On the maximum term of an integral function

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1. Let $\mu(r)=|a_{\nu(r)}|r^{\nu(r)}$ denote the maximum term of the integral function $f(z)=\sum_{n=0}^{\infty} a_n z^n$ of order ϱ and lower order λ and $\nu(r)$ its rank. Also, let $\mu(r, f^{(s)})$ and $\nu(r, f^{(s)})$, $(s=1, 2, \ldots)$, denote the maximum term and its rank for the function $f^{(s)}(z)$, the s-th derivative of $f(z)$. Then $\mu(r)$, $\mu(r, f^{(s)})$, $\nu(r)$ and $\nu(r, f^{(s)})$ are all positive and non-decreasing functions of r. We shall denote by $\mu'(r)$, $\mu'(r, f^{(1)})$, $\ldots$, $\mu'(r, f^{(s)})$ the derivatives of $\mu(r)$, $\mu(r, f^{(1)})$, $\ldots$, $\mu(r, f^{(s)})$ respectively with respect to r. The following relation connecting the maximum term and the rank has been given by G. VALIRON ([1], p. 31)

$$\log \mu(r) = \log \mu(r_0) + \int_{r_0}^{r} \frac{\nu(x)}{x}\, dx \tag{1. 1}$$

for $0<r_0<r$ and the relation corresponding to (1. 1) for the case $f^{(s)}(z)$ follows from (2. 20) of SRIVASTAVA ([2], p. 107)

$$\log \mu(r, f^{(s)}) = \log \mu(r_0, f^{(s)}) + \int_{r_0}^{r} \left\{ \frac{\nu(x, f^{(s)}) - s}{x} \right\} dx \tag{1. 2}$$

for $0<r_0<r$. Further, SRIVASTAVA ([2], p. 106) has shown that

$$\mu(r, f^{(s)}) \sim \mu(r) \left\{ \frac{\nu(r)}{r} \right\}^{s}, \qquad s=1, 2, \ldots, \tag{1. 3}$$

provided $f(z)$ is not a polynomial and r lies outside a set of measure zero.

We know ([3], p. 309)

$$\lim_{r\to\infty} \frac{\sup}{\inf} \frac{\log \left[r \left\{ \frac{\mu(r, f^{(s)})}{\mu(r)} \right\}^{\frac{1}{s}} \right]}{\log r} = \frac{\varrho}{\lambda}. \tag{1. 4}$$

In this paper we have established a relation between the order of an integral function and the derivatives of the maximum terms and have studied the case when the integral function is of regular growth. We have also obtained some of the properties of $\mu(r, f^{(s)})$ and $\mu'(r, f^{(s)})$.

2. Theorem 1. *Let $f(z)$ be an integral function, other than a polynomial, of regular growth. The order of $f(z)$ will be $1+\frac{1}{\alpha}$, where α is a number not lying between 0 and -1, if and only if*

$$\log\left\{\frac{\mu'(r,f^{(s)})}{\mu'(r)}\right\}\sim \log r^{\frac{s}{\alpha}} \tag{2.1}$$

for almost all large r.

We shall first prove the following lemma:

Lemma 1. *If $f(z)$ be an integral function, other than a polynomial, of finite order ϱ and lower order λ, then*

$$\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\left[r\left\{\frac{\mu'(r,f^{(s)})}{\mu'(r)}\right\}^{\frac{1}{s}}\right]}{\log r}=\frac{\varrho}{\lambda}, \tag{2.2}$$

where r tends to infinity through values outside a set of measure zero.

Proof. Differentiating (1. 1) and (1. 2), we get

$$\frac{\mu'(r)}{\mu(r)}=\frac{\nu(r)}{r} \tag{2.3}$$

and

$$\frac{\mu'(r,f^{(s)})}{\mu(r,f^{(s)})}=\frac{\nu(r,f^{(s)})-s}{r} \tag{2.4}$$

for almost all values of $r>r_i$ and $r>r_j$ respectively. Taking the logarithm of (2. 3) and (2. 4) and then subtracting, we obtain

$$\frac{\log\left[r\left\{\frac{\mu'(r,f^{(s)})}{\mu'(r)}\right\}^{\frac{1}{s}}\right]}{\log r}=\frac{\log\left[r\left\{\frac{\mu(r,f^{(s)})}{\mu(r)}\right\}^{\frac{1}{s}}\right]}{\log r}+\frac{1}{s}\frac{\log\left\{\frac{\nu(r,f^{(s)})-s}{\nu(r)}\right\}}{\log r}$$

for almost all values of $r>r_0=\max(r_i,r_j)$. Proceeding to limits and using (1. 4), we get

$$\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\left[r\left\{\frac{\mu'(r,f^{(s)})}{\mu'(r)}\right\}^{\frac{1}{s}}\right]}{\log r}=\frac{\varrho}{\lambda}.$$

We may note here that for functions of regular growth and of order ϱ

$$\lim_{r\to\infty}\frac{\log\left[r\left\{\frac{\mu'(r,f^{(s)})}{\mu'(r)}\right\}^{\frac{1}{s}}\right]}{\log r}=\varrho. \tag{2.5}$$

Also from the above result, we can easily deduce, for functions of lower order λ $(\lambda>1)$,

$$\mu'(r)<\mu'(r,f^{(1)})<\dots\dots<\mu'(r,f^{(p)})$$

for almost all values of r, $r>r_0\geqq 1$, where $r_0=\max(r_1,r_2,\dots,r_p)$.

PROOF OF THEOREM 1. From (2. 2), follows easily

$$\lim_{r\to\infty} \frac{\log\left\{\frac{\mu'(r, f^{(s)})}{\mu'(r)}\right\}^{\frac{1}{s}}}{\log r} = \varrho - 1$$

or,

$$\lim_{r\to\infty} \frac{\log\left\{\frac{\mu'(r, f^{(s)})}{\mu'(r)}\right\}}{\log r} = s(\varrho - 1).$$

Hence, if $\varrho = 1 + \frac{1}{\alpha}$, where α is either < -1 or > 0, so that $0 < \varrho < \infty$ then

$$\log\left\{\frac{\mu'(r, f^{(s)})}{\mu'(r)}\right\} \sim \left(\frac{s}{\alpha}\right) \log r.$$

Again, if (2. 1) holds, we have

$$\lim_{r\to\infty} \frac{\log\left\{\frac{\mu'(r, f^{(s)})}{\mu'(r)}\right\}}{\log r^{\frac{s}{\alpha}}} = 1$$

Hence,

$$\lim_{r\to\infty} \frac{\log\left[r\left\{\frac{\mu'(r, f^{(s)})}{\mu'(r)}\right\}^{\frac{1}{s}}\right]}{\log r} = 1 + \frac{1}{\alpha} = \varrho,$$

from (2. 2).

Corollary 1. If $f(z)$ is an integral function of regular growth and order ϱ and $\log\left\{\frac{\mu'(r, f^{(s)})}{\mu'(r)}\right\} \sim \log r$, then $1 < \varrho \leqq 2$. For, with $\alpha = s$ and s being a positive integer $1 < 1 + \frac{1}{\alpha} \leqq 2$. The second inequality becomes an equality only for $\alpha = 1$.

Corollary 2. If $f(z)$ is an integral function of regular growth and order 1 then

$$\log\left\{\frac{\mu'(r, f^{(s)})}{\mu'(r)}\right\} = o(\log r)$$

for almost all large r.

Applications. The following results can easily be derived from Theorem 1.

(i) If $f(z)$ is an integral function of regular growth and its order $\varrho < 1$ then $\mu'(r), \mu'(r, f^{(1)}), \mu'(r, f^{(2)}), \ldots, \mu'(r, f^{(p)})$ form a decreasing sequence for almost all values of r, $r > r_0 = r_0(f) \geqq 1$, where $r_0 = \max(r_1, r_2, \ldots, r_p)$.

(ii) If $\mu'(r), \mu'(r, f^{(1)}), \mu'(r, f^{(2)}), \ldots, \mu'(r, f^{(p)})$ form a decreasing sequence for almost all values of $r, r > r_0 = r_0(f) \geqq 1$, where $r_0 = \max(r_1, r_2, \ldots r_p)$, and $f(z)$ is an integral function of regular growth, then its order $\varrho < 1$.

(iii) If $f(z)$ is an integral function of regular growth and its order $\varrho > 1$, then $\mu'(r), \mu'(r, f^{(1)}), \mu'(r, f^{(2)}), \ldots, \mu'(r, f^{(p)})$ form an increasing sequence for almost all values of $r, r > r_0 = r_0(f) \geqq 1$, where $r_0 = \max(r_1, r_2, \ldots r_p)$.

(iv) If $\mu'(r), \mu'(r, f^{(1)}), \mu'(r, f^{(2)}), \ldots, \mu'(r, f^{(p)})$ form an increasing sequence for almost all values of r, $r > r_0 = r_0(f) \geqq 1$, where $r_0 = \max(r_1, r_2, \ldots, r_p)$, and $f(z)$ is an integral function of regular growth, then its order $\varrho > 1$.

3. Theorem 2. *Let $f(z)$ be an integral function and let $\nu(r)$ and $\nu(r, f^{(s)})$ denote the ranks of the maximum terms $\mu(r)$ and $\mu(r, f^{(s)})$, $(s = 1, 2, \ldots)$, of $f(z)$ and its s-th derivative $f^{(s)}(z)$ respectively for $|z| = r$. Then*

(i)

$$\left(\frac{r_2}{r_1}\right)^{\{\nu(r_1, f^{(s)}) - s\}} \leqq \frac{\mu(r_2, f^{(s)})}{\mu(r_1, f^{(s)})} \leqq \left(\frac{r_2}{r_1}\right)^{\{\nu(r_2, f^{(s)}) - s\}}, \tag{3. 1}$$

provided $0 < r_1 < r_2$;

(ii) $\mu(r), \mu(r, f^{(1)}), \mu(r, f^{(2)}), \ldots, \mu(r, f^{(p)})$ *form an increasing sequence provided, $f(z)$ is not a polynomial,* $\dfrac{\nu(r)}{r \log r} > 1$, $r > r_0 \geqq e$ *and r lies outside a set of measure zero.*

Proof. (i) From (1. 2), we have

$$\log \mu(r_2, f^{(s)}) \leqq \log \mu(r_1, f^{(s)}) + \{\nu(r_2, f^{(s)}) - s\} \log \frac{r_2}{r_1} \tag{3. 2}$$

and

$$\log \mu(r_2, f^{(s)}) \geqq \log \mu(r_1, f^{(s)}) + \{\nu(r_1, f^{(s)}) - s\} \log \frac{r_2}{r_1} \tag{3. 3}$$

Combining (3. 2) and (3. 3), the result follows.

(ii) Taking $s = 1$ in (1. 3), we get

$$\frac{\mu(r, f^{(1)})}{\mu(r)} \sim \frac{\nu(r)}{r}.$$

Therefore,

$$\left\{\frac{\mu(r, f^{(1)})}{\mu(r) \log r}\right\} \sim \frac{\nu(r)}{r \log r}.$$

Hence, if $\dfrac{\nu(r)}{r \log r} > 1$, we have

$$\mu(r, f^{(1)}) > \mu(r) \log r > \mu(r),$$

for $r > r_0 \geqq e$ and r lies outside a set of measure zero. Again, for $s = 2$,

$$\frac{\mu(r, f^{(2)})}{\mu(r)} \sim \left\{\frac{\nu(r)}{r}\right\}^2$$

or,

$$\frac{\mu(r, f^{(2)})}{\mu(r, f^{(1)})} \sim \frac{\nu(r)}{r}$$

Hence,

$$\mu(r, f^{(2)}) > \mu(r, f^{(1)}) \log r > \mu(r, f^{(1)})$$

for $r > r_0 \geqq e$ and r lies outside a set of measure zero. Similarly, we can obtain results for subsequent derivatives.

Corollary. If $f(z)$ is an integral function other than a polynomial and α is a constant, $0 < \alpha < 1$, then

$$\lim_{r\to\infty} \frac{\mu(\alpha r, f^{(s)})}{\mu(r, f^{(s)})} = 0.$$

If we put $r_1 = \alpha r$ and $r_2 = r$ in (3. 1), then

$$(\alpha)^{\{\nu(r,\, f^{(s)}) - s\}} \leqq \frac{\mu(\alpha r, f^{(s)})}{\mu(r, f^{(s)})} \leqq (\alpha)^{\{\nu(\alpha r,\, f^{(s)}) - s\}}$$

and the result follows on taking limits.

4. Theorem 3. *If $\nu(r, f^{(s)})$ and $\nu(r, f^{(s+1)})$ denote the ranks of the maximum terms $\mu(r, f^{(s)})$ and $\mu(r, f^{(s+1)})$ of $f^{(s)}(z)$ and $f^{(s+1)}(z)$, the s-th and $(s+1)$-th derivatives, respectively, of the integral function $f(z)$ for $|z| = r$, then at the points of existence of $\mu'(r, f^{(s)})$, $r > r_0$*

$$\text{(4. 1)} \qquad \frac{\nu(r, f^{(s)})}{\nu(r, f^{(s+1)})} \leqq \frac{1}{\mu(r, f^{(s+1)})} \left\{\mu'(r, f^{(s)}) + \frac{s}{r}\mu(r, f^{(s)})\right\} \leqq 1,$$

where $\mu'(r, f^{(s)})$ denotes the derivative of $\mu(r, f^{(s)})$.

PROOF. Differentiating (1. 2), we get

$$\text{(4. 2)} \qquad \frac{\mu'(r, f^{(s)})}{\mu(r, f^{(s)})} = \frac{\nu(r, f^{(s)}) - s}{r}$$

for almost all values of $r > r_0$.

We know ([4], p. 21) that

$$\text{(4. 3)} \qquad \nu(r) \leqq r\,\frac{\mu(r, f^{(1)})}{\mu(r)} \leqq \nu(r, f^{(1)}).$$

Writing (4. 3) for $(s+1)$-th derivative, we get

$$\text{(4. 4)} \qquad \nu(r, f^{(s)}) \leqq r\,\frac{\mu(r, f^{(s+1)})}{\mu(r, f^{(s)})} \leqq \nu(r, f^{(s+1)}).$$

From (4. 2) and (4. 4), we get

$$\text{(4. 5)} \qquad \frac{\mu'(r, f^{(s)})}{\mu(r, f^{(s)})} = \frac{\nu(r, f^{(s)}) - s}{r} \leqq \frac{\mu(r, f^{(s+1)})}{\mu(r, f^{(s)})} - \frac{s}{r},$$

and

$$\frac{\mu'(r, f^{(s)})}{\mu(r, f^{(s+1)})} = \frac{\{\nu(r, f^{(s)}) - s\}}{\left\{r\,\dfrac{\mu(r, f^{(s+1)})}{\mu(r, f^{(s)})}\right\}} \geqq \frac{\nu(r, f^{(s)})}{\nu(r, f^{(s+1)})} - \frac{s}{r}\,\frac{\mu(r, f^{(s)})}{\mu(r, f^{(s+1)})},$$

for almost all $r > r_0$. Therefore,

$$\frac{\nu(r, f^{(s)})}{\nu(r, f^{(s+1)})} \leqq \frac{1}{\mu(r, f^{(s+1)})} \left\{ \mu'(r, f^{(s)}) + \frac{s}{r}\, \mu(r, f^{(s)}) \right\} \leqq 1,$$

for almost all $r > r_0$. The last inequality follows easily from (4. 5).

Corollary: $\left\{ \mu'(r, f^{(s)}) + \frac{s}{r}\, \mu(r, f^{(s)}) \right\} \sim \mu(r, f^{(s+1)})$ as $r \to \infty$ through values excluding a set of measure zero at which $\mu'(r, f^{(s)})$ does not exist.

This follows from (4. 1) and the fact that $\nu(r, f^{(s)}) \sim \nu(r, f^{(s+1)})$.

5. Let $\Phi(r)$ be a "slowly changing" function; that is $\Phi(r) > 0$ and continuous for $r > r_0$ and $\Phi(lr) \sim \Phi(r)$ as $r \to \infty$, for every constant $l > 0$. Also, let

(5. 1) $$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \mu(r, f^{(s)})}{r^{\varrho}\Phi(r)} = \begin{matrix} p \\ q \end{matrix}$$

and

(5. 2) $$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\nu(r, f^{(s)})}{r^{\varrho}\Phi(r)} = \begin{matrix} c \\ d \end{matrix}.$$

We shall now prove the following results:

Theorem 4. *If $f(z)$ be an integral function of order $\varrho\,(0 < \varrho < \infty)$, then*

(i) $$d \leqq \frac{c}{e}\, e^{\frac{d}{c}} \leqq \varrho p \leqq c;$$

(ii) $$d \leqq \varrho q \leqq d\left(1 + \log \frac{c}{d}\right) \leqq c;$$

(iii) $$c \leqq e\varrho p;$$

(iv) $$d \leqq \varrho p;$$

(v) $$c + d \leqq e\varrho p;$$

and

(vi) *equality can not simultaneously hold in* (iv) *and* (v).

Theorem 5.

(vii) $$e\varrho q \leqq \varrho p + ed;$$

(viii) $$c + \varrho q \leqq e\varrho p.$$

Proof of Theorem 4. Writing (1. 2) as

$$\log \mu(rk^{\frac{1}{\varrho}}, f^{(s)}) = \log \mu(r_0, f^{(s)}) + \int_{r_0}^{r} \frac{\nu(x, f^{(s)})}{x}\, dx + \int^{rk^{\frac{1}{\varrho}}} \frac{\nu(x, f^{(s)})}{x}\, dx - \int_{r_0}^{rk^{\frac{1}{\varrho}}} \frac{s}{x} dx <$$

$$< 0(1) + (c+\varepsilon) \int_{r_0}^{r} x^{\varrho - 1}\Phi(x)\, dx + \frac{\nu(rk^{\frac{1}{\varrho}}, f^{(s)})}{\varrho} \log k - s \log r \sim$$

$$\sim (c+\varepsilon) \frac{r^{\varrho}}{\varrho}\, \Phi(r) + \frac{\nu(rk^{\frac{1}{\varrho}}, f^{(s)})}{\varrho} \log k - s \log r,\ k \geqq 1, \text{ by [5], Lemma 5.}$$

Hence, taking limits using (5. 1) and (5. 2), we have

$$kp \leqq \frac{c+ck\log k}{\varrho}; \tag{5. 3}$$

and

$$kq \leqq \frac{c+dk\log k}{\varrho}. \tag{5. 4}$$

Similarly, we obtain

$$kp \geqq \frac{d+c\log k}{\varrho}; \tag{5. 5}$$

and

$$kq \geqq \frac{d+d\log k}{\varrho}. \tag{5. 6}$$

Putting $k=1$ in (5. 3) and (5. 6), we get

$$p\varrho \leqq c \quad \text{and} \quad d \leqq q\varrho. \tag{5. 7}$$

Further, putting $k=c/d$ in (5. 4) we obtain

$$\varrho q \leqq d\left(1+\log\frac{c}{d}\right) = d\log\frac{ec}{d} \leqq d\cdot\frac{c}{d} = c. \tag{5. 8}$$

Next we put $k=\exp\left(\frac{c-d}{c}\right)$ in (5. 5). This gives

$$\varrho p \geqq \frac{d+c\log e^{\frac{c-d}{c}}}{ee^{-\frac{d}{c}}} = \frac{c}{e}e^{\frac{d}{c}} \geqq d \tag{5. 9}$$

since, for $x \geqq 0$, $\frac{e^x}{x} \geqq e$, so that $\frac{ce^{d/c}}{e} \geqq d$.

This proves (i) and (ii).

(iii) and (iv) follow immediately from (i) and (ii).

(v) is easily obtained on putting $k=e$ in (5. 5).

We now show that equality can not simultaneously hold in (iv) and (v). Let $d=\varrho p$. Then from (5. 5)

$$p \geqq \frac{\varrho p+c\log k}{\varrho k}$$

so that

$$c \leqq \frac{\varrho p(k-1)}{\log k}.$$

Putting $k = (1+\eta)$, where $\eta \to 0$, then

$$c \leqq \frac{\varrho p\eta}{\eta+0(\eta^2)} \leqq \varrho p.$$

But $d \leqq c$. Hence $c = \varrho p$. Therefore,

$$c + d = 2\varrho p < e\varrho p.$$

Next, suppose that $c + d = e\varrho p$, then $d < \varrho p$, for if $d = \varrho p$, then by the above $c + d$ will have to be less than $e\varrho p$.

PROOF OF THEOREM 5. We know that

$$(5.10) \qquad \nu(re^{\frac{1}{\varrho}}, f^{(s)}) \geqq \varrho \int_r^{re^{\frac{1}{\varrho}}} \frac{\nu(x, f^{(s)})}{x}\, dx$$

Adding $\left\{\varrho \log \mu(r, f^{(s)}) - \varrho \int_r^{re^{\frac{1}{\varrho}}} \frac{s}{x}\, dx\right\}$ on both the sides, we obtain

$$\nu(re^{\frac{1}{\varrho}}, f^{(s)}) + \varrho \log \mu(r, f^{(s)}) - \varrho \int_r^{re^{\frac{1}{\varrho}}} \frac{s}{x}\, dx \geqq \varrho \int_r^{re^{\frac{1}{\varrho}}} \frac{\nu(x, f^{(s)})}{x}\, dx + $$

$$+ \varrho \log \mu(r, f^{(s)}) - \varrho \int_r^{re^{\frac{1}{\varrho}}} \frac{s}{x}\, dx$$

or (5. 11)

$$\nu(re^{\frac{1}{\varrho}}, f^{(s)}) + \varrho \log \mu(r, f^{(s)}) - s \geqq \varrho \log \mu(re^{\frac{1}{\varrho}}, f^{(s)}).$$

Similarly we have

$$(5.12) \qquad \nu(r, f^{(s)}) + \varrho \log \mu(r, f^{(s)}) - s \leqq \varrho \log \mu(re^{\frac{1}{\varrho}}, f^{(s)}).$$

Dividing (5. 11) and (5. 12) by $r^{\varrho}\Phi(r)$, taking limits and using (5. 1) and (5. 2), the results (vii) and (viii) follow.

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