On exceptional values of meromorphic functions

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1. Let $f(z)$ be a meromorphic function. For $0 \leqq |\alpha| < \infty$ let $n(r, \alpha)$ denote the number of roots of $f(z) - \alpha$ in $|z| \leqq r$. Further let $\bar{n}(r, \alpha)$ denote the number of roots of $f(z) - \alpha = 0$ in $|z| \leqq r$ each root being counted only once. For $\alpha = \infty$ let $n(r, \infty) = n(r, f)$ be the number of poles of $f(z)$ in $|z| \leqq r$ and let $\bar{n}(r, \infty) = \bar{n}(r, f)$ be the number of poles of $f(z)$ in $|z| \leqq r$ each pole being counted only once. Let

$$N\left(r, \frac{1}{f-\alpha}\right) = N(r, \alpha) = \int_0^r \frac{n(t, \alpha) - n(0, \alpha)}{t}\, dt + n(0, \alpha) \log r$$

and

$$N(r, \infty) = N(r, f) = \int_0^r \frac{n(t, \infty) - n(0, \infty)}{t}\, dt + n(0, \infty) \log r.$$

Let

$$\bar{N}(r, \alpha) = \int_0^r \frac{\bar{n}(t, \alpha) - \bar{n}(0, \alpha)}{t}\, dt + \bar{n}(0, \alpha) \log r$$

and

$$\bar{N}(r, \infty) = \bar{N}(r, f) = \int_0^r \frac{\bar{n}(t, \infty) - \bar{n}(0, \infty)}{t}\, dt + \bar{n}(0, \infty) \log r.$$

Then clearly $n(r, f') = n(r, f) + \bar{n}(r, f)$ and $N(r, f') = N(r, f) + \bar{N}(r, f)$ see NEVANLINNA [1, 105].

Let $T(r, f) = T(r)$ be the NEVANLINNA characteristic function. Let E denote the set of positive non-decreasing functions Φ such that

$$\int_A^\infty \frac{dx}{x\Phi(x)} \text{ is convergent.}$$

Then we say that α is an e. v. E if

$$\liminf_{r \to \infty} \frac{T(r)}{n(r, \alpha)\Phi(r)} > 0 \quad \text{for some } \Phi \in \mathrm{E}.$$

See S. M. SHAH [2].

If
$$\delta(r, \alpha) = \delta(\alpha) = 1 - \limsup_{r \to \infty} \frac{N(r, \alpha)}{T(r, f)} > 0,$$

then α will be called e. v. N (exceptional value in the sense of NEVANLINNA), see [1, p. 79]; and if

$$\Delta(r, \alpha) = \Delta(\alpha) = 1 - \liminf_{r\to\infty} \frac{N(r, \alpha)}{T(r,f)} > 0$$

then α will be called e. v. V. (in the sense of Valiron). Also let

$$\theta(\alpha) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r, \alpha)}{T(r)}$$

and

$$\mu(r, \alpha) = \mu(\alpha) = \liminf_{r\to\infty} \frac{N(r, \alpha) - \bar{N}(r, \alpha)}{T(r)}$$

2. We prove the following theorem.

Theorem 1. *Let $f(z)$ be a meromorphic function such that*

$$\sum_1^\infty \delta(a_i) + \sum_1^\infty \mu(a_i) = 1,$$

where $a_1, a_2, \ldots$ are any finite constants pairwise different, then, if "∞" is an e. v. E for $f(z)$, it is also an e. v. E for $f'(z)$ and conversely.

PROOF. ∞ is an e. v. E for $f(z)$, so $\delta(\infty) = 1$, see S. M. SHAH [2]. But, it is known that

$$\limsup_{r\to\infty} \frac{T(r,f')}{T(r,f)} \leqq 2 - \delta(\infty) - \mu(\infty) \ldots = 1, \tag{2. 1}$$

see NEVANLINNA [1, 104]. Also it is known that

$$\liminf_{r\to\infty} \frac{T(r,f')}{T(r,f)} \geqq \sum_1^\infty \delta(a_i) + \sum_1^\infty \mu(a_i) = 1, \tag{2. 2}$$

see WITTICH [6,21]. From (2. 1) and (2. 2) it follows that

$$T(r,f') \sim T(r,f).$$

Now

$$n(r,f') = n(r,f) + \bar{n}(r,f) \leqq 2n(r,f)$$

Hence

$$\frac{T(r,f')}{n(r,f')\Phi(r)} \geqq \frac{1}{2}\, \frac{T(r,f')}{n(r,f)\Phi(r)} > \frac{1}{2}\, \frac{(1-\varepsilon)\, T(r,f)}{n(r,f)\Phi(r)} \quad \text{for} \quad r \geqq r_0.$$

Since "∞" is an e. v. E for $f(z)$, it follows

$$\liminf_{r\to\infty} \frac{T(r,f')}{n(r,f')\Phi(r)} > 0.$$

Hence ∞ is an e. v. E for $f'(z)$.

Conversely if ∞ is an e. v. E for $f'(z)$, we prove that it is e. v. E for $f(z)$ also. Since $\delta(f', \infty)=1$,

$$\lim_{r\to\infty}\frac{N(r,f')}{T(r,f')}=0.$$

But $N(r,f') = N(r,f)+\bar{N}(r,f)\geqq 2\bar{N}(r,f)$, so

$$\frac{\bar{N}(r,f)}{T(r,f)}\leqq\frac{1}{2}\,\frac{N(r,f')}{T(r,f')}\,\frac{T(r,f')}{T(r,f)}.$$

$$\limsup_{r\to\infty}\frac{\bar{N}(r,f)}{T(r,f)}\leqq\frac{1}{2}\limsup_{r\to\infty}\frac{N(r,f')}{T(r,f')}\limsup_{r\to\infty}\frac{T(r,f')}{T(r,f)}=0,$$

so

$$\theta(\infty)=1-\limsup_{r\to\infty}\frac{\bar{N}(r,f)}{T(r,f)}=1.$$

But it is known that [6, 21],

$$\limsup_{r\to\infty}\frac{T(r,f')}{T(r,f)}\leqq 2-\theta(\infty)=1.$$

But, as before,

$$\liminf_{r\to\infty}\frac{T(r,f')}{T(r,f)}\geqq\sum_{1}^{\infty}\delta(r,a_j)+\sum_{1}^{\infty}\mu(r,a_j)=1.$$

Hence $T(r,f')\sim T(r,f)$. Now $n(r,f')\geqq n(r,f)$. So for $r\geqq r_0$,

$$\frac{T(r,f)}{n(r,f)\Phi(r)}>\frac{(1-\varepsilon)\,T(r,f')}{n(r,f')\Phi(r)}.$$

Now since ∞ is an e. v. E for $f'(z)$, so the right hand side of the above inequality is positive, hence

$$\liminf_{r\to\infty}\frac{T(r,f)}{n(r,f)\Phi(r)}>0.$$

So ∞ is an e. v. E of $f(z)$.

Theorem 2. *If ∞ is an e. v. E for a meromporhic function $f(z)$ and $\sum_{\alpha_i\neq\infty}\delta(\alpha_i)=1$, then*

$$\delta(f^{(k)},0)+\delta(f^{(k)},\infty)=2\quad\text{for}\quad k=1,2,\ldots.$$

PROOF. Since ∞ is an e. v. E,

$$\delta(f,\infty)=1.$$

But

$$\limsup_{r\to\infty}\frac{T(r,f')}{T(r,f)}\leqq 2-\delta(\infty)-\mu(\infty)=1.$$

Again

$$\liminf_{r\to\infty}\frac{T(r,f')}{T(r,f)}\geqq\sum_{1}^{\infty}\delta(\alpha_i)+\sum_{1}^{\infty}\mu(\alpha_i)=1.$$

So $T(r,f')\sim T(r,f)$.

As in Theorem 1, since ∞ is an e. v. E for $f'(z)$, $\delta(f', \infty)=1$. Again, we have [6, 18],

$$N\left(r, \frac{1}{f'}\right)+\sum_{1}^{q} m(r, a_j)+S(r) \leqq T(r, f').$$

So

$$N\left(r, \frac{1}{f'}\right)+m\left(r, \frac{1}{f'}\right)-m\left(r, \frac{1}{f'}\right)+\sum_{1}^{q} m(r, a_j)+S(r) \leqq T(r, f'),$$

$$\sum_{1}^{q} m(r, a_j)+S(r) \leqq m\left(r, \frac{1}{f'}\right)+O(1).$$

Hence

$$\liminf \sum_{1}^{q} \frac{m(r, a_j)}{T(r, f')}+\frac{S(r)}{T(r, f')} \leqq \delta(f', 0).$$

So

$$\frac{1}{\limsup\limits_{r\to\infty} \dfrac{T(r, f')}{T(r, f)}} \sum_{1}^{q} \delta(r, a_j) \leqq \delta(f', 0).$$

But

$$\limsup_{r\to\infty} \frac{T(r, f')}{T(r, f)} \leqq 1,$$

and

$$\sum_{1}^{\infty} \delta(r, a_j)=1.$$

Hence

$$\delta(f', 0)=1,$$

so

$$\delta(f', 0)+\delta(f', \infty)=2.$$

Repeating the argument we get

$$\delta(f^{(k)}, 0)+\delta(f^{(k)}, \infty)=2.$$

Corollary. *If ∞ is an e. v. E for a meromorphic function $f(z)$ and if further $\sum_{1}^{\infty} \delta(\alpha_i)=1$, then ϱ, the order of the function, must be a positive integer.*

For an alternative proof of the corollary see S. M. Shah and S. K. Singh [5].

Proof. By theorem 2,

$$\delta(f', 0)=1, \quad \delta(f', \infty)=1.$$

Hence

$$\lim_{r\to\infty} \frac{N(r, 1/f')+N(r, f')}{T(r, f')}=0.$$

Hence the order of $f'(z)$ must be a positive integer, because for non-integral order,

$$\limsup_{\to\infty} \frac{N(r, a)+N(r, b)}{T(r)}>0 \text{ for all } a \text{ and } b, \quad (a \neq b),$$

see NEVANLINNA [1, 51]. But the order of a meromorphic function is the same as the order of its derivative. This proves the corollary.

Theorem 3. *If there is at least one e. v. N for a meromorphic function $f(z)$, then* 0 *must be an exceptional value of $f'(z)$ in the sense of* VALIRON.

PROOF. Suppose, if possibile 0 is a normal value for $f'(z)$ in the sense of Valiron. Then $\Delta(f', 0)=0$. But, it is known that

$$\Delta(f', 0) \liminf_{r\to\infty} \frac{T(r, f')}{T(r, f)} \geqq \sum_{1}^{\infty} \delta(a_i),$$

see [4]. So, $\sum_{1}^{\infty} \delta(a_i)=0$, contradicting the hypothesis that $\delta(a_i)>0$ for at least one a_i. Hence 0 is an exceptional value of $f'(z)$ in the sense of VALIRON.

References

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