The extended Hermite—Fejér interpolation formula with application to the theory of generalized almost-step parabolas

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Introduction

Let

$$x_1, x_2, \ldots, x_n; \ \xi_1, \xi_2, \ldots, \xi_m \tag{1}$$

be $n+m$ distinct points on the real axis, and let us denote, correspondingly, by

$$y_1, y_2, \ldots, y_n; \ \eta_1, \eta_2, \ldots, \eta_m \tag{2}$$

arbitrary real values, further let for the first n points the corresponding real numbers

$$y_1', y_2', \ldots, y_n' \tag{3}$$

be prescribed as derivative values or *slopes. The extended Hermite—Fejér interpolation formula* (as I wish to call it) yields a polynomial $S(x)$ of degree $2n+m-1$ at most with the property

$$S(x_\nu)=y_\nu, \quad S(\xi_k)=\eta_k, \quad S'(x_\nu)=y_\nu' \qquad (\nu=1, 2, \ldots, n; \ k=1, 2, \ldots, m). \tag{4}$$

As well-known from the theory of interpolation due to CH. HERMITE[1]), this polynomial exists always and is unique. We shall find below (§ 1) for it the explicit form (24**). But for this purpose we shall not make use of the general theory.

In particular in the case $y_1'=\ldots=y_n'=0$ the polynomial $q(x)$ of degree $2n+m-1$ at most with the property

$$q(x_\nu)=y_\nu, \quad q(\xi_k)=\eta_k, \quad q'(x_\nu)=0 \qquad (\nu=1, 2, \ldots, n; \ k=1, 2, \ldots, m) \tag{5}$$

will be called *extended step parabola* belonging to the *fundamental points* under (1) and the ordinates (2). This notion is similar to that of the step parabola due to L. FEJÉR[2]). Conversely, in the case $y_1=\ldots=y_n=0$, $\eta_1=\ldots=\eta_m=0$ the polynomial $Q(x)$ of degree $2n+m-1$ at most for which

$$Q(x_\nu)=0, \quad Q(\xi_k)=0, \quad Q'(x_\nu)=y_\nu' \qquad (\nu=1, 2, \ldots, n; \ k=1, 2, \ldots, m) \tag{6}$$

holds, will be named *extended wave parabola* determined by the fundamental points (1) and the slopes (3), corresponding to the notion of wave parabola introduced by L. FEJÉR.[3]) In general the above polynomial $S(x)$ of degree $2n+m-1$ at most

[1]) See [5], 70 or 432, respectively.
[2]) See [2], 210; further [3], 66.
[3]) See [4], 714.

characterized by the relations under (4) will be called *generalized extended step parabola* belonging to the fundamental points (1) and the ordinates and slopes under (2), (3) respectively.

In the case $m=2$, $\xi_1=1$, $\xi_2=-1$ the extended Hermite—Fejér interpolation is identical with the quasi-Hermite—Fejér one introduced by the author.[4]) P. Turán[5]) proposed that the case $m=1$, $\xi=1$ be investigated also. In §§ 2, 3 of the present paper we deal with this case.

In the just mentioned case $m=1$, $\xi=1$ we shall speak of *almost Hermite—Fejér interpolation formula*. It yields the unique polynomial $S(x)$ of degree $2n$ at most with the property

$$S(x_\nu)=y_\nu,\quad S(1)=\eta,\quad S'(x_\nu)=y'_\nu \qquad (\nu=1,2,\ldots,n). \tag{7}$$

In particular in the case $y'_1=\ldots=y'_n=0$ the polynomial $S(x)$ of degree $2n$ at most characterized by (7) will be called the *almost-step parabola*, while in the case $y_1=\ldots=y_n=0$, $\eta=0$ it will be named the *almost-wave parabola*, determined beside the points

$$x_1, x_2, \ldots, x_n, 1 \tag{8}$$

in the first case by the ordinates

$$y_1, y_2, \ldots, y_n, \eta \tag{9}$$

and in the second by the slopes (3). In general it will be called *the generalized almost-step parabola* belonging to the fundamental points (8) and the corresponding ordinates and slopes (9), (3) respectively.

Our Theorem I below (§ 2) states the uniform convergence of the generalized almost-step parabola to the function $f(x)$ continuous in the closed interval $-1\leqq x\leqq 1$, in the case when the fundamental points are beside 1 the zeros of the Jacobi polynomial $P_n^{(\frac{1}{2},-\frac{1}{2})}(x)$ with the notation of G. Szegő[6]) and the ordinates the corresponding values of $f(x)$, provided that the slopes are uniformly bounded if $n\to+\infty$. Theorem II (§ 3) contains a similar statement for the case of the roots of the Legendre polynomial $P_n(x)$, the convergence however, is then assured only in the interval $-1<x\leqq 1$ open from below, and uniform in each subinterval $-1+\delta\leqq x\leqq 1$ $(0<\delta<2)$.

§ 1. The extended Hermite—Fejér interpolation formula

Using the fundamental points under (1) let us introduce the following notations:

$$\omega(x) = C(x-x_1)(x-x_2)\ldots(x-x_n), \qquad C\neq 0 \tag{10}$$

and

$$\Omega(x) = K(x-\xi_1)(x-\xi_2)\ldots(x-\xi_m), \quad K\neq 0 \tag{11}$$

[4]) P. Szász, [7], 414.

[5]) Oral communication for which I am thankful to P. Turán.

[6]) See [8], 58.

further

$$l_\nu(x) = \frac{\omega(x)}{\omega'(x_\nu)(x-x_\nu)} \qquad (\nu = 1, 2, \dots, n) \tag{12}$$

and

$$L_k(x) = \frac{\Omega(x)}{\Omega'(\xi_k)(x-\xi_k)} \qquad (k = 1, 2, \dots, m). \tag{13}$$

Then we define

$$r_\nu(x) = \frac{\Omega(x)}{\Omega(x_\nu)}[1 + c_\nu(x-x_\nu)]\, l_\nu(x)^2 \qquad (\nu = 1, 2, \dots, n) \tag{14}$$

and

$$\varrho_k(x) = \frac{\omega(x)^2}{\omega(\xi_k)^2} L_k(x) \qquad (k = 1, 2, \dots, m) \tag{15}$$

where

$$c_\nu = \sum_{k=1}^{m} \frac{1}{\xi_k - x_\nu} - \frac{\omega''(x_\nu)}{\omega'(x_\nu)} \qquad (\nu = 1, 2, \dots, n) \tag{16}$$

to be the *fundamental polynomials of the first kind*, and

$$s_\nu(x) = \frac{\Omega(x)}{\Omega(x_\nu)}(x-x_\nu)\, l_\nu(x)^2 \qquad (\nu = 1, 2, \dots, n) \tag{17}$$

to be *those of the second kind*. With regard to (10), (11), (12), (13) it follows by (14), (15), (16), (17) at once

$$r_\nu(x_\nu) = 1, \quad r_\nu(x_\mu) = 0 \ (\mu \neq \nu), \ r_\nu(\xi_k) = 0 \qquad (\nu, \mu = 1, 2, \dots, n;\ k = 1, 2, \dots, m) \tag{18}$$

and

$$r'_\nu(x_\mu) = 0 \qquad (\nu, \mu = 1, 2, \dots, n), \tag{19}$$

further

$$\varrho_k(\xi_k) = 1, \ \varrho_k(\xi_l) = 0 \quad (l \neq k), \ \varrho_k(x_\nu) = 0 \qquad (k, l = 1, 2, \dots, m;\ \nu = 1, 2, \dots, n) \tag{20}$$

and

$$\varrho'_k(x_\nu) = 0 \qquad (\nu = 1, 2, \dots, n;\ k = 1, 2, \dots, m), \tag{21}$$

while

$$s_\nu(x_\mu) = 0, \quad s_\nu(\xi_k) = 0 \qquad (\nu, \mu = 1, 2, \dots, n;\ k = 1, 2, \dots, m) \tag{22}$$

and

$$s'_\nu(x_\nu) = 1, \ s'_\nu(x_\mu) = 0 \ (\mu \neq \nu) \qquad (\nu, \mu = 1, 2, \dots, n). \tag{23}$$

By reason of the relations (18)–(23), if the arbitrary real values under (2) and (3) are prescribed corresponding to the fundamental points (1), the polynomial

$$S(x) = \sum_{\nu=1}^{n} y_\nu r_\nu(x) + \sum_{k=1}^{m} \eta_k \varrho_k(x) + \sum_{\nu=1}^{n} y'_\nu s_\nu(x) \tag{24}$$

of degree $2n+m-1$ at most has obviously the property (4). Moreover this polynomial $S(x)$ is unique. Indeed, $S^*(x)$ being a polynomial of degree $2n+m-1$ at most and with the same property, i. e.

$$S^*(x_\nu)=y_\nu,\ \ S^*(\xi_k)=\eta_k,\ \ S^{*\prime}(x_\nu)=y'_\nu \qquad (\nu=1,2,\dots,n;\ k=1,2,\dots,m),$$

for the polynomial

$$p(x) = S^*(x)-S(x)$$

of degree $2n+m-1$ at most there holds

$$p(x_\nu)=0,\quad p(\xi_k)=0,\quad p'(x_\nu)=0$$
$$(\nu=1,2,\dots,n;\ k=1,2,\dots,m)$$

and consequently for all x

$$p(x) = S^*(x)-S(x) = 0,$$

thus $S^*(x)$ is identical with $S(x)$.

As we see, the unique polynomial $S(x)$ of degree $2n+m-1$ at most with the property (4) is given by (24). Therefore (24) *presents the generalized extended step parabola* defined in the introduction. In particular the *extended step parabola* $q(x)$ characterized by (5) is given by

$$q(x)=\sum_{\nu=1}^{n} y_\nu r_\nu(x)+\sum_{k=1}^{m}\eta_k\varrho_k(x) \tag{25}$$

while the *extended wave parabola* $Q(x)$ determined by (6) is

$$Q(x)=\sum_{\nu=1}^{n} y'_\nu s_\nu(x). \tag{26}$$

Thus, (24) can be written in the abbreviated form

$$S(x) = q(x)+Q(x). \tag{24*}$$

With respect to (12), (13), (14), (15) we obtain by (25) for $q(x)$ the explicit form

$$q(x)=\sum_{\nu=1}^{n} y_\nu\frac{\Omega(x)}{\Omega(x_\nu)}\,[1+c_\nu(x-x_\nu)]\left(\frac{\omega(x)}{\omega'(x_\nu)(x-x_\nu)}\right)^2+ \tag{25*}$$
$$+\sum_{k=1}^{m}\eta_k\frac{\Omega(x)^2}{\Omega(\xi_k)^2}\,\frac{\Omega(x)}{\Omega'(\xi_k)(x-\xi_k)}$$

where $\omega(x)$, $\Omega(x)$ and c_ν are given by (10), (11) and (16). Further, with regard to (12) and (17), $Q(x)$ can be written by reason of (26) in the detailed form

$$Q(x)=\sum_{\nu=1}^{n} y'_\nu\frac{\Omega(x)}{\Omega(x_\nu)}\,(x-x_\nu)\left(\frac{\omega(x)}{\omega'(x_\nu)(x-x_\nu)}\right)^2. \tag{26*}$$

Consequently, the generalized extended step parabola given by (24*) has the explicit form

$$(24^{**})\qquad S(x)=\sum_{\nu=1}^{n} y_\nu \frac{\Omega(x)}{\Omega(x_\nu)}[1+c_\nu(x-x_\nu)]\left(\frac{\omega(x)}{\omega'(x_\nu)(x-x_\nu)}\right)^2+$$

$$+\sum_{k=1}^{m}\eta_k\frac{\omega(x)^2}{\omega(\xi_k)^2}\frac{\Omega(x)}{\Omega'(\xi_k)(x-\xi_k)}+\sum_{\nu=1}^{n} y'_\nu\frac{\Omega(x)}{\Omega(x_\nu)}(x-x_\nu)\left(\frac{\omega(x)}{\omega'(x_\nu)(x-x_\nu)}\right)^2$$

where $\omega(x)$, $\Omega(x)$ and c_ν are given again by (10), (11), (16). (24) or (24**) is the *extended Hermite—Fejér interpolation formula* as we have named it in the introduction.

In particular, in the case $m=1$, $\xi=1$ we can take $\Omega(x)=1-x$, and with respect to (12) by (24**) we get

$$(27)\qquad S(x)=\sum_{\nu=1}^{n} y_\nu\frac{1-x}{1-x_\nu}[1+c_\nu^*(x-x_\nu)\ l_\nu(x)^2+\eta\frac{\omega(x)^2}{\omega(1)^2}+$$

$$+\sum_{k=1}^{m} y'_\nu\frac{1-x}{1-x_\nu}(x-x_\nu)l_\nu(x)^2$$

where

$$(28)\qquad c_\nu^*=\frac{1}{1-x_\nu}-\frac{\omega''(x_\nu)}{\omega'(x_\nu)}\qquad(\nu=1,2,\ldots,n).$$

This is the *almost-Hermite—Fejér interpolation formula* which yields the *generalized almost-step parabola* as we have called it in the introduction.

From the unicity of this generalized almost-step parabola given by (27) it follows that for each polynomial $\Pi(x)$ of degree $2n$ at most

$$\Pi(x)=\sum_{\nu=1}^{n}\Pi(x_\nu)\frac{1-x}{1-x_\nu}[1+c_\nu^*(x-x_\nu)]l_\nu(x)^2+\Pi(1)\frac{\omega(x)^2}{\omega(1)^2}+$$

$$+\sum_{k=1}^{m}\Pi'(x_\nu)(x-x_\nu)l_\nu(x)^2$$

holds. This gives for $\Pi(x)=1$ the identity

$$(29)\qquad \sum_{\nu=1}^{n}\frac{1-x}{1-x_\nu}[1+c_\nu^*(x-x_\nu)]l_\nu(x)^2+\frac{\omega(x)^2}{\omega(1)^2}=1$$

fundamental in our following treatment; let it be called the *fundamental identity*.

§ 2. The generalized almost-step parabola corresponding to the zeros of the Jacobi polynomial $P_n^{(\frac{1}{2},-\frac{1}{2})}(x)$. Weierstrass approximation by means of the same

Disregarding a constant factor the Jacobi polynomial $P_n^{(\frac{1}{2},-\frac{1}{2})}(x)$ is identical with[7])

$$\omega(x)=\left[\frac{\sin\,(2n+1)\,\dfrac{\vartheta}{2}}{\sin\dfrac{\vartheta}{2}}\right]_{x=\cos\vartheta}. \tag{30}$$

It has the zeros

$$x_{\nu n}=\cos\frac{2\nu\pi}{2n+1}\qquad(\nu=1,2,\ldots,n). \tag{31}$$

To produce the fundamental polynomials of the first and the second kind in the case when $m=1$, $\xi=1$ and the fundamental points beside 1 are those under (31), we start from the well-known differential equation

$$(1-x^2)\,\omega''(x)-(1+2x)\,\omega'(x)+n(n+1)\,\omega(x)=0. \tag{32}$$

Applying this for the root $x_{\nu n}$ of $\omega(x)$, we obtain (setting x_ν instead of $x_{\nu n}$)

$$\frac{\omega''(x_\nu)}{\omega'(x_\nu)}=\frac{1+2x_\nu}{1-x_\nu^2}$$

consequently the coefficient c_ν^* under (28) is

$$c_\nu^*=\frac{1}{1-x_\nu}-\frac{1+2x_\nu}{1-x_\nu^2}=-\frac{x_\nu}{1-x_\nu^2}$$

and so we get

$$1+c_\nu^*(x-x_\nu)=\frac{1-x_\nu x}{1-x_\nu^2}\qquad(\nu=1,2,\ldots,n). \tag{33}$$

Thus, by taking $\Omega(x)=1-x$, the fundamental polynomials of the first kind given by (14) and (15) are

$$\begin{cases} r_\nu(x)=\dfrac{1-x}{1-x_\nu}\,\dfrac{1-x_\nu x}{1-x_\nu^2}\,l_\nu(x)^2 & (\nu=1,2,\ldots,n),\\[2mm] \varrho(x)=\dfrac{\omega(x)^2}{\omega(1)^2} & \end{cases} \tag{34}$$

while those of the second kind under (17) are

$$s_\nu(x)=\frac{1-x}{1-x_\nu}(x-x_\nu)\,l_\nu(x)^2\qquad(\nu=1,2,\ldots,n). \tag{35}$$

[7]) G. Szegő, [8], 60, form. (4. 1. 8.).

As we see, *in this case of the zeros of the Jacobi polynomial* (30) *the fundamental polynomials of the first kind under* (34) *are all non-negative in the interval* $-1 \leq x \leq 1$. This fact has a fundamental importance in the following treatment.

With respect to (34) and (35) in the present case the almost-step parabola (25) and the almost-wave parabola (26) are

$$q^*(x) = \sum_{\nu=1}^{n} y_\nu \frac{1-x}{1-x_\nu} \frac{1-x_\nu x}{1-x_\nu^2} l_\nu(x)^2 + \eta \frac{\omega(x)^2}{\omega(1)^2} \tag{36}$$

and

$$Q^*(x) = \sum_{\nu=1}^{n} y'_\nu \frac{1-x}{1-x_\nu} (x-x_\nu) l_\nu(x)^2, \tag{37}$$

respectively, while the generalized almost-step parabola according to (24*) has the abbreviated form

$$S(x) = q^*(x) + Q^*(x). \tag{38}$$

Now, in what follows we need the following

Lemma I. *For the zeros under* (31) *of the polynomial* (30) *we have*

$$\sum_{\nu=1}^{n} \frac{1}{(1-x_{\nu n})(1-x_{\nu n}^2)\omega'(x_{\nu n})^2} = \frac{2n}{(2n+1)^2}. \tag{39}$$

For the proof we start from the identity

$$\sin \frac{\vartheta}{2} \omega(\cos \vartheta) = \sin (2n+1) \frac{\vartheta}{2}.$$

By deriving we get

$$\cos \frac{\vartheta}{2} \omega(\cos \vartheta) - 2 \sin \frac{\vartheta}{2} \sin \vartheta \ \omega'(\cos \vartheta) = (2n+1) \cos (2n+1) \frac{\vartheta}{2}.$$

Putting here $x = \cos \vartheta$, for the root $x_{\nu n}$ of $\omega(x)$ there results

$$\omega'(x_{\nu n}) = (-1)^{\nu+1} \frac{2n+1}{2 \sin \dfrac{\nu\pi}{2n+1} \sin \dfrac{2\nu\pi}{2n+1}}. \tag{40}$$

Since with respect to (31)

$$1 - x_{\nu n} = 2 \sin^2 \frac{\nu\pi}{2n+1}, \quad 1 - x_{\nu n}^2 = \sin^2 \frac{2\nu\pi}{2n+1},$$

by the aid of (40) follows (39), Lemma I is proved.

Next, Weierstrass approximation is expressed by the following[8])

Theorem I. *Let* $f(x)$ *be a continuous function in the closed interval* $-1 \leq x \leq 1$.

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[8]) First exposed in my lecture „*Különböző típusú lépcsőparabolákról*" (in Hungarian) on the 6th April 1962 in Debrecen (Hungary), later in the lecture of mine "*On generalized quasi-step and almost-step parabolas, respectively*" delivered on the 18th August, at the *International Congress of Mathematicians Stockholm 1962.*

Furthermore let beside 1 *the zeros under* (31) *of the Jacobi polynomial* $P_n^{(\frac{1}{2},-\frac{1}{2})}(x)$ *be chosen as fundamental points, and let the arbitrary given numbers*

$$y'_{1n}, y'_{2n}, \ldots, y'_{nn} \qquad (n=1,2,3,\ldots)$$

be taken for slopes, provided that

$$|y'_{\nu n}| \leqq \Delta \tag{41}$$

where Δ *is independent of* n *and* ν. *Then for the generalized almost-step parabola* $S_n(x)$ *of degree* $2n$ *at most belonging to* $f(x)$, *i. e. with the property*

$$S_n(x_{\nu n})=f(x_{\nu n}),\ \ S_n(1)=f(1),\ \ S'_n(x_{\nu n})=y'_{\nu n} \qquad (\nu=1,2,\ldots,n;\ n=1,2,3,\ldots)$$

holds

$$S_n(x) \to f(x) \ \ \text{if} \ \ n \to +\infty$$

the convergence being uniform in the whole closed interval.

Proof. We prove first of all that the almost-step parabola $q_n^*(x)$ of degree $2n$ at most corresponding to $f(x)$, that is according to (36)

$$q_n^*(x)=\sum_{\nu=1}^{n} f(x_{\nu n})\,\frac{1-x}{1-x_{\nu n}}\,\frac{1-x_{\nu n}x}{1-x_{\nu n}^2}\,l_{\nu n}(x)^2+f(1)\,\frac{\omega_n(x)^2}{\omega_n(1)^2} \tag{42}$$

where $\omega_n(x)$ is identical with the polynomial $\omega(x)$ under (30), and by (12)

$$l_{\nu n}(x)=\frac{\omega_n(x)}{\omega'_n(x_{\nu n})(x-x_{\nu n})}, \tag{43}$$

converges uniformly to $f(x)$ if $n \to +\infty$.

With regard to (33), in the present case the fundamental identity (29) has the form

$$\sum_{\nu=1}^{n} \frac{1-x}{1-x_{\nu n}}\,\frac{1-x_{\nu n}x}{1-x_{\nu n}^2}\,l_{\nu n}(x)^2+\frac{\omega_n(x)^2}{\omega_n(1)^2}=1. \tag{44}$$

Consequently it may be written

$$f(x)=\sum_{\nu=1}^{n} f(x)\,\frac{1-x}{1-x_{\nu n}}\,\frac{1-x_{\nu n}x}{1-x_{\nu n}^2}\,l_{\nu n}(x)^2+f(x)\,\frac{\omega_n(x)^2}{\omega_n(1)^2} \tag{45}$$

and by (42), (45) there follows

$$f(x)-q_n^*(x)=\sum_{\nu=1}^{n}[f(x)-f(x_{\nu n})]\,\frac{1-x}{1-x_{\nu n}}\,\frac{1-x_{\nu n}x}{1-x_{\nu n}^2}\,l_{\nu n}(x)^2+[f(x)-f(1)]\,\frac{\omega_n(x)^2}{\omega_n(1)^2}. \tag{46}$$

But

$$[f(x)-f(1)]\,\frac{\omega_n(x)^2}{\omega_n(1)^2} \to 0$$

uniformly for $-1 \leqq x \leqq 1$. This is an easy consequence of the fact that the first factor [...] vanishes for $x=1$ and is continuous for $-1\leqq x\leqq 1$, while with regard to (30)

$$\frac{\omega_n(x)}{\omega_n(1)}=\frac{1}{2n+1}\left[\frac{\sin\,(2n+1)\,\dfrac{\vartheta}{2}}{\sin\,\dfrac{\vartheta}{2}}\right]_{x=\cos\vartheta}\to 0$$

uniformly in each subinterval $-1\leqq x\leqq 1-\sigma$ $(0<\sigma<2)$ and is bounded in the whole closed interval since as well-known

$$\left|\frac{\sin\,(2n+1)\,\dfrac{\vartheta}{2}}{\sin\,\dfrac{\vartheta}{2}}\right|\leqq 2n+1.$$

Therefore (46) implies the uniform convergence of $q_n^*(x)$ to $f(x)$ if one shows that

$$\sum_{\nu=1}^{n}[f(x)-f(x_{\nu n})]\,\frac{1-x}{1-x_{\nu n}}\,\frac{1-x_{\nu n}x}{1-x_{\nu n}^2}\,l_{\nu n}(x)^2\to 0 \tag{47}$$

uniformly for $-1\leqq x\leqq 1$. This may be seen as follows.

By reason of the theorem of Weierstrass let for $-1\leqq x\leqq 1$ be

$$|f(x)|\leqq M. \tag{48}$$

The positive number ε having been chosen as small as we please, there exists a positive number δ such that

$$|f(x')-f(x'')|\leqq\varepsilon\ \text{ when }\ |x'-x''|\leqq\delta, \tag{49}$$

the continuous function $f(x)$ being, as well-known, also uniformly continuous for $-1\leqq x\leqq 1$. Owing to the non-negativity of the fundamental polynomials under (34), by the aid of (44), (49) we have for a fixed x in the interval $-1\leqq x\leqq 1$

$$\left|\sum_{|x_{\nu n}-x|\leqq\delta}[f(x)-f(x_{\nu n})]\,\frac{1-x}{1-x_{\nu n}}\,\frac{1-x_{\nu n}x}{1-x_{\nu n}^2}\,l_{\nu n}(x)^2\right|\leqq \tag{50}$$

$$\leqq\varepsilon\sum_{|x_{\nu n}-x|\leqq\delta}\frac{1-x}{1-x_{\nu n}}\,\frac{1-x_{\nu n}x}{1-x_{\nu n}^2}\,l_{\nu n}(x)^2\leqq\varepsilon.$$

On the other hand, in view of (48), (43) and the obvious inequality $0<1-x_{\nu n}x<2$ we have

$$\left|\sum_{|x_{\nu n}-x|>\delta}[f(x)-f(x_{\nu n})]\,\frac{1-x}{1-x_{\nu n}}\,\frac{1-x_{\nu n}x}{1-x_{\nu n}^2}\,l_{\nu n}(x)^2\right|\leqq \tag{51}$$

$$\leqq\frac{4M}{\delta^2}\sum_{|x_{\nu n}-x|>\delta}\frac{(1-x)\,\omega_n(x)^2}{(1-x_{\nu n})(1-x_{\nu n}^2)\,\omega_n'(x_{\nu n})^2}.$$

However, with regard to (30) it is clear that

(52) $$0 \leqq (1-x)\omega_n(x)^2 \leqq 2 \quad \text{for} \quad -1 \leqq x \leqq 1.$$

Hence by the aid of (51) and of Lemma I under (39) there follows that

(53) $$\left| \sum_{|x_{\nu n}-x|>\delta} [f(x)-f(x_{\nu n})] \frac{1-x}{1-x_{\nu n}^2} \frac{1-x_{\nu n}x}{1-x_{\nu n}^2} l_{\nu n}(x)^2 \right| \leqq \frac{8M}{\delta^2} \frac{2n}{(2n+1)^2} \leqq \varepsilon$$

if n is large enough, i. e. $n \geqq N$, say. Now, by (50) and (53) we have for $-1 \leqq x \leqq 1$

$$\left| \sum_{\nu=1}^{n} [f(x)-f(x_{\nu n})] \frac{1-x}{1-x_{\nu n}} \frac{1-x_{\nu n}x}{1-x_{\nu n}^2} l_{\nu n}(x)^2 \right| \leqq 2\varepsilon$$

when $n \geqq N$, i. e. (47) is valid. Thus, the uniform convergence of the almost-step parabola (42) to $f(x)$ is established.

The generalized almost-step parabola $S_n(x)$ being by (37) and (38)

$$S_n(x) = q_n^*(x) + \sum_{\nu=1}^{n} y'_{\nu n} \frac{1-x}{1-x_{\nu n}} (x-x_{\nu n}) l_{\nu n}(x)^2,$$

for the proof of the present theorem there remains still to show that

$$\sum_{\nu=1}^{n} y'_{\nu n} \frac{1-x}{1-x_{\nu n}} (x-x_{\nu n}) l_{\nu n}(x)^2 \to 0$$

uniformly for $-1 \leqq x \leqq 1$ if $n \to +\infty$. But by the assumption (41) we have

$$\left| \sum_{\nu=1}^{n} y'_{\nu n} \frac{1-x}{1-x_{\nu n}} (x-x_{\nu n}) l_{\nu n}(x)^2 \right| \leqq \varDelta \sum_{\nu=1}^{n} \frac{1-x}{1-x_{\nu n}} |x-x_{\nu n}| l_{\nu n}(x)^2.$$

Hence, for our purpose it is sufficient to show that

(54) $$\sum_{\nu=1}^{n} \frac{1-x}{1-x_{\nu n}} |x-x_{\nu n}| l_{\nu n}(x)^2 \to 0$$

uniformly for $-1 \leqq x \leqq 1$.

First, the identity (44) implies

(55) $$\sum_{\nu=1}^{n} \frac{1-x}{1-x_{\nu n}} \frac{1-x_{\nu n}x}{1-x_{\nu n}^2} l_{\nu n}(x)^2 \leqq 1.$$

However, by the obvious inequalities

$$0 < 1+x_{\nu n} < 2, \quad 0 < 1-x_{\nu n} < 2,$$

one has

$$\left[\frac{1-x_{\nu n}x}{1-x_{\nu n}^2}\right]_{x=1} = \frac{1}{1+x_{\nu n}} > \frac{1}{2}, \quad \left[\frac{1-x_{\nu n}x}{1-x_{\nu n}^2}\right]_{x=-1} = \frac{1}{1-x_{\nu n}} > \frac{1}{2},$$

consequently

$$\frac{1-x_{\nu n}x}{1-x_{\nu n}^2} > \frac{1}{2} \quad \text{for} \quad -1 \leqq x \leqq 1,$$

and so by reason of (55) we have

$$\sum_{\nu=1}^{n} \frac{1-x}{1-x_{\nu n}} l_{\nu n}(x)^2 \leqq 2 \qquad (-1 \leqq x \leqq 1) \tag{56}$$

in view to the non-negativity of the terms. Now, let an arbitrary positive number ε be given. Owing to the inequality (56) we have for a fixed x

$$\sum_{|x_{\nu n}-x| \leqq \varepsilon} \frac{1-x}{1-x_{\nu n}} |x-x_{\nu n}| l_{\nu n}(x)^2 \leqq \varepsilon \sum_{|x_{\nu n}-x| \leqq \varepsilon} \frac{1-x}{1-x_{\nu n}} l_{\nu n}(x)^2 \leqq 2\varepsilon. \tag{57}$$

On the other hand, in view of (43), (52) and the obvious inequality $0 < 1 - x_{\nu n}^2 < 1$, by Lemma I under (39) there follows

$$\sum_{|x_{\nu n}-x| > \varepsilon} \frac{1-x}{1-x_{\nu n}} |x-x_{\nu n}| l_{\nu n}(x)^2 \leqq 2 \sum_{|x_{\nu n}-x| > \varepsilon} \frac{1}{(1-x_{\nu n})\omega_n'(x_{\nu n})^2 |x-x_{\nu n}|} <$$
$$< \frac{2}{\varepsilon} \sum_{|x_{\nu n}-x| > \varepsilon} \frac{1}{(1-x_{\nu n})(1-x_{\nu n}^2)\omega_n'(x_{\nu n})^2} \leqq \frac{2}{\varepsilon} \frac{2n}{(2n+1)^2}.$$

Consequently,

$$\sum_{|x_{\nu n}-x| > \varepsilon} \frac{1-x}{1-x_{\nu n}} |x-x_{\nu n}| l_{\nu n}(x)^2 < \varepsilon \tag{58}$$

when n is large enough, i. e. $n \geqq N'$, say. Combining (57) and (58) we obtain for $-1 \leqq x \leqq 1$

$$\sum_{\nu=1}^{n} \frac{1-x}{1-x_{\nu n}} |x-x_{\nu n}| l_{\nu n}(x)^2 < 3\varepsilon \quad \text{if only} \quad n \geqq N'$$

which expresses just the validity of (54), thus completing the proof of the theorem.

§ 3. Weierstrass approximation in each subinterval $-1+\delta \leqq x \leqq 1$ $(1<\delta<2)$ by means of the generalized almost-step parabola belonging to the zeros of the Legendre polynomial $P_n(x)$

The Legendre polynomial $\omega(x) = P_n(x)$ of degree n (exactly) satisfies the differential equation

$$(1-x^2)\omega''(x) - 2x\omega'(x) + n(n+1)\omega(x) = 0;$$

for the zeros

$$x_{1n}, x_{2n}, \ldots, x_{nn} \tag{59}$$

of the same we have (setting x_ν instead of $x_{\nu n}$)

$$\frac{\omega''(x_\nu)}{\omega'(x_\nu)} = \frac{2x_\nu}{1-x_\nu^2} \qquad (\nu = 1, 2, \ldots, n).$$

Consequently, in the present case for the coefficient c_ν^* by (28) results

$$c_\nu^* = \frac{1}{1-x_\nu} - \frac{2x_\nu}{1-x_\nu^2} = \frac{1}{1+x_\nu}$$

and so we obtain

$$1+c_\nu^*(x-x_\nu)=\frac{1+x}{1+x_\nu}\qquad(\nu=1,2,\ldots,n). \tag{60}$$

Thus, with respect to $P_n(1)=1$, the fundamental polynomials of the first kind under (14) and (15) are (taking $\Omega(x)=1-x$ again)

$$\left\{\begin{aligned} &r_\nu(x)=\frac{1-x}{1-x_\nu}\frac{1+x}{1+x_\nu}l_\nu(x)^2=\frac{1-x^2}{1-x_\nu^2}\left(\frac{P_n(x)}{P_n'(x_\nu)(x-x_\nu)}\right)^2\\ &\qquad\qquad(\nu=1,2,\ldots,n),\\ &\qquad\qquad\varrho(x)=P_n(x)^2,\end{aligned}\right. \tag{61}$$

while those of the second kind given by (17) are

$$s_\nu(x)=\frac{1-x}{1-x_\nu}(x-x_\nu)l_\nu(x)^2=\frac{1-x}{1-x_\nu}(x-x_\nu)\left(\frac{P_n(x)}{P_n'(x_\nu)(x-x_\nu)}\right)^2 \tag{62}$$

$$(\nu=1,2,\ldots,n).$$

As we see, *also in this case* $\omega(x)=P_n(x)$ *the fundamental polynomials of the first kind are all non-negative in the interval* $-1\leq x\leq 1$. This fact is again of fundamental importance in what follows.

With regard to (61), in the present case the fundamental identity (29) has the form

$$\sum_{\nu=1}^{n}\frac{1-x^2}{1-x_\nu^2}\left(\frac{P_n(x)}{P_n'(x_\nu)(x-x_\nu)}\right)^2+P_n(x)^2=1 \tag{63}$$

already discovered by E. EGERVÁRY and P. TURÁN[9]) but in another way.

By the aid of (61) and (62) the almost-step parabola (25*) and the almost-wave parabola (26*) have at present the forms

$$q^*(x)=\sum_{\nu=1}^{n}y_\nu\frac{1-x^2}{1-x_\nu^2}\left(\frac{P_n(x)}{P_n'(x_\nu)(x-x_\nu)}\right)^2+\eta P_n(x)^2 \tag{64}$$

and

$$Q^*(x)=\sum_{\nu=1}^{n}y_\nu'\frac{1-x}{1-x_\nu}(x-x_\nu)\left(\frac{P_n(x)}{P_n'(x_\nu)(x-x_\nu)}\right)^2, \tag{65}$$

respectively.

Now, in what follows we need the following

Lemma II. *For the roots under* (59) *of the Legendre polynomial* $P_n(x)$ *we have*

$$\sum_{\nu=1}^{n}\frac{1-x^2}{1-x_{\nu n}^2}\,|x-x_{\nu n}|\left(\frac{P_n(x)}{P_n'(x_{\nu n})(x-x_{\nu n})}\right)^2\to 0 \tag{66}$$

uniformly for $-1\leq x\leq 1$.

[9]) See [1], 264, form. (6. 7).

This fact was already proved by the author in the paper quoted above.[10] For the sake of completeness we repeat here the proof with a slight modification.

Let the positive number ε be chosen as small as we please. Owing to (63) and the non-negativity of the fundamental polynomials of the first kind under (61), we have for fixed x

$$\sum_{|x_{\nu n}-x|\leqq\varepsilon}\frac{1-x^2}{1-x_{\nu n}^2}|x-x_{\nu n}|\left(\frac{P_n(x)}{P_n'(x_{\nu n})(x-x_{\nu n})}\right)^2\leqq \tag{67}$$

$$\leqq\varepsilon\sum_{|x_{\nu n}-x|\leqq\varepsilon}\frac{1-x^2}{1-x_{\nu n}^2}\left(\frac{P_n(x)}{P_n'(x_{\nu n})(x-x_{\nu n})}\right)^2\leqq\varepsilon.$$

On the other hand, by the aid of the relation

$$\sum_{\nu=1}^{n}\frac{1}{(1-x_{\nu n}^2)P_n'(x_{\nu n})^2}=1 \tag{68}$$

shown by L. FEJÉR[11])

$$\sum_{|x_{\nu n}-x|>\varepsilon}\frac{1-x^2}{1-x_{\nu n}^2}|x-x_{\nu n}|\left(\frac{P_n(x)}{P_n'(x_{\nu n})(x-x_{\nu n})}\right)^2=$$

$$=(1-x^2)P_n(x)^2\sum_{|x_{\nu n}-x|>\varepsilon}\frac{1}{(1-x_{\nu n}^2)P_n'(x_{\nu n})^2}\,\frac{1}{|x-x_{\nu n}|}\leqq\frac{1}{\varepsilon}(1-x^2)P_n(x)^2$$

holds. Thus, taking into account T. J. STIELTJES's estimation[12])

$$|P_n(x)|<\frac{c}{\sqrt{n}}\,\frac{1}{\sqrt[4]{1-x^2}}\quad\text{for}\quad -1<x<1 \tag{69}$$

where c is a numerical constant, we have obviously

$$\sum_{|x_{\nu n}-x|>\varepsilon}\frac{1-x^2}{1-x_{\nu n}^2}|x-x_{\nu n}|\left(\frac{P_n(x)}{P_n'(x_{\nu n})(x-x_{\nu n})}\right)^2\leqq\varepsilon \tag{70}$$

if n is large enough, i. e. $n\geqq N$, say. Combining (67) and (70) there follows for $-1\leqq\leqq x\leqq 1$

$$\sum_{\nu=1}^{n}\frac{1-x^2}{1-x_{\nu n}^2}|x-x_{\nu n}|\left(\frac{P_n(x)}{P_n'(x_{\nu n})(x-x_{\nu n})}\right)^2\leqq 2\varepsilon\quad\text{if only}\quad n\geqq N,$$

i. e. (66) is valid, Lemma II is proved.

Next, similarly to our Theorem I (§ 2) Weierstrass approximation is expressed, but only in each subinterval $-1+\delta\leqq x\leqq 1$ $(0<\delta<2)$ by the following

Theorem II. *Let $f(x)$ be a continuous function in the closed interval $-1\leqq x\leqq 1$. Furthermore let beside* 1 *the zeros under* (59) *of the Legendre polynomial $P_n(x)$ be chosen as fundamental points, and let the arbitrary given numbers*

$$y_{1n}',\ y_{2n}',\ \dots,\ y_{nn}'\qquad(n=1,2,3,\dots)$$

[10]) P. SZÁSZ, [7], 424.
[11]) See [2], 221, form. (45); or [3], 78, form. (39).
[12]) See [6], 241–242.

be taken for slopes, provided that

$$|y'_{\nu n}| \leqq \Delta \qquad (\nu = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots) \tag{71}$$

Δ being independent of n and ν. Then for the generalized almost-step parabola $S_n(x)$ of degree $2n$ at most corresponding to $f(x)$, i. e. with the property

$$S_n(x_{\nu n}) = f(x_{\nu n}),\ S_n(1) = f(1),\ S'_n(x_{\nu n}) = y'_{\nu n}$$
$$(\nu = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots)$$

the relation

$$S_n(x) \to f(x) \ \textit{for} \ -1 < x \leqq 1$$

holds if $n \to +\infty$, the convergence being uniform in each subinterval $-1+\delta \leqq x \leqq 1$ $(0 < \delta < 2)$.

PROOF. First we prove that the almost-step parabola $q_n^*(x)$ of degree $2n$ at most corresponding to $f(x)$, that is by (64)

$$q_n^*(x) = \sum_{\nu=1}^{n} f(x_{\nu n}) \frac{1-x^2}{1-x_{\nu n}^2} \left(\frac{P_n(x)}{P'_n(x_{\nu n})(x-x_{\nu n})} \right)^2 + f(1) P_n(x)^2 \tag{72}$$

converges uniformly to $f(x)$ for $-1+\delta \leqq x \leqq 1$ $(0 < \delta < 2)$ if $n \to +\infty$.

By the aid of (63) it may be written

$$f(x) = \sum_{\nu=1}^{n} f(x) \frac{1-x^2}{1-x_{\nu n}^2} \left(\frac{P_n(x)}{P'_n(x_{\nu n})(x-x_{\nu n})} \right)^2 + f(x) P_n(x)^2 \tag{73}$$

and by (72), (73) follows

$$f(x) - q_n^*(x) = \tag{74}$$
$$= \sum_{\nu=1}^{n} [f(x) - f(x_{\nu n})] \frac{1-x^2}{1-x_{\nu n}^2} \left(\frac{P_n(x)}{P'_n(x_{\nu n})(x-x_{\nu n})} \right)^2 + [f(x) - f(1)] P_n(x)^2.$$

But

$$[f(x) - f(1)] P_n(x)^2 \to 0 \tag{75}$$

uniformly for $-1+\delta \leqq x \leqq 1$ $(0 < \delta < 2)$. This is an easy consequence of the fact that the first factor [...] vanishes for $x = 1$ and is continuous for $-1 \leqq x \leqq 1$, while $P_n(x) \to 0$ uniformly in each internal subinterval $-1+\sigma \leqq x \leqq 1-\sigma$ $(0 < \sigma < 1)$ and is bounded in the whole closed interval. Hence it is sufficient to prove that

$$\sum_{\nu=1}^{n} [f(x) - f(x_{\nu n})] \frac{1-x^2}{1-x_{\nu n}^2} \left(\frac{P_n(x)}{P'_n(x_{\nu n})(x-x_{\nu n})} \right)^2 \to 0 \tag{76}$$

uniformly[13]) for $-1 \leqq x \leqq 1$ if $n \to +\infty$.

[13]) This fact was already shown by E. EGERVÁRY and P. TURÁN ([1], 264—265) and later a little differently by the author ([7], 422—423). For the sake of completeness we repeat the proof in the text.

By reason of the theorem of Weierstrass there exists a positive number M for which

$$|f(x)| \leqq M \quad (-1 \leqq x \leqq 1). \tag{77}$$

Further the continuous function $f(x)$ is also uniformly continuous in the interval $-1 \leqq x \leqq 1$, therefore the positive number ε having been chosen as small as we please, there exists a positive number τ such that

$$|f(x')-f(x'')| \leqq \varepsilon \text{ when } |x'-x''| < \tau. \tag{78}$$

By the aid of (78), (63) and the non-negativity of the fundamental polynomials of the first kind under (61), we have for fixed x

$$\left| \sum_{|x_{\nu n}-x| \leqq \tau} [f(x)-f(x_{\nu n})] \frac{1-x^2}{1-x_{\nu n}^2} \left(\frac{P_n(x)}{P_n'(x_{\nu n})(x-x_{\nu n})} \right)^2 \right| \leqq \tag{79}$$

$$\leqq \varepsilon \sum_{|x_{\nu n}-x| \leqq \tau}' \frac{1-x^2}{1-x_{\nu n}^2} \left(\frac{P_n(x)}{P_n'(x_{\nu n})(x-x_{\nu n})} \right)^2 \leqq \varepsilon.$$

On the other hand, by (77) and (68)

$$\left| \sum_{|x_{\nu n}-x| > \tau} [f(x)-f(x_{\nu n})] \frac{1-x^2}{1-x_{\nu n}^2} \left(\frac{P_n(x)}{P_n'(x_{\nu n})(x-x_{\nu n})} \right)^2 \right| \leqq \tag{80}$$

$$\leqq \frac{2M}{\tau^2}(1-x^2)P_n(x)^2 \sum_{|x_{\nu n}-x| > \tau} \frac{1}{(1-x_{\nu n}^2)P_n'(x_{\nu n})^2} \leqq \frac{2M}{\tau^2}(1-x^2)P_n(x)^2$$

holds. Taking (69) into account again, we obtain by (80) at once

$$\left| \sum_{|x_{\nu n}-x| > \tau} [f(x)-f(x_{\nu n})] \frac{1-x^2}{1-x_{\nu n}^2} \left(\frac{P_n(x)}{P_n'(x_{\nu n})(x-x_{\nu n})} \right)^2 \right| \leqq \varepsilon \tag{81}$$

if n is large enough, i. e. $n \geqq N'$, say. Finally, by (79), (81) there follows

$$\left| \sum_{\nu=1}^{n} [f(x)-f(x_{\nu n})] \frac{1-x^2}{1-x_{\nu n}^2} \left(\frac{P_n(x)}{P_n'(x_{\nu n})(x-x_{\nu n})} \right)^2 \right| \leqq 2\varepsilon$$

for $-1 \leqq x \leqq 1$ if $n \leqq N'$. Thus, (76) is established, i. e. the almost-step parabola (72) converges uniformly to $f(x)$, for $-1+\delta \leqq x \leqq 1$ if $n \to +\infty$.

Now, since the generalized almost-step parabola $S_n(x)$ is according to (38) the sum of $q_n^*(x)$ and $Q_n^*(x)$, that is by (65)

$$S_n(x) = q_n^*(x) + \sum_{\nu=1}^{n} y_{\nu n}' \frac{1-x}{1-x_{\nu n}} (x-x_{\nu n}) \left(\frac{P_n(x)}{P_n'(x_{\nu n})(x-x_{\nu n})} \right)^2,$$

for a proof of our theorem there remains to be shown that

$$\sum_{\nu=1}^{n} y_{\nu n}' \frac{1-x}{1-x_{\nu n}} (x-x_{\nu n}) \left(\frac{P_n(x)}{P_n'(x_{\nu n})(x-x_{\nu n})} \right)^2 \to 0 \tag{82}$$

uniformly for $-1+\delta \leqq x \leqq 1$. This may be seen as follows.

Since

$$y'_{\nu n}\frac{1-x}{1-x_{\nu n}}=y'_{\nu n}\frac{1+x_{\nu n}}{1+x}\,\frac{1-x^2}{1-x_{\nu n}^2},$$

by reason of the assumption under (71) and the obvious inequality

$$\left|\frac{1+x_{\nu n}}{1+x}\right|<\frac{2}{\delta}\quad\text{for}\quad -1+\delta\leqq x\leqq 1,$$

we have in the same interval

$$\left|\sum_{\nu=1}^{n} y'_{\nu n}\frac{1-x}{1-x_{\nu n}}(x-x_{\nu n})\left(\frac{P_n(x)}{P'_n(x_{\nu n})(x-x_{\nu n})}\right)^2\right|\leqq$$

$$\leqq\frac{2\Delta}{\delta}\sum_{\nu=1}^{n}\frac{1-x^2}{1-x_{\nu n}^2}|x-x_{\nu n}|\left(\frac{P_n(x)}{P'_n(x_{\nu n})(x-x_{\nu n})}\right)^2.$$

Thus, from Lemma II under (66) results (82), with which the proof of the theorem is complete.

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