - g(hZ,W)g(hX,Y)\\
&+ g(hZ,Y)g(hX,W) + g(\phi X,Z)g(\phi hW,Y)\\
&+ g(\phi hX,Z)g(\phi W,Y) - g(\phi hY,Z)g(\phi hX,W)\\
&+ g(\phi hX,Y)g(\phi hZ,W) - 2\eta(Z)\eta(Y)g(hX,W)\\
&+ 2\eta(X)\eta(Y)g(hZ,W) + 2\eta(Z)\eta(W)g(hX,Y)\\
&- 4\eta(X)\eta(W)g(hZ,Y) + \eta(W)\eta(Y)g(hX,Z)\\
&+ \eta(X)\eta(Z)g(hY,W).
\end{aligned}
$$

Combining (3.13) and (3.14), and using (3.4) and the property of the curvature tensor, we obtain

$$
\begin{aligned}
4g(R(X,Y)Z,W) &= (H+3)(g(X,W)g(Y,Z)\\
&- g(X,Z)g(Y,W)) + (H-1)(\eta(X)\eta(Z)g(Y,W)\\
&+ \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z)\\
&- \eta(Y)\eta(Z)g(X,W) + g(\phi X,Z)g(\phi W,Y)\\
&- g(\phi X,W)g(\phi Z,Y) + 2g(\phi X,Y)g(\phi W,Z))\\
&+ 4(k-1)(\eta(X)\eta(W)g(Y,Z) + \eta(Y)\eta(Z)g(X,W)\\
&- \eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(W)g(X,Z))\\
&+ 4(g(hY,Z)g(X,W) + g(hX,W)g(Z,Y)\\
&- g(hY,W)g(X,Z) - g(Y,W)g(hX,Z)\\
&+ \eta(Y)\eta(W)g(hX,Z) + \eta(X)\eta(Z)g(hY,W)\\
&- \eta(X)\eta(W)g(hY,Z) - \eta(Y)\eta(Z)g(hX,W))
\end{aligned}
\tag{3.15}
$$

$$+ 2(g(hY,Z)g(hX,W) - g(hY,W)g(hX,Z)$$
$$+ g(\phi hW,Y)g(\phi hX,Z) - g(\phi hX,W)g(\phi hY,Z)).$$

Therefore we get (3.1).

Next we show that H is constant on M if $n \neq 1$. From (3.15) we easily get

$$\begin{aligned}\operatorname{Ric}(Y,Z) &= \left(\frac{n(H+3)+(H-1)}{2} + (k-1)\right) g(Y,Z) \\ &+ \left((2n-1)(k-1) - \frac{(n+1)(H-1)}{2}\right)\eta(Y)\eta(Z) \\ &+ 2(n-1)g(hY,Z),\end{aligned} \tag{3.16}$$

and

$$S = n(n+1)H + 3n^2 + n + 4n(k-1). \tag{3.17}$$

On the other hand, from the Bianchi identity we get

$$2\sum_{i=1}^{2n+1} (\nabla_{E_i}\operatorname{Ric})(E_i,Z) - \nabla_Z S = 0 \tag{3.18}$$

($\{E_i\}$ is an orthonormal frame). Substituting (3.16) and (3.17) into (3.18), we obtain from (2.1)

$$\begin{aligned}&(n+1)(\nabla_Z H) - (n+1)\eta(Z)\nabla_\xi H \\ &+4(n-1)\sum_{i=1}^{2n+1} g\left((\nabla_{E_i}h)E_i, Z\right) - n(n+1)\nabla_Z H = 0.\end{aligned}$$

Here we find from (2.4) and $\operatorname{Tr} h = 0$ that

$$\begin{aligned}\sum_{i=1}^{2n+1} g\left((\nabla_{E_i}h)E_i, Z\right) &= \sum_{i=1}^{2n+1} g\left(E_i, (\nabla_{E_i}h)Z\right) \\ &= \sum_{i=1}^{2n+1} g(E_i,(\nabla_Z h)E_i) + (1-k)\sum_{i=1}^{2n+1}(2g(Z,\phi E_i)\eta(E_i) \\ &+ \eta(Z)g(\phi E_i,E_i) - \eta(E_i)g(\phi Z,E_i)) \\ &+ \sum_{i=1}^{2n+1}\eta(Z)g(\phi hE_i,E_i) - \sum_{i=1}^{2n+1}\eta(E_i)g(\phi hZ,E_i) \\ &= \operatorname{Tr}\nabla_Z h = 0.\end{aligned}$$

Thus we have

$$(n+1)(\nabla_Z H) - (n+1)\eta(Z)\nabla_\xi H - n(n+1)\nabla_Z H = 0, \tag{3.19}$$

from which, putting $Z=\xi$, we get $\nabla_\xi H = 0$. Therefore (3.19) yields

$$(n-1)\nabla_Z H = 0,$$

which completes the proof.

4. Invariant submanifolds of contact metric manifolds with constant $\bar\phi$-sectional curvature

Let M^{2n+1} be a $(2n+1)$-dimensional C^∞-submanifold of a $(2r+1)$-dimensional contact metric manifold $\bar M^{2r+1}(\bar\phi,\bar\xi,\bar\eta,\bar g)$. M^{2n+1} is said to be invariant if the structure vector field $\bar\xi$ is tangent to M^{2n+1} everywhere on M^{2n+1} and $\bar\phi X$ is tangent to M^{2n+1} for any vector field X tangent to M^{2n+1} at every point of M^{2n+1}. If M^{2n+1} is an invariant submanifold of $\bar M^{2r+1}$, we can put

$$\bar\phi X = \phi X, \quad \bar\xi = \xi, \quad \bar\eta(X) = \eta(X).$$

Then it is well-known that if $\bar M^{2r+1}$ $(\bar\phi,\ \bar\xi,\ \bar\eta,\ \bar g)$ is a contact metric manifold (respectively Sasakian manifold), then M^{2n+1} is also a contact metric manifold with respect to the induced structure (ϕ,ξ,η,g) (g is the induced metric) (respectively Sasakian manifold)(e.g., see [14] or [15]). Here, if we define an operator $h = -\frac{1}{2}\mathcal{L}_\xi\phi$ in an invariant submanifold $M^{2n+1}(\phi,\xi,\eta,g)$ of the contact metric manifold $\bar M^{2r+1}$, then we have the results that h is symmetric, h anti-commutes with ϕ (i.e., $\phi h + h\phi = 0$), $\eta\circ h = 0$ and $h\xi = 0$. Also, by the definition of $\bar h$, we can see that $\bar hX$ is tangent to M^{2n+1} and $\bar hX = hX$ for any vector field $X\in T_m(M^{2n+1})$ (see [4]).

The Gauss equation is given by

$$\begin{aligned}\bar g(\bar R(X,Y)Z,W) &= g(R(X,Y)Z,W) - \sum_B g(H_BY,Z)g(H_BX,W)\\ &\quad + \sum_B g(H_BX,Z)g(H_BY,W)\end{aligned} \tag{4.1}$$

for any vector field X,Y,Z and W on M^{2n+1}, where H_B $(B=1,2,\dots,2(r-n))$ denote the second fundamental forms of M^{2n+1}.

We need the following two theorems later.

Theorem A ([2] and [3]). *Let M^{2n+1} be an invariant submanifold of a contact metric manifold $\bar{M}^{2r+1}$. Then M^{2n+1} is minimal and we have*

$$g(H_A X, Y) = -g(H_A \phi X, \phi Y) \quad (A = 1, 2, \ldots, 2(r-n))$$

for any vector field X and Y on M^{2n+1}. Especially $H_A \xi = 0$ $(A = 1, 2, \ldots, 2(r-n))$.

From (3.15) and (4.1) we easily get the following.

Theorem 4.1. *Let $\bar{M}^{2r+1}$ be a contact metric manifold with $\bar{\xi}$ belonging to the $\bar{k}$-nullity distribution and with a constant $\bar{\phi}$-sectional curvature $\bar{H}$. If M^{2n+1} is an invariant submanifold of $\bar{M}^{2r+1}$, then the curvature tensor of M^{2n+1} is given by*

$$\begin{aligned}
(4.2) \quad g(R(X,Y)Z,W) = {} & \frac{(\bar{H}+3)}{4}(g(X,W)g(Y,Z) \\
& - g(X,Z)g(Y,W)) + \frac{(\bar{H}-1)}{4}(\eta(X)\eta(Z)g(Y,W) \\
& + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z) \\
& - \eta(Y)\eta(Z)g(X,W) + g(\phi X, Z)g(\phi W, Y) \\
& - g(\phi X, W)g(\phi Z, Y) + 2g(\phi X, Y)g(\phi W, Z)) \\
& + (\bar{k}-1)(\eta(X)\eta(W)g(Y,Z) + \eta(Y)\eta(Z)g(X,W) \\
& - \eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(W)g(X,Z)) \\
& + (g(hY,Z)g(X,W) + g(hX,W)g(Z,Y) \\
& - g(hY,W)g(X,Z) - g(Y,W)g(hX,Z) \\
& + \eta(Y)\eta(W)g(hX,Z) + \eta(X)\eta(Z)g(hY,W) \\
& - \eta(X)\eta(W)g(hY,Z) - \eta(Y)\eta(Z)g(hX,W)) \\
& + \frac{1}{2}(g(hY,Z)g(hX,W) - g(hY,W)g(hX,Z) \\
& + g(\phi hW, Y)g(\phi hX, Z) - g(\phi hX, W)g(\phi hY, Z)) \\
& + \sum_B g(H_B Y, Z)g(H_B X, W) - \sum_B g(H_B X, Z)g(H_B Y, W).
\end{aligned}$$

Under the assumptions of Theorem 4.1, using Theorem A we get

$$\text{(4.3)}\quad \operatorname{Ric}(Y,Z) = \left(\frac{n(\bar{H}+3)+(\bar{H}-1)}{2} + (\bar{k}-1)\right)g(Y,Z) + \left((2n-1)(\bar{k}-1) - \frac{(n+1)(\bar{H}-1)}{2}\right)\eta(Y)\eta(Z) + 2(n-1)g(hY,Z) - \sum_B g(H_BY, H_BZ)$$

$$\text{(4.4)}\quad S = n(n+1)\bar{H} + 3n^2 + n + 4n(\bar{k}-1) - \sum_B \operatorname{Tr} H_B^2$$

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