On the summability of the Fourier series of L^2 integrable functions, I.

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§ 1. Introduction

Let π_n be the class of trigonometrical polynomials of order n and

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx) \tag{1.1}$$

an element of the class π_n. The partial sums of $f(x)$ of order l will be denoted by $s_l(x)$ or $s_l(x;f)$. (Of course $s_n(x)=f(x)$.)

Let A_n, B_n, C_n be the least quantities for which the inequalities

$$\int_0^{2\pi}\max_{0\le l\le n}\{|s_l(x)|^2\}dx\le A_n, \tag{1.2}$$

$$\int_0^{2\pi}\max_{0\le l\le n}\{|s_l(x)|\}dx\le B_n, \tag{1.3}$$

$$\max_{n(x)}\left\{\left|\int_0^{2\pi}s_{n(x)}(x)\,dx\right|\right\}\le C_n, \tag{1.4}$$

respectively, hold for any $f\in\pi_n$ of unit square norm:

$$\int_0^{2\pi}|f(x)|^2\,dx=1.$$

Here $n(x)$ may be any measurable function, assuming on $(0, 2\pi)$ only the values $0, 1, \ldots, n$.

These quantities play an important role in investigations into the convergence of Fourier series of L^2 integrable functions. If any of the sets $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ would be bounded, it would follow that the Fourier series of every L^2 integrable function would converge almost everywhere. If, moreover $A_n=O(1)$ would be true, then the envelope of the partial sums of the Fourier series of any L^2 integrable function would be itself L^2 integrable [2].

Actually, however, considerably less is proved, namely that

$$A_n=O(\log n),\quad B_n=O(\sqrt{\log n}),\quad C_n=O(\sqrt{\log n}) \tag{1.5}$$

and it is not known whether these estimations can be improved or not beyond exchanging the symbol O into o.

In this paper there will be given another proof of formulae (1. 5) by investigating the behaviour of the following "finite" analogues of A_n, B_n, C_n, i. e. of the least quantities $A_n^{(m)}$, $B_n^{(m)}$, $C_n^{(m)}$ for which

$$(1.6)\qquad \frac{1}{m}\sum_{r=1}^{m}\max_{l=0,1,\ldots,n}\left\{\left|s_l\left(\frac{2\pi}{m}r\right)\right|^2\right\}\leqq A_n^{(m)}\left\{\frac{a_0\bar{a}_0}{2}+\sum_{k=1}^{n}(a_k\bar{a}_k+b_k\bar{b}_k)\right\},$$

$$(1.7)\qquad \frac{1}{m}\sum_{r=1}^{m}\max_{l=0,1,\ldots,n}\left\{\left|s_l\left(\frac{2\pi}{m}r\right)\right|\right\}\leqq B_n^{(m)}\left\{\frac{a_0\bar{a}_0}{2}+\sum_{k=1}^{n}(a_k\bar{a}_k+b_k\bar{b}_k)\right\}^{\frac{1}{2}},$$

$$(1.8)\qquad \max_{\substack{n_r=0,1,\ldots,n\\ r=1,2,\ldots,m}}\frac{1}{m}\left|\sum_{r=1}^{m}s_{n_r}\left(\frac{2\pi}{m}r\right)\right|\leqq C_n^{(m)}\left\{\frac{a_0\bar{a}_0}{2}+\sum_{k=1}^{n}(a_k\bar{a}_k+b_k\bar{b}_k)\right\}^{\frac{1}{2}}$$

hold for any $f\in\pi_n$.

We shall prove

Theorem 1. *Between corresponding pairs of the six quantities* $A_n, B_n, C_n, A_n^{(m)}$, $B_n^{(m)}, C_n^{(m)}$ *the inequalities*

$$(1.9)\qquad A_n\leqq 2A_n^{(m)},\quad B_n\leqq 2\sqrt{\pi}\,B_n^{(m)},\quad C_n\leqq 2\sqrt{\pi}\,C_n^{(m)}$$

hold.

Theorem 2. *If* $c_1n<m<c_2n$ *where* c_1 *and* c_2 *are independent of* n, *e. g.*

$$(1.10)\qquad m=n,\quad m=2n,\quad m=2n+1$$

then

$$(1.11)\qquad A_n^{(m)}=O(\log n),\quad B_n^{(m)}=O(\sqrt{\log n}),\quad C_n^{(m)}=O(\sqrt{\log n}).$$

Formulae (1. 11) are equivalent to the estimations (1. 5) since each of these six estimations can be deduced from the corresponding estimations of the other set. Yet it seems worth while to prove the asymptotic behaviour of $A_n^{(m)}$, $B_n^{(m)}$ and $C_n^{(m)}$ without having recourse to the asymptotics of A_n, B_n or C_n and so to furnish another proof of the known asymptotical behaviour of these latter quantities.

Moreover the quantities $A_n^{(m)}$, $B_n^{(m)}$ and $C_n^{(m)}$ are in some respects more easily treatable than the A_n's, B_n's and C_n's. So if one wants to compute numerically the first few terms of the sequences $A_1, A_2, \ldots, B_1, B_2, \ldots$, or $C_1, C_2, \ldots$ one faces nearly unsurmountable difficulties[1]). In contrast to this one can find in the cases

[1]) Even in the simplest case $n=1$ the calculations yield the not very informative result

$$A_1=1+\frac{\sqrt{\pi^2+32}}{\pi\left(\frac{\pi}{4}c+1\right)}\quad\text{with}\quad c=\frac{\pi+\sqrt{\pi^2+32}}{4}.$$

The corresponding extremal functions in (1.2) are the polynomials $a\{1+c\cos(x-x_0)\}$ where x_0 is an arbitrary constant.

(1. 10) for small n's and m's by more or less straightforward calculations relatively simple numerical data of which the following ones seem to be worth mentioning:

$$A_n^{(2n)}=1,\ B_n^{(2n)}=1 \quad \text{for} \quad n=1,2,3; \tag{1. 12}$$

$$A_n^{(n)}=\frac{3}{2},\ B_n^{(n)}=\sqrt{\frac{3}{2}} \quad \text{for} \quad n\leqq 6,\ C_n^{(n)}=\sqrt{\frac{3}{2}} \quad \text{for} \quad n\leqq 10 \quad \text{and} \quad n\leqq 16; \tag{1. 13}$$

$$A_n^{(2n+1)}<1,\ B_n^{(2n+1)}<1,\ C_n^{(2n+1)}<1, \quad \text{for} \quad n\leqq 4; \tag{1. 14}$$

$$A_n^{(m)}<1,\ B_n^{(m)}<1,\ C_n^{(m)}<1 \quad \text{for } m \text{ even and } >2n,\ n=1,2,3. \tag{1. 15}$$

In the cases corresponding to (1. 12) and (1. 13) one can discuss the cases of equality too, in (1. 6) through (1. 8). We have for $m=2n$ $(n=1,2,3)$ that equality occurs in (1. 6) and (1. 7) with the value $A_n^{(2n)}=B_n^{(2n)}=1$ if and only if

$$f(x)=a_n\cos nx \qquad (n=1,2,3).$$

In the case $m=n$ corresponding to (1. 13) equality occurs both in (1. 6) and (1. 7) for $n\leqq 6$ and in (1. 8) for $n\leqq 10$ and $n=16$ (these are the only cases checked) if and only if

$$f(x)=a(\tfrac{1}{2}+\cos nx).$$

The results of these computations point to the estimations (1. 11) [and hence (1. 5)] being not the best ones, at least in the cases (1. 10).

But more than this can be said. A survey of the numerical values of many $A_n^{(m)}$'s, $B_n^{(m)}$'s and $C_n^{(m)}$'s led to the result that in none of the cases calculated did the values of the quantities $A_n^{(m)}$, $[B_n^{(m)}]^2$ and $[C_n^{(m)}]^2$ exceed $(n/m)+1$. Particularly simple numerical values were obtained when m was a divisor of n. In these cases the extremal functions in (1.6) through (1.8) could be found too, i. e. the functions for which equality is attained in these formulas. The numerical data obtained suggest the following

Conjectures. 1. *If* $m=2n$, *then* $A_n^{(2n)}=B_n^{(2n)}=1$ *and the only extremal functions are both in* (1. 6) *and in* (1. 7) $a_n\cos nx$.

2. *If* $m|n$ *(including the case* $n=0$ *when of course* m *may be any natural number) then*

$$A_n^{(m)}=(B_n^{(m)})^2=(C_n^{(m)})^2=\frac{n}{m}+\frac{1}{2}$$

and the only extremal functions are in any of the inequalities (1. 6), (1. 7), (1. 8) *the polynomials*

$$\tfrac{1}{2}+\cos mx+\cos 2mx+\cos 3mx+\ldots+\cos nx \tag{1.16}$$

apart from a constant factor[2]).

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[2]) It should be remarked that by using Lemmas 1 and 2 the checking or disproof of these conjectures for not too large values of n and m can be mechanized and programmed for a computing machine.

Remark added on proof, September 4, 1964. In the meantime it could be shown by using the computer FINAC of the Istituto Nazionale per le Applicazioni del Calcolo, Rome, that if $m|n$

If any of these conjectures would be true this would imply in view of Theorem I the boundedness of the sequences $\{A_n\}$, $\{B_n\}$, $\{C_n\}$.

The conjectured extremal functions $\hat{f}$ have the properties

$$\left|s_l\left(\frac{2\pi}{m}r;\hat{f}\right)\right|<\hat{f}\left(\frac{2\pi}{m}r\right) \quad (l=0, 1, \dots, n-1;\ r=1, 2, \dots, m)$$

and

$$\left|\hat{f}\left(\frac{2\pi r}{m}\right)\right|=|\hat{f}(0)| \quad (r=1, 2, \dots, m)$$

and if Conjecture 2 is true, the "best" set $\{n_r\}$ in (1. 8) i. e. the only case for which equality can occur there for a certain trigonometric polynomial is $n_1=n_2=\dots$ $\dots=n_m=n$.

It should be noted that

$$A_n^{(2n)}\geqq 1,\ A_n^{(m)}\geqq\frac{n}{m}+\frac{1}{2},\ \ B_n^{(m)}\geqq\sqrt{\frac{n}{m}+\frac{1}{2}},\ \ C_n^{(m)}\geqq\sqrt{\frac{n}{m}+\frac{1}{2}} \quad (m|n)$$

is readily seen to be true by inserting into the inequalities (1. 6) through (1. 8) the functions indicated as extremal functions in the conjectures.

§ 2. Proof of Theorem 1

It is easily seen that if

$$A_n^{(m)}(x_0)=\max_{\substack{f\in\pi_n\\ f\not\equiv 0}}\frac{\frac{1}{m}\sum_{r=1}^{m}\max_{l}\left|s_l\left(x_0+\frac{2\pi}{m}r\right)\right|^2}{\frac{1}{\pi}\int_0^{2\pi}|f(x)|^2dx} \tag{2. 1}$$

then

$$A_n^{(m)}(x_0)=A_n^{(m)}. \tag{2. 2}$$

For if $f_{x_0}=f_{x_0}(x)=f(x+x_0)$ then

$$\frac{\frac{1}{m}\sum_{r=1}^{m}\max_{l}\left|s_l\left(x_0+\frac{2\pi}{m}r;\ f\right)\right|^2}{\frac{1}{\pi}\int_0^{2\pi}|f(x)|^2dx}=\frac{\frac{1}{m}\sum_{r=1}^{m}\max_{l}\left|s_l\left(\frac{2\pi}{m}r;\ f_{x_0}\right)\right|^2}{\frac{1}{\pi}\int_0^{2\pi}|f_{x_0}(x)|^2dx}.$$

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and $m\leqq 53$, then $C_n^{(m)}=\sqrt{\frac{n}{m}+\frac{1}{2}}$ further equality stands in (1.8) only if $n_1=n_2=\dots=n_m=n$ and $f(x)$ is of the form (1.16). Cfr. Report n. 1439 of the INAC, December 30, 1963, further Part II of this paper and a note of A. GHIZZETTI: On the evaluation of quantities concerning the almost everywhere convergence of the Fourier series of L^2 integrable functions. Quaderni dell' INAC, in the press.)

I express my sincere thanks to the Istituto Nazionale del Calcolo and particularly to its director, A. GHIZZETTI, for their kindness of having performed the numerical calculations in connection with this conjecture.

Further if $f^*(x)$ is a trigonometric polynomial with partial sums $s_0^*, s_1^*(x), \ldots$ having the extremal property that

$$\int_0^{2\pi} \max_l |s_l^*(x)|^2 dx = A_n \int_0^{2\pi} |f^*(x)|^2 dx$$

then

$$A_n = \frac{\sum_{r=0}^{m-1} \int_0^{2\pi/m} \max_l \left| s_l^*\left(x + \frac{2\pi}{m} r\right)\right|^2 dx}{\int_0^{2\pi} |f^*(x)|^2 dx} = \int_0^{2\pi/m} \frac{m}{\pi} \frac{\frac{1}{m} \sum_{r=1}^{m} \max_l \left| s_l^*\left(x + \frac{2\pi}{m} r\right)\right|^2}{\frac{1}{\pi}\int_0^{2\pi} |f^*(t)|^2 dt} dx \leqq$$

$$\leqq \int_0^{2\pi/m} \frac{m}{\pi} A_n^{(m)}(x)\, dx = \int_0^{2\pi/m} \frac{m}{\pi} A_n^{(m)} dx = 2A_n^{(m)}.$$

A similar argument shows the validity of the other inequalities (1. 9).

§ 3. Some more general problems

Let $x_1, x_2, \ldots, x_m$ be a set of strictly increasing real numbers $(x_0 > 0, x_m \leqq 2\pi)$ and let $n_1, n_2, \ldots, n_m$ be a set of nonnegative integers none of which exceeds n. We consider the following problems.

PROBLEM 1a. *To find the least constant $A_n(x_1, x_2, \ldots, x_m)$ for which*

$$(3.1) \qquad \frac{1}{m} \sum_{r=1}^{m} \max_{l=0,1,\ldots,n} |s_l(x_r; f)|^2 \leqq A_n(x_1, x_2, \ldots, x_m) \left\{ \frac{a_0 \bar{a}_0}{2} + \sum_{k+1}^{n} (a_k \bar{a}_k + b_k \bar{b}_k) \right\}$$

holds, if $f \in \pi_n$.

PROBLEM 1b. *To find the least constant $\Lambda_n(x_1, \ldots, x_m; n_1, \ldots, n_m)$ for which*

$$(3.2) \qquad \sum_{r=1}^{m} |s_{n_r}(x_r; f)|^2 \leqq \Lambda_n(x_1, \ldots, x_m;\ n_1, \ldots, n_m) \left\{ \frac{a_0 \bar{a}_0}{2} + \sum_{k=1}^{n} (a_k \bar{a}_k + b_k \bar{b}_k) \right\}$$

holds, if $f \in \pi_n$.

The solution of Problem 1a depends on the solution of Problem 1b since

$$(3.3) \qquad A_n(x_1, \ldots, x_m) = \frac{1}{m} \max_{\substack{n_r = 0, 1, \ldots, n \\ r = 1, 2, \ldots, m}} \Lambda_n(x_1, \ldots, x_m;\ n_1, \ldots, n_m).$$

Indeed if $f^*(x)$ is a polynomial for which (3. 1) holds with sign of equality and n_r^* is one of those indices for which

$$\max_l |s_l(x_r; f^*)|^2 = |s_{n_r^*}(x_r; f^*)|^2,$$

then

$$A_n(x_1, \dots, x_m) \leqq \frac{1}{m} \Lambda_n(x_1, \dots, x_m;\ n_1^*, \dots, n_m^*). \tag{3. 4}$$

Again if f^{**} is a polynomial for which (3. 2) holds with sign of equality then one has by inserting f^{**} into (3. 1) and comparing (3. 1) and (3. 2) that

$$\frac{1}{m} \Lambda_n(x_1, \dots, x_m;\ n_1, \dots, n_m) \leqq A_n(x_1, \dots, x_m). \tag{3. 5}$$

The last two inequalities are equivalent to (3. 3), which reduces the solution of Problem 1a to that of Problem 1b and to the finding of the maximum of a finite number of quantities.

Now let

$$D_l(x) = \frac{1}{2} + \cos x + \dots + \cos lx = \frac{1}{2} \sum_{\nu=-l}^{l} e^{i\nu x} \tag{3. 6}$$

be Dirichlet's kernel. We state the following

Lemma 1. *The solution $\Lambda = \Lambda_n(x_1, \dots x_m; n_1, \dots, n_m)$ of Problem 1b is equal to the largest eigenvalue of the positive semi-definite and real symmetric matrix $D = [d_{pq}]_{p,q=1}^{m}$ with elements*

$$d_{pq} = D_{\min(n_p,\, n_q)}(x_p - x_q). \tag{3. 7}$$

We introduce the notation

$$\varepsilon_{g,\,h} = \begin{cases} 1 & \text{if} \quad g \geqq |h|, \\ 0 & \text{if} \quad g < |h| \end{cases} \tag{3. 8}$$

and represent the polynomial $f(x)$ in the form $\sum \xi_\nu e^{i\nu x}$ where

$$\xi_0 = \frac{a_0}{2},\ \xi_\nu = \frac{1}{2}(a_\nu - ib_\nu) \quad \text{if} \quad \nu > 0 \quad \text{and} \quad \xi_\nu = \frac{1}{2}(a_\nu + ib_\nu) \quad \text{if} \quad \nu < 0$$

so that

$$2 \sum_{\nu=-n}^{n} \xi_\nu \bar{\xi}_\nu = \frac{a_0 \bar{a}_0}{2} + \sum_{\nu=1}^{n} (a_\nu \bar{a}_\nu + b_\nu \bar{b}_\nu). \tag{3. 9}$$

In view of the representation

$$s_{n_r}(x_r; f) = \sum_{\nu=-n}^{n} \xi_\nu \varepsilon_{n_r,\,\nu} e^{i\nu x_r} \tag{3. 10}$$

Problem 1b can be reformulated as follows: to find the least upper bound Λ of the quotient

$$\frac{\sum_{r=1}^{m} |s_{n_r}(x_r)|^2}{\frac{a_0 \bar{a}_0}{2} + \sum_{k=1}^{n} (a_k \bar{a}_k + b_k \bar{b}_k)} = \frac{\sum_{\nu=-n}^{n} \sum_{\mu=-n}^{n} c_{\nu\mu} \xi_\nu \bar{\xi}_\nu}{\sum_{\nu=-n}^{n} \xi_\nu \bar{\xi}_\nu}$$

$(\Sigma \xi_\nu \bar{\xi}_\nu \neq 0)$ where

$$c_{\nu\mu} = \frac{1}{2} \sum_{r=1}^{m} \varepsilon_{n_r, \nu} e^{i\nu x_r} \cdot \varepsilon_{n_r, \mu} e^{-i\mu x_r}.$$

Now this is a classical problem in the theory of quadratic forms, for both the numerator and the denominator are quadratic forms of the variables ξ_ν. Since $c_{\nu\mu} = \bar{c}_{\mu\nu}$, both of these forms are positive definite or semi-definite Hermitian forms and it is well known that Λ is equal to the largest eigenvalue of the Hermitian matrix $C = [c_{\nu\mu}]_{\nu, \mu = -n}^{n}$.

We introduce now the $2n+1$ by m matrix $G = [g_{\nu p}]$ $(-n \leqq \nu \leqq n, 1 \leqq p \leqq m)$ where $g_{\nu p} = 2^{-1/2} \varepsilon_{n_p, \nu} e^{i\nu x_p}$. If $m \leqq 2n+1$, we add to the matrix G $2n+1-m$ columns of zeros. So we get the quadratic matrix F:

$$F = \frac{1}{\sqrt{2}} \begin{bmatrix} \varepsilon_{n_1, -n} e^{-inx_1} & \dots & \varepsilon_{n_m, -n} e^{-inx_m} & 0 \dots 0 \\ \varepsilon_{n_1, -n+1} e^{-i(n-1)x_1} & \dots & \varepsilon_{n_m, -n+1} e^{-i(n-1)x_m} & 0 \dots 0 \\ \vdots & & & \vdots \\ \varepsilon_{n_1, n} e^{inx_1} & \dots & \varepsilon_{n_m, n} e^{inx_m} & 0 \dots 0 \end{bmatrix}.$$

The conjugate transposed matrix of F will be denoted, as usual, by F^*.

It is easily seen that $C = FF^*$.

We refer now to a theorem of Matrix Algebra according to which the spectra of the matrices $C = FF^*$ and $\Delta = F^*F$ are the same [3]. But $\Delta = [\delta_{pq}]$ $(p, q = 1, 2, \dots, 2n+1)$ is of the form

$$\Delta = \begin{bmatrix} d_{11} & d_{12} \dots d_{1m} & 0 \dots 0 \\ d_{21} & d_{22} \dots d_{2m} & 0 \quad 0 \\ \vdots & & \vdots \\ d_{m1} & d_{m2} \dots d_{mm} & 0 \quad 0 \\ 0 & 0 \quad\quad 0 & 0 \quad 0 \\ \vdots & & \vdots \\ 0 & 0 \dots 0 & \underbrace{0 \quad 0}_{2n+1-m \text{ columns}} \end{bmatrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} 2n+1-m \text{ rows}$$

where the quantities d_{pq} are defined by (3. 7).

Indeed, it is easy to see that δ_{pq} vanishes whenever $p > m$ or $q > m$. In the remaining cases in view of (3. 6) through (3. 8)

$$\delta_{pq} = \frac{1}{2} \sum_{\nu=-n}^{n} \varepsilon_{n_p, \nu} e^{i\nu x_p} \cdot \varepsilon_{n_q, \nu} e^{-i\nu x_q}$$

$$= \frac{1}{2} \sum_{|\nu| \leqq \min(n_p, n_q)} e^{i\nu(x_p - x_q)} = D_{\min(n_p, n_q)}(x_p - x_q) = d_{pq}.$$

Finally the spectrum of Δ differs from that of D only by a number of zeros so that their largest eigenvalues coincide.

We mention that in the case $m>2n+1$ a similar argument shows again the validity of the lemma. The matrix F is formed in this case by adding to the matrix G $m-2n-1$ rows of zeros. Δ is again equal to F^*F.

§ 4. Some more general problems (continued)

Let $z_1, z_2, \ldots, z_m$ be any complex numbers, $x_1, \ldots, x_m$; $n_1, \ldots, n_m$ the quantities defined at the beginning of the foregoing section. The sets $\{z_r\}, \{x_r\}, \{n_r\}$ $(r=1, 2, \ldots, m)$ will be denoted briefly by $\mathbf{z}, \mathbf{x}$ and $\mathbf{n}$, respectively.

Problems analogous to those dealt with in § 3 are

Problem 2a. *To find the least constant* $C_n^{(m)}(\mathbf{z}, \mathbf{x})$ *for which*

$$\max_{\substack{n_r=0,1,\ldots,n\\ r=1,2,\ldots,m}} \frac{1}{m}\left|\sum_{r=1}^{m} z_r s_{n_r}(x_r; f)\right| \leqq C_n^{(m)}(\mathbf{z}, \mathbf{x})\left\{\frac{|a_0|^2}{2}+\sum_{k=1}^{n}(a_k\bar{a}_k+b_k\bar{b}_k)\right\}^{\frac{1}{2}} \tag{4. 1}$$

holds, if $f\in\pi_n$ *and*

Problem 2b. *To find the least constant* $\lambda_n=\lambda_n(\mathbf{z}, \mathbf{x}, \mathbf{n})$ *for which*

$$\left|\sum_{r=1}^{m} z_r s_{n_r}(x_r; f)\right| \leqq \lambda_n(\mathbf{z}, \mathbf{x}, \mathbf{n})\left\{\frac{a_0\bar{a}_0}{2}+\sum_{k=1}^{n}(a_k\bar{a}_k+b_k\bar{b}_k)\right\}^{\frac{1}{2}} \tag{4. 2}$$

holds if $f\in\pi_n$.

Just as in § 3 one can show that the solution of Problem 2a reduces to that of Problem 2b by virtue of the relation

$$C_n^{(m)}(\mathbf{z}, \mathbf{x}) = \frac{1}{m}\max_{\substack{n_r=0,1,\ldots,n\\ r=1,2,\ldots,m}} \lambda_n(\mathbf{z}, \mathbf{x}, \mathbf{n}). \tag{4. 3}$$

Concerning the quantity λ_n we have

Lemma 2. *The solution* λ_n *of Problem 2b is equal to*

$$\left\{\sum_{p,q=1}^{m} d_{pq} z_p \bar{z}_q\right\}^{\frac{1}{2}}$$

with the notations of (3. 7) *and* (3. 6).

Indeed, using in turn (3. 10), Cauchy's inequality and (3. 9) we have

$$\left|\sum_{r=1}^{m} z_r s_{n_r}(x_r; f)\right|^2 = \left|\sum_{\nu=-n}^{n}\xi_\nu \sum_{r=1}^{m}\varepsilon_{n_r,\nu} z_r e^{i\nu x_r}\right|^2 \leqq$$

$$\leqq \sum_{\nu=-n}^{n}|\xi_\nu|^2\cdot\sum_{\nu=-n}^{n}\left(\sum_{p=1}^{m}\varepsilon_{n_p,\nu} z_p e^{i\nu x_p}\sum_{q=1}^{m}\varepsilon_{n_q,\nu}\bar{z}_q e^{-i\nu x_q}\right)=$$

$$=\sum_{\nu}|\xi_\nu|^2\cdot\sum_{p,q=1}^{m} z_p\bar{z}_q\sum_{\nu}\varepsilon_{n_p,\nu}\varepsilon_{n_q,\nu}e^{i\nu(x_p-x_q)}=\sum_{\nu}|\xi_\nu|^2\cdot\sum_{p,q} z_p\bar{z}_q\sum\varepsilon_{\min(n_p,n_q),\nu}e^{i\nu(x_p-x_q)}=$$

$$=\frac{1}{2}\left\{\frac{a_0\bar{a}_0}{2}+\sum_{k=1}^{n}(a_k\bar{a}_k+b_k\bar{b}_k)\right\}\cdot\sum_{p,q}2d_{pq}z_p\bar{z}_q$$

and equality holds if and only if

$$\xi_\nu = \gamma \sum_{r=1}^{m} \varepsilon_{n_r, \nu} e^{-i\nu x_r} \bar{z}_r \qquad (\nu = -n, -n+1, \ldots, n)$$

where γ is an arbitrary constant.

§ 5. Proof of Theorem 2

To prove the assertion of Theorem 2 with respect to $\Lambda_n^{(m)}$ we specify Problem 1a and 1b (§ 3) by putting

$$x_r = \frac{2\pi}{m} r \qquad (r = 1, 2, \ldots, m)$$

and estimate the largest eigenvalue Λ of the corresponding matrix D with elements

$$d_{pq} = D_{\min(n_p, n_q)} \left(\frac{2\pi}{m} (p-q) \right) \qquad (p, q = 1, 2, \ldots, m).$$

We use for this estimation the theorem of GERSHGORIN (e. g. [4]) which states that

$$|d_{pp} - \Lambda| \leqq \sum_{\substack{q=1 \\ q \neq p}}^{m} |d_{pq}| \qquad (p = 1, 2, \ldots, m)$$

for at least one p, or

(5. 1) $$\Lambda_n(x_1, \ldots, x_m; n_1, \ldots, n_m) = \Lambda \leqq |d_{pp}| + \sum_{\substack{q=1 \\ q \neq p}}^{m} |d_{pq}|$$

for at least one p.

Now

$$d_{pp} = D_{n_p}(0) = n_p + \tfrac{1}{2} \leqq n + \tfrac{1}{2}$$

and since

$$\frac{1}{\sin x} < \frac{1}{x} + \frac{1}{\pi - x} \qquad (0 < x < \pi)$$

we have for $p \neq q$

$$|d_{pq}| \leqq \frac{1}{2 \sin \frac{\pi}{m} |p-q|} < \frac{1}{2} \left\{ \frac{1}{\frac{\pi}{m} |p-q|} + \frac{1}{\pi - \frac{\pi}{m} |p-q|} \right\} = \frac{m}{2\pi} \left\{ \frac{1}{|p-q|} + \frac{1}{m - |p-q|} \right\}$$

and further

$$\frac{\Lambda_n(x_1, \ldots, x_m; n_1, \ldots, n_m)}{m} < \frac{n + \frac{1}{2}}{m} + \frac{1}{\pi} \left(\frac{1}{1} + \frac{1}{2} + \cdots \frac{1}{m-1} \right)$$

$$< \frac{2}{c_1} + \frac{1}{\pi} (1 + \log c_2 n) = O(\log n),$$

since $m > c_1 n > \frac{c_1}{2}\left(n+\frac{1}{2}\right)$. Hence by (3. 3) the statement of Theorem 1 concerning $A_n^{(m)}$ follows.

Since for any set of nonnegative integers $n_1, n_2, \dots, n_m$

$$\frac{1}{m}\left|\sum_{r=1}^{m} s_{n_r}(x_r)\right| \leqq \frac{1}{m}\sum_{r=1}^{m}\max_{l}|s_l(x_r)| \leqq \left\{\frac{1}{m}\sum_{r=1}^{m}\max|s_l(x_r)|^2\right\}^{\frac{1}{2}} \tag{5. 2}$$

we have by the definition of $A_n^{(m)}$, $B_n^{(m)}$ and $C_n^{(m)}$ that

$$C_n^{(m)} \leqq B_n^{(m)} \leqq \sqrt{A_n^{(m)}} \tag{5. 3}$$

and so Theorem 2 is fully proved.

There exists another way of proving $C_n^{(m)} = O(\sqrt{\log n})$ independently of the inequalities (5. 3), namely by the use of Lemma 2 in the particular case $x_\nu = 2\pi r/m$, $z_r = 1$. Then we have to estimate from above

$$\max_{\substack{n_r=0,1,\dots,n\\ r=1,2,\dots,m}} \sum_{p,q=1}^{m} D_{\min(n_p,n_q)}\left(\frac{2\pi}{m}(p-q)\right).$$

This can be done by using the upper bound we have found for the right hand side of (5. 1). If we perform this estimation we have again $C_n^{(m)} = O(\sqrt{\log n})$. Incidentally this procedure is nothing else, but the finite version of the reasoning of Kolmogoroff and Seliverstoff [1] in their proof of $C_n = O(\sqrt{\log n})$.

§ 6. The constants $A_n^{(m)}$ and the corresponding extremal functions

In the following we shall use the well known

Lemma 3. *If the elements of the Hermitian matrices $\Gamma' = [\gamma'_{ij}]$ and $\Gamma'' = [\gamma_{ij}]$ satisfy the conditions*

$$\gamma'_{ij} = \gamma''_{ij} \qquad (i \neq j); \qquad \gamma'_{ii} \geqq \gamma''_{ii}$$

for each i and j, then the largest eigenvalue of Γ' is not less than that of the matrix Γ''. If moreover $\gamma'_{ii} - \gamma''_{ii} = \gamma > 0$ $(i = 1, 2, \dots, n)$, then the difference of the two maximal eigenvalues is just γ.

We consider now the problem expressed in (1. 6), that is the finding of $A_n^{(m)}$. In connection with Problem 1a and 1b of Section 3, particularly with (3. 3) in the case $x_r = 2\pi r/m$ we define U as the aggregate of all "maximal" sets of indices $n_1, n_2, \dots, n_m$. This means that if $\{n_r\} \in U$, then and only then

$$A_n^{(m)} = \frac{1}{m} A_n\left(\frac{2\pi}{m}, \frac{4\pi}{m}, \dots, 2\pi;\ n_1, \dots, n_m\right) = \frac{1}{m} A_n(n_1, \dots, n_m).$$

We state

Lemma 4. *U contains a set of indices* $\{n_r^{(0)}\}$ *such that* $\max n_r^{(0)} = n$ *and if the elements of* $\{n_r^{(0)}\}$ *are re-written in a non-increasing order* $n_{\alpha_1}^{(0)}, n_{\alpha_2}^{(0)}, \ldots, n_{\alpha_m}^{(0)}$ *then*

$$(6.1) \qquad n_{\alpha_h}^{(0)} - n_{\alpha_{h+1}}^{(0)} < m \quad \textit{for} \quad h = 1, 2, \ldots, m-1.$$

We remark that this lemma implies that $\min n_r^{(0)} \geqq n - m(m-1)$.

Let us choose an element $\{n_r\}$ of U. If $\max n_r = n_\sigma < n$ then consider the set $\{n_r^{(1)}\}$ defined by

$$n_r^{(1)} = n_r, \quad \text{if} \quad r \neq \sigma, \; n_\sigma^{(n)} = n.$$

If the set $\{n_r^{(1)}\}$ does not fulfil the condition (6.1) given for $\{n_r^{(0)}\}$ then after reordering its elements in a non-increasing order, the sequence of integers

$$(6.2) \qquad n_{\alpha_1}^{(1)}, n_{\alpha_2}^{(1)}, \ldots, n_{\alpha_m}^{(1)}$$

contains g gaps of at least $m-1$ numbers:

$$n_{\alpha_{h_1}}^{(1)} - n_{\alpha_{h_1+1}}^{(1)} \geqq m, \ldots, n_{\alpha_{h_g}}^{(1)} - n_{\alpha_{h_g+1}}^{(1)} \geqq m$$

where $h_1 < h_2 < \ldots < h_g$.

Let us suppose that

$$(\varkappa+1)m \geqq n_{\alpha_{h_g}}^{(1)} - n_{\alpha_{h_g+1}}^{(1)} \geqq \varkappa m$$

where $\varkappa$ is an integer.

Then we construct the sequence $\{n_r^{(2)}\}$ in such a way that its α_{h_g} largest elements coincide with the α_{h_g} largest elements of $\{n_r^{(1)}\}$ whereas the remaining elements are greater by $\varkappa m$ than the corresponding elements of $\{n_r^{(1)}\}$. The set of indices $\{n_r^{(2)}\}$ contains one gap less of the type defined above, than the set $\{n_r^{(1)}\}$.

If one compares the elements of the matrix $D(n_1^{(1)}, \ldots, n_m^{(1)})$ with those of $D(n_1^{(2)}, \ldots, n_m^{(2)})$ [where $D = D(n_1, \ldots, n_m)$ is defined by (3.7)] then one sees that the off-diagonal elements are the same while the diagonal elements of the latter matrix are not less than the corresponding elements of the former one. For if, say, $n_p \leqq n_q$, then

$$(6.3) \qquad D_{\min(n_p+m,\, n_q+m)}(x_p - x_q) - D_{\min(n_p,\, n_q)}(x_p - x_q) =$$

$$= \sum_{\mu=1}^{m} \cos\,(n_p + \mu)\frac{2\pi}{m}(p-q) = \begin{cases} 0 & \text{if} \quad p \neq q, \\ m & \text{if} \quad p = q. \end{cases}$$

Hence by Lemma 3

$$\Lambda_n(n_1^{(1)}, \ldots, n_m^{(1)}) \leqq \Lambda_n(n_1^{(2)}, \ldots, n_m^{(2)}).$$

Repeating the process of reducing the number of gaps g times one arrives at a set of indices $\{n_r^{(g+1)}\}$ which fulfils the conditions imposed on $\{n_r^{(0)}\}$.

Concerning the behaviour of the $A_n^{(m)}$'s as functions of n we can now state

Theorem 3. *$A_{n+m}^{(m)} \geqq A_n^{(m)} + 1$ and there exists a function $\varphi(m)$ of m such that if $n > \varphi(m)$ then $A_{n+m}^{(m)} = A_n^{(m)} + 1$. For such n's the coefficients of any extremal function $f(x)$ in (1. 6) exhibit the periodicity $a_k = a_{k+m}$, $b_k = b_{k+m}$ if $k+m < n - \varphi(m)$; in particular $b_m = b_{2m} = \ldots = 0$.*

In proving the theorem we shall use (3. 3) with $x_r = 2\pi r/m$ and Lemma 3. Let $\{n_r^{(0)}\}$ be the set of indices defined above. Then

$$A_n^{(m)} = \frac{1}{m} \Lambda_n(n_1^{(0)}, \ldots, n_m^{(0)})$$

holds, where $\Lambda_n(n_1^{(0)}, \ldots, n_m^{(0)})$ is the largest eigenvalue of the matrix $D(n_1^{(0)}, \ldots, n_m^{(0)})$.

Then one has by (6. 3)

$$D(n_1^{(0)} + m, \ldots, n_m^{(0)} + m) = D(n_1^{(0)}, \ldots, n_m^{(0)}) + mI, \tag{6. 4}$$

where I is the identity matrix. Hence by Lemma 3 the largest eigenvalue of the left hand matrix in (6. 4) is equal to

$$\Lambda_n(n_1^{(0)}, \ldots, n_m^{(0)}) + m.$$

By (3. 3) we get

$$A_{n+m}^{(m)} \geqq \frac{1}{m} [\Lambda_n(n_1^{(0)}, \ldots, n_m^{(0)}) + m] = A_n^{(m)} + 1. \tag{6. 5}$$

We choose now $\varphi(m) = m(m-1)$ and suppose that $n \geqq m(m-1)$. Then by Lemma 4 the set U' of all maximal indices corresponding to $A_{n+m}^{(m)}$ contains a set $\{n_r'\}$ with the property $\min n_r' \geqq m$:

$$A_{n+m}^{(m)} = \frac{1}{m} \Lambda_{n+m}(n_1', \ldots, n_m').$$

Now

$$D(n_1', \ldots, n_m') = D(n_1' - m, \ldots, n_m' - m) + mI,$$

hence

$$\Lambda_{n+m}(n_1', \ldots, n_m') = \Lambda_n(n_1' - m, \ldots, n_m' - m) + m$$

or by (3. 3)

$$A_n^{(m)} \geqq \frac{1}{m} \Lambda_n(n_1' - m, \ldots, n_m' - m) = \frac{1}{m} \Lambda_{n+m}(n_1', \ldots, n_m') - 1 = A_{n+m}^{(m)} - 1. \tag{6. 6}$$

(6. 5) and (6. 6) yield $A_{n+m}^{(m)} = A_n^{(m)} + 1$ for $n \geqq \varphi(m)$.

Concerning the statement of Theorem 3 about the coefficients of the extremal polynomial (or polynomials) in (1. 6) we first prove that all the coefficients b_k $(m|k)$ of an extremal polynomial must vanish. Indeed, let

$$f^*(x) = \frac{a_0^*}{2} + \sum_1^n (a_k^* \cos kx + b_k^* \sin kx)$$

be an extremal polynomial and

$$f_1(x)=\frac{a_0^*}{2}+\sum_1^n a_k^* \cos kx+\sum_{m\nmid k} b_k^* \sin kx.$$

Then, denoting the quotient

$$\frac{1}{m}\sum_{r=1}^{m}\left|s_{n_r^{(0)}}\left(\frac{2\pi r}{m};f\right)\right|^2 \Big/ \left\{\frac{a_0\bar{a}_0}{2}+\sum_1^n (a_k\bar{a}_k+b_k\bar{b}_k)\right\}$$

by $Q(f)$ where $n_r^{(0)}$ has the same meaning as in Lemma 4 one has by the extremal property

$$A_n^{(m)}=\max_{f\in\pi_n} Q(f)=Q(f^*).$$

Further, since $\sin k\cdot 2\pi r/m=0$ if $m|k$, the numerators in $Q(f^*)$ and $Q(f_1)$ are equal, whereas the denominator of $Q(f^*)$ is not less than that of $Q(f_1)$. This implies

$$A_n^{(m)}=Q(f^*)\leqq Q(f_1).$$

The sign $<$ cannot stand here for this would contradict the extremal property of the polynomial $f^*(x)$ and the sign $=$ stands only if $b_m=b_{2m}=\ldots=b_{[n/m]m}=0$.

Further let us denote by $f_2(x)=a_{2,0}/2+\sum(a_{2,k}\cos kx+b_{2,k}\sin kx)$ the trigonometrical polynomial of order n the coefficients of which are defined as follows.

If $\sigma m\leqq n-\varphi(m)<(\sigma+1)m$, σ an integer, then

$$\left(\sigma+\frac{1}{2}\right)a_{2,0}=\frac{a_0^*}{2}+a_m^*+a_{2m}^*+\ldots+a_{\sigma m}^*;$$

if $0<k<m$, then $a_{2,k}$ and $b_{2,k}$ are the arithmetical means of all those coefficients $a_k^*, a_{k+m}^*, a_{k+2m}^*, \ldots$ and $b_k^*, b_{k+m}^*, b_{k+2m}^*, \ldots$ respectively, the indices of which do not exceed $n-\varphi(m)$;

if $m\leqq k\leqq n-\varphi(m)$, then $a_{2,k}=a_{2,k-m}$, $b_{2,k}=b_{2,k-m}$;

if $k>n-\varphi(m)$ then $a_{2,k}=a_{2,k}^*$, $b_{2,k}=b_{2,k}^*$.

Comparing now the quantities $Q(f)$ and $Q(f_2)$ we see that the numerators are again equal since $\exp\{ik\cdot 2\pi r/m\}=\exp\{i(k+m)\cdot 2\pi r/m\}$. By the inequality between weighted arithmetical and quadratic means the denominator of the latter quotient is less than that of the former, except in the case when $f^*(x)$ fulfils the requirements of Theorem 3. On the other hand this latter case has to occur, otherwise $f^*(x)$ would be no extremal polynomial.

§ 7. The constants $C_n^{(m)}$

Let $\mathbf{n}^*=\{n_1^*, n_2^*, \ldots, n_m^*\}$ be a set of maximal numbers in Problems 2a and 2b i. e. for which

$$C_n^{(m)}(\mathbf{z},\mathbf{x})=\frac{1}{m}\lambda_n(\mathbf{z},\mathbf{x},\mathbf{n}^*).$$

We state the following theorems.

Theorem 4. *There exists a maximal set* $\mathbf{n}^*$ *with the property* $\max_{r=1,2,\dots,m} n_r^* = n$.

Indeed if it were not so, let n_1^* be such that it satisfies the inequalities

$$n > n_1^* \geqq n_r^* \qquad (r = 1, 2, \dots, m)$$

and let the set $\{n, n_2^*, n_3^*, \dots, n_m^*\}$ be denoted by $\tilde{\mathbf{n}}$. Then by Lemma 2

$$[\lambda_n(\mathbf{z}, \mathbf{x}, \tilde{\mathbf{n}})]^2 - [\lambda_n(\mathbf{z}, \mathbf{x}, \mathbf{n}^*)]^2 = D_n(0) z_1 \bar{z}_1 - D_{n_1^*}(0) z_1 \bar{z}_2 = (n - n_1^*) z_1 \bar{z}_1 \geqq 0,$$

a contradiction if $z_1 \neq 0$; if however $z_1 = 0$ then $\tilde{\mathbf{n}}$ is a maximal set.

Theorem 5. *If*

(7. 1) $$z_1 = z_2 = \dots = z_m = 1 \text{ and } x_r = 2\pi r/m \qquad (r = 1, 2, \dots, m)$$

further $\{n_1^*, \dots, n_m^*\}$ *is a maximal set then* $\min n_r^* \equiv 0 \pmod m$.

Supposing the contrary, namely

(7. 2) $$n_1^* = \min n_r^* \not\equiv 0 \pmod m$$

without loss of generality we denote by $\underline{\mathbf{n}}$ the set $\{n_1^* - 1, n_2^*, n_3^*, \dots, n_m^*\}$ and form the difference

$$[\lambda_n(\mathbf{z}, \mathbf{x}, \mathbf{n}^*)]^2 - [\lambda_n(\mathbf{z}, \mathbf{x}, \underline{\mathbf{n}})]^2 =$$

$$= \left\{D_{n_1^*}(0) + 2\sum_{r=1}^{m-1} D_{n_1^*}(x_r)\right\} - \left\{D_{n_1^*-1}(0) + 2\sum_{r=1}^{m-1} D_{n_1^*-1}(x_r)\right\} =$$

$$= 1 + 2\sum_{r=1}^{m-1} \cos n_1^* \frac{2\pi}{m} r = -1 < 0,$$

since $m \nmid n_1^*$. This shows that $\mathbf{n}^*$ cannot be a maximal set unless (7. 2) is not true.

Theorem 6. *If* (7.1) *is fulfilled, moreover* $m|n$, *then no element of a maximal set* $\mathbf{n}^*$ *can be equal to* $n-1$.

Supposing the contrary, say $n_1^* = n-1$, we are again led to a contradiction. Denoting namely again by $\tilde{\mathbf{n}}$ the set $\{n, n_2, \dots, n_m\}$ and by $\alpha_1, \alpha_2, \dots, \alpha_s$ the indices of all those elements of $\mathbf{n}^*$ which are equal to n we have

$$\{\lambda_n(\mathbf{z}, \mathbf{x}, \tilde{\mathbf{n}})\}^2 - \{\lambda_n(\mathbf{z}, \mathbf{x}, \mathbf{n}^*)\}^2 =$$

$$= \left\{D_n(0) + 2\sum_{\sigma=1}^{s} D_n(x_{\alpha_\sigma} - x_1)\right\} - \left\{D_{n-1}(0) + 2\sum_{\sigma=1}^{s} D_{n-1}(x_{\alpha_\sigma} - x_1)\right\}$$

$$= 1 + 2\sum_{\sigma=1}^{s} \cos \frac{n}{m} \cdot 2\pi(\alpha_\sigma - 1) = 1 + 2s > 0.$$

While the foregoing theorems are but feeble contributions supporting the conjecture of the Introduction concerning the maximum of the quantities $\lambda_n(\mathbf{z}, \mathbf{x}, \mathbf{n})$

in the special case (7. 1), the converse problem, namely to find the minimum of the same quantities can be completely solved. This solution is the content of

Theorem 7. *Denoting by* $\mathbf{n}_0$ *the set of* m *numbers* $\{0, 0, \ldots, 0\}$ *we have*

$$\min_{\substack{n_r=0,1,\ldots,n \\ r=1,2,\ldots,m}} \lambda_n(\mathbf{z}, \mathbf{x}, \mathbf{n}) = \lambda_n(\mathbf{z}, \mathbf{x}, \mathbf{n}_0) = \left\{\frac{1}{2} \sum_{p,q=1}^{m} z_p \bar{z}_q\right\}^{\frac{1}{2}}.$$

In the proof we restrict ourselves to the case $m \leqq 2n$. By deleting the middle row and last column of the matrix F of Section 3 we form the $2n$ by $2n$ quadratic matrix F_0 and denote by Δ_0 the matrix $F_0^* F_0$. We have with the notations of Section 3:

$$\Delta_0 = \begin{bmatrix} d_{11}-\frac{1}{2} & d_{12}-\frac{1}{2} & \ldots & d_{1m}-\frac{1}{2} & 0 & \ldots & 0 \\ \vdots & & & & & & \vdots \\ d_{m1}-\frac{1}{2} & d_{m2}-\frac{1}{2} & \ldots & d_{mm}-\frac{1}{2} & 0 & \ldots & 0 \\ 0 & 0 & & 0 & 0 & & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} 2n-m \text{ rows} \end{matrix}$$

$$\underbrace{\qquad\qquad}_{2n-m \text{ columns}}$$

The matrix Δ_0, being the product of a pair of conjugate transposed matrices, is the matrix of a positive semidefinite Hermitian quadratic form. Hence

$$\sum_{p,q=1}^{m} d_{pq} z_p \bar{z}_q \geqq \sum_{p,q=1}^{m} \frac{1}{2} z_p \bar{z}_q \geqq 0$$

which is the assertion of Theorem 7. (Cf. Lemma 2.)

§ 8. The cases of small m's and n's

The calculations leading to the conjectures of the Introduction are of no particular interest. There were used three different methods based either on the definitions of $A_n^{(m)}$, $B_n^{(m)}$, $C_n^{(m)}$, or on Lemmas 1 and 2.

In each case a survey of several and sometimes of a great many particular cases was necessary. So we confine us to give two not very tedious specimens of these computations, namely (a) the discussion of the case $m=4$, n any even number, (b) the calculation of $C_n^{(8)}$ if n is a multiple of 8.

(a) In dealing with $m=4$, n even, first we will verify that

$$A_n^{(4)} = \frac{n}{4} + \frac{1}{2} \qquad (n \text{ even}). \tag{8. 1}$$

Lemma 1 can now be applied conveniently, since, if $p \neq q$, the quantities d_{pq} $(p \neq q)$ occurring there are either $1/2$ or $-1/2$ and $d_{pp} = n_p + 1/2$. We have to show that the eigenvalues of the matrix D are not greater than $n+2$ or if I is the unit matrix of order 4, the eigenvalues of the matrix $\tilde{D} = D - \frac{1}{2} I = [\tilde{d}_{pq}]$ are not greater than $n + \frac{3}{2}$.

Let $y=\{y_1, y_2, y_3, y_4\}$ be any real valued column vector and $y'=\{y_1', \ldots, y_4'\}$ the column vector $\tilde{D}y$. We have to show that $y'^2 \leqq (n+\frac{3}{2})^2 y^2$, or that

$$Q(y_1, \ldots, y_4)=\left(n+\frac{3}{2}\right)^2 y^2-(\tilde{D}y)^2=\sum_{r=1}^{4}\left\{\left(n+\frac{3}{2}\right)^2 y_r^2-y_r'^2\right\}$$

is a positive semidefinite quadratic form.

We have

$$y_1'^2=\left(\sum_{q=1}^{4}\tilde{d}_{1q}y_q\right)^2=n_1^2 y_1^2+2n_1\sum_{q=2}^{4}y_1 d_{1q}y_q+\left(\sum_{q=2}^{4}d_{1q}y_q\right)^2\leqq$$

$$\leqq n_1^2 y_1^2+n_1\sum_{q=2}^{4}|d_{1q}|(y_1^2+y_q^2)+\frac{3}{4}\sum_{q=2}^{4}y_q^2=$$

$$=\left(n_1^2+n_1-\frac{3}{4}\right)y_1^2+\left(\frac{n_1}{2}+\frac{3}{4}\right)(y_1^2+y_2^2+y_3^2+y_4^2).$$

From similar inequalities for the quantities y_2', y_3', y_4' we infer that

$$Q(y_1, \ldots, y_4)\geqq\left(n+\frac{3}{2}\right)^2\sum_r y_r^2-\sum_r\left(n_r^2+n_r-\frac{3}{4}\right)y_r^2-\sum_r\left(\frac{n_r}{2}+\frac{3}{4}\right)\sum_r y^2=$$

$$=\sum_r\left\{(n^2-n_r^2+n-n_r+\frac{4n-\sum n_r}{2}\right\}y_r^2=Q_1(y_1, \ldots, y_4)$$

and $Q_1(y_1, \ldots, y_4)$ is positive definite if $4n-\sum n_r>0$ i. e. it is definite unless

(8. 2) $$n_1=n_2=n_3=n_4=n.$$

So the eigenvalues of the matrices D are all definitely less than $n+2$ unless (8. 2) holds.

It rests to envisage the matrices D in the remaining case (8. 2). There are two types of them:

(8. 3) $$\begin{bmatrix} n+\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & n+\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & n+\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & n+\frac{1}{2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} n+\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & n+\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & n+\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & n+\frac{1}{2} \end{bmatrix}$$

according to whether $n\equiv 0 \pmod 4$ or $n\equiv 2 \pmod 4$. In both cases the eigenvalues are $n+2, n, n, n$ and so (8. 1) is verified.

The extremal functions in (1. 6) satisfy in our case the relations

$$|f(x_r)|^2 \geqq |s_l(x_r; f)|^2 \qquad (l=0, 1, \ldots, n;\ r=1, 2, \ldots, m)$$

otherwise $A_n^{(4)}$ would be definitely less than $\frac{n}{4}+\frac{1}{2}$. To find them we have to find according to (3. 3) the functions for which

(8. 4) $$\frac{1}{4}\sum_{r=1}^{4}|f(x_r)|^2=\left(\frac{n}{4}+\frac{1}{2}\right)\left\{\frac{|a_0|^2}{2}+\sum_{k=1}^{n}(|a_k|^2+|b_k|^2)\right\}$$

holds. (This is formula (3. 2) with sign of equality and with $n_1=n_2=n_3=n_4=n$.) Since the largest eigenvalues of both matrices (8. 3) are simple, there will be according to Section 3 only one polynomial satisfying (8. 4). [3])

It is easy to verify that this is apart from a constant factor the polynomial

$$\frac{1}{2}+\cos 4x+\cos 8x+\cos 12x+\ldots+\cos nx \tag{8. 5a}$$

for $n\equiv 0 \pmod 4$ and

$$\cos 2x+\cos 6x+\cos 10x+\ldots+\cos nx \tag{8. 5b}$$

for $n\equiv 2 \pmod 4$.

The special case $n=2$ yields $A_2^{(4)}=1$ and the corresponding extremal function is $\cos 2x$, as already mentioned in the Introduction.

By (5. 3) it follows that

$$B_n^{(4)}\leqq\sqrt{\frac{n}{4}+\frac{1}{2}},\quad C_n^{(4)}\leqq\sqrt{\frac{n}{4}+\frac{1}{2}}\qquad (n\equiv 0 \bmod 2).$$

Actually the sign of equality holds in the first relation if $n\equiv 0 \pmod 2$ and in the second one if $n\equiv 0 \pmod 4$. This is readily seen by inserting (8. 5a) or (8. 5b) in (1. 7) and (1. 8).

Again the polynomials (8. 5) are the unique extremal functions of both inequalities (1. 7) and (1. 8). For if there would exist another extremal of these inequalities linearly independent of (8. 5), then in view of (5. 2) the relation (1. 6) would hold for the same function with sign of equality, a possibility excluded earlier.

(b) For the determination of $C_{8h}^{(8)}$ (h any natural number) we want to show that if $\mathbf{z}=\{1, 1, \ldots, 1\}$, $x_r=2\pi r/8$ (i. e. $\mathbf{x}=\{\pi/4, 2\pi/4, \ldots, 8\pi/4\}$), $\mathbf{n}=\{n_1, \ldots, n_8\}$ where the n_r's are integers not exceding $n=8h$ and

$$n_1+n_2+\ldots+n_8<8n, \tag{8. 6}$$

finally $\mathbf{n}^*=\{8h, 8h, \ldots, 8h\}$, then

$$\{\lambda_n(\mathbf{z}, \mathbf{x}, \mathbf{n})\}^2<\{\lambda_n(\mathbf{z}, \mathbf{x}, \mathbf{n}^*)\}^2=\left\{\sum_{p,q=1}^{8} d_{pq}\right\}_{\substack{d_{pp}=8h+\frac{1}{2}\\ d_{pq}=\frac{1}{2},\ \text{if}\ p\neq q}} \tag{8. 7}$$

(Cfr. Section 4.)

In case of arbitrary n_r's a survey of every possible numerical value that can be assumed by any of the d_{pq}'s shows that

$$d_{pp}=n_p+\tfrac{1}{2},\ d_{pq}\leqq\tfrac{1}{2}\quad\text{if}\quad p-q\equiv\pm 2,\ \pm 3, 4 \pmod 8 \tag{8. 8}$$

and, if $p-q\equiv\pm 1 \pmod 8$ then

$$d_{pq}=\frac{1}{2}+\frac{1}{\sqrt{2}}\quad\text{if}\quad\min(n_p, n_q)\equiv 1 \text{ or } 2 \pmod 8,\quad d_{pq}\leqq\frac{1}{2}\ \text{otherwise}. \tag{8. 9}$$

[3]) The spectra of C and Δ are the same and the largest eigenvalue of Δ is in our case simple. So there exists only one eigenvector belonging to the largest eigenvalue of C; its coordinates are the complex Fourier coefficients ξ_ν of the extremal polynomial.

This means that of the off-diagonal elements of the matrix in Lemma 1 only those "neighbouring" diagonal elements may exceed 1/2 and these in turn only if one of the neighbouring diagonal elements is of the form $8g+1+1/2$ or $8g+2+1/2$. Conversely each index n_r of the form $8g+1$ or $8g+2$ is responsible for at most 4 off-diagonal elements d_{pq} greather than 1/2. (If e. g. $n_3=2$, then the elements $d_{3,3\pm1}$, $d_{3\pm1,3}$ may exceed 1/2.)

Suppose now, that α elements of the set $\{n_r\}$ are congruent to 1 (mod 8) and β of them are congruent to 2 (mod 8). Then by (8. 8) and (8. 9)

$$\sum_{p,q=1}^{8} d_{pq} \leqq \sum_{p=1}^{8} n_p + 8^2\cdot\frac{1}{2} + 4(\alpha+\beta)\frac{1}{\sqrt{2}} \leqq$$
$$\leqq (8-\alpha-\beta)8h + (\alpha+\beta)[8(h-1)+2] + 32 + 4(\alpha+\beta)\cdot\frac{1}{\sqrt{2}} =$$
$$= 64h + 32 + (\alpha+\beta)\left(-8+2+\frac{4}{\sqrt{2}}\right)$$

hence

$$\sum_{p,q=1}^{8} d_{pq} \leqq 64h+32$$

and equality is attained if and only if $\alpha=\beta=0$ and $n_1=n_2=\ldots=n_8=8h$. Finally by (4. 3) there follows

$$C_{8h}^{(8)} = \sqrt{h+\frac{1}{2}} = \sqrt{\frac{n}{8}+\frac{1}{2}}.$$

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