Kernels of purity in Abelian groups

By CHARLES K. MEGIBBEN (Lubbock, Tex.)

1. Introduction

All groups in this paper are assumed to be additively written abelian groups. We follow the notation and terminology of [1]. In addition, we denote the torsion subgroup of the group G by G_t. Recall that a subgroup H of the group G is neat in G if $H \cap pG = pH$ for all primes p and that H is pure in G if $H \cap nG = nH$ for all integers n. It is well-known (see [1], p. 92) that every subgroup H of an abelian group G is contained in a subgroup minimal among the neat subgroups of G containing H. It is not difficult to see that if K is a neat subgroup of G containing H, then K is minimal neat containing H if and only if $\{k\} \cap H \neq 0$ for all non-zero k in K. By the p-adic topology on the group G, we mean the topology on G obtained by taking as neighborhoods of 0 the subgroups p^nG.

If H is a subgroup of G and M is a subgroup of G which is maximal with respect to $H \cap M = 0$, then M is neat though not necessarily pure in G. Following REID ([5]), we call the subgroup H a center of purity if all such subgroups M are pure in G. IRWIN ([3]) first raised the question of determining all centers of purity in primary groups and PIERCE ([4]), influenced by partial results due to Reid ([5]), settled the question not only for primary groups but for arbitrary abelian groups. The present author, independent of Reid and Pierce's work, also solved the problem posed by Irwin and by an approach that suggested a problem in a certain sense dual (see [1], exercise 41, p. 95) to that of characterizing centers of purity. Imitating Reid, we call a subgroup H of G a kernel of purity if every minimal neat subgroup of G containing H is pure. In Theorem 2, we give a complete characterization of kernels of purity in arbitrary abelian groups. Although the characterization is not complicated, the applications of it which follow may be more enlightening than the characterization itself. For example, we determine those groups all of whose subgroups are kernels of purity.

2. The main theorem

We consider first the primary case.

Theorem 1. *The subgroup H of the p-primary group G is a kernel of purity in G if and only if it satisfies the following condition for all positive integers i:*

(*) *If $p^{i+1}g \in H$, then either $p^ig + z \in H \cap p^iG$ for some $z \in G[p]$ or else $G[p]/H[p] \subseteq p^i(G/H[p])$.*

PROOF. First we establish the necessity of the condition (*). Suppose there is a positive integer i such that $G[p]/H[p]$ is not contained in $p^i(G/H[p])$. Then there exists an $x \in G[p]$ such that $x-h \notin p^iG$ for all $h \in H[p]$. Suppose further that there is a $g \in G$ such that $p^{i+1}g \in H$, but that $p^ig+z \notin H \cap p^iG$ for all $z \in G[p]$. We distinguish two cases: (1) $p^ig+y \in H$ for some $y \in G[p]$ and (2) $p^ig+y \notin H$ for all $y \in G[p]$. Assume then that $p^ig+y \in H$ for some $y \in G[p]$. Observe then that $y-h \notin p^iG$ for all $h \in H[p]$. Indeed if $y-h = p^ig'$ for some $h \in H[p]$, then $p^ig+p^ig' \in H$ which contradicts our assumption that p^ig+z cannot be an element of $H \cap p^iG$ for any $z \in G[p]$. We show that no subgroup K of G containing H and having $H[p]$ as socle can be pure in G. Indeed, suppose K is such a subgroup of G which is pure. Then there is a $k \in K$ such that $p^{i+1}k = p^{i+1}g$. Hence $p^ik-p^ig \in G[p]$ and $y+p^ig-p^ik \in G[p] \cap K = H[p]$, which contradicts the fact that $y-h \notin p^iG$ for all $h \in H[p]$.

Consider now the case where $p^ig+y \notin H$ for all $y \in G[p]$. It follows then that $p^{i+1}g \notin pH$. Set $H' = \{H, p^ig+x\}$. We show that $H'[p] = H[p]$. Indeed if $h+t(p^ig+x) \in G[p]$ with $h \in H$, we first observe that $(t,p) \neq 1$; for otherwise it would follow that $p^{i+1}g \in pH$. But if p divides t, then the element $h+t(p^ig+x)$ is already in H. We need only observe now, as in the preceeding case, that there can be no pure subgroup K of G containing H' such that $K[p]=H[p]$.

We now prove the sufficiency of the condition (*). Assume that H satifies (*) for all positive integers i and that K is minimal neat in G containing H. Suppose we have established that $p^nG \cap K = p^nK$, and let $p^{n+1}g \in K$. By neatness, $p^{n+1}g = pk$ for some $k \in K$ and therefore $p^ng-k \in G[p]$. If $G[p]/H[p] \subseteqq p^n(G/H[p])$, then $p^ng-k-h \in p^nG$ for some $h \in H[p]$ and by induction $k+h = p^nk'$ for some $k' \in K$. Hence $p^{n+1}g = p^{n+1}k'$. Suppose however that $p^ng-k-h \notin p^nG$ for all $h \in H[p]$ and consequently that $G[p]/H[p] \nsubseteq p^n(G/H[p])$. It will then follow that $p^{n+1}g \in H$. Indeed, otherwise there would be a minimal $m \geqq 1$ such that $p^{n+m+1}g \in H$. Then an application of (*) yields a $z \in G[p]$ such that $p^{n+m}g+z \in H$. But the assumption that $m \geqq 1$ implies $p^{n+m}g \in K$ and hence $z \in K \cap G[p] = H[p]$. This however implies that $p^{n+m}g \in H$, which contradicts the minimality of m. We conclude that $p^{n+1}g \in H$ and by (*), $p^ng+z \in H$ for some $z \in (p^nG)[p]$. Therefore, by induction, there is a $k' \in K$ such that $p^nk' = p^ng+z$ and thus $p^{n+1}k' = p^{n+1}g$.

As a close scrutiny indicates, a very slight modification of the foregoing proof yields

Theorem 2. *The subgroup H of the abelian group G is a kernel of purity in G if and only if for each prime p, H satisfies the condition (*) for all positive integers i.*

3. Applications

As a simple corollary of Theorem 2, we have

Theorem 3. *If for every prime p either* (1) $pG \subseteqq H$ *or* (2) $H[p]$ *is dense (relative to the p-adic topology on G) in $G[p]$, then H is a kernel of purity in G.*

PROOF. If $p^{i+1}g \in H$ for some positive integer i and $pG \subseteqq H$, then $p^{i+1}g = p^ih$ for some $h \in pG$. Therefore $p^ig+z = p^{i-1}h \in H \cap p^iG$ for some $z \in G[p]$. On the otherhand, if $H[p]$ is dense in $G[p]$, then $G[p]/H[p] \subseteqq p^i(G/H[p])$ for all positive integers i.

Corollary. *If $G_t \subseteqq H$, then H is a kernel of purity in G.*

For subsocles we have the following complete characterization of kernels of purity.

Theorem 4. *Let H be a subgroup of $G[p]$. Then a necessary and sufficient condition that H be a kernel of purity in G is that it satisfy one of the following two conditions:*

(1) *$H \cap p^n G \neq 0$ for all positive integers n and H is dense in $G[p]$.*

(2) *There is a minimal positive integer n such that $p^{n+1}G \cap H = 0$ and $G[p] = \{H, (p^{n-1}G)[p]\}$.*

Proof. Consider first the case where the elements of H are unbounded in p-height, that is, $H \cap p^n G \neq 0$ for all positive integers n. H being dense in $G[p]$ is, of course sufficient. If however H is not dense in $G[p]$, then $G[p]/H$ is not contained in $p^i(G/H)$ for some positive integer i. And if $p^{i+1}g$ is a non-zero element in H, then $p^i g + z$ has order p^2 for all $z \in G[p]$ and consequently is not in H. Therefore if H is not dense in $G[p]$, then H does not satisfy (*) for some positive integer i.

Suppose however that the elements of H are bounded in p-height. Then for some minimal $n \geqq 1$, $H \cap p^{n+1}G = 0$ and hence $p^n G \cap H \neq 0$ provided $n \neq 1$. The condition (*) is then automatically satisfied for all $i \geqq n$. And if $G[p] = \{H, (p^{n-1}G)[p]\}$, then $G[p]/H \subseteqq p^i(G/H)$ for all $i < n$. If however $G[p] \neq \{H, (p^{n-1}G)[p]\}$, then evidently $n \neq 1$ and $p^n g$ is a non-zero element in H for some $g \in G$. But then $G[p]/H \nsubseteqq p^{n-1}(G/H)$, and $p^{n-1}g + z$ has order p^2 for all $z \in G[p]$ and consequently is not in H, that is, H does not satisfy (*) for $i = n-1$.

Remark. From Theorem 4 it is not difficult to establish Pierce's (see [4]) characterization of centers of purity in primary groups.

The proof of the next theorem should now be evident.

Theorem 5. *Let H be a subgroup of the p-primary group G such that $H \cap p^n G \subsetneqq pH$ for infinitely many positive integers n. Then H is a kernel of purity in G if and only if $H[p]$ is dense in $G[p]$.*

Another instance where the density of $H[p]$ in $G[p]$ is essential is given by

Theorem 6. *If G/H is divisible, then H is a kernel of purity in G if and only if $H[p]$ is dense in $G[p]$ for each prime p.*

Proof. Suppose that H is a kernel of purity in G but that $H[p]$ is not dense in $G[p]$. Then there is an $x \in G[p]$ and an integer i such that $x - h \notin p^i G$ for all $h \in H[p]$. Since however G/H is divisible, $x - h_0 = p^i g$ for some $h_0 \in H$ and $g \in G$. Since $G[p]/H[p] \nsubseteqq p^i(G/H[p])$ and $ph_0 \in H \cap p^{i+1}G$, there must be a $z \in G[p]$ such that $p^i g + z = x - h_0 + z = h_1 \in H \cap p^i G$. But then $x - (h_0 + h_1) = z \in p^i G$ and clearly $h_0 + h_1 \in H[p]$, which contradicts the choice of x and i.

Corollary. *If G/H is divisible and H is a pure subgroup of G, then $H[p]$ is dense in $G[p]$ for each prime p.*

Remark. G/H being divisible does not in itself imply that $H[p]$ is dense in $G[p]$ (see [2]).

For primary groups we have the following reduction theorem.

Theorem 7. *Let H be a subgroup of the primary group G and let A be a basic subgroup of H. Then H is a kernel of purity in G if and only if A is a kernel of purity in G.*

The proof follows from the next two lemmas.

Lemma 1. *Let H and A be subgroups of the group G such that A is a neat subgroup of H and for every prime p, $A[p]$ is dense (relative to the p-adic topology on G) in $H[p]$. Then if H is a kernel of purity in G, A is also a kernel of purity in G.*

PROOF. Suppose $G[p]/A[p] \subsetneqq p^i(G/A[p])$ and that $p^{i+1}g \in A$. Since $A[p]$ is dense in $H[p]$, $G[p]/H[p] \subsetneqq p^i(G/H[p])$; and since H is a kernel of purity, $p^ig+z = h \in H \cap p^iG$ for some $z \in G[p]$. Since $ph \in A$ and A is neat in H, $h = a+z'$ for some $a \in A$ and $z' \in G[p]$. But clearly $z' \in H[p]$ and since $A[p]$ is dense in $H[p]$, we may assume that $z' \in p^iG$. Then $z_0 = z-z'$ is an element of $G[p]$ such that $p^ig+z_0 \in A \cap p^iG$, and we conclude that A is a kernel of purity in G.

Lemma 2. *Let H be a subgroup of G and A a subgroup of H such that H/A is divisible. Then if A is a kernel of purity in G, H is also a kernel of purity in G.*

PROOF. Suppose that $G[p]/H[p] \subsetneqq p^i(G/H[p])$. Then since $A \subseteqq H$, $G[p]/A[p] \subsetneqq p^i(G/A[p])$. Let $p^{i+1}g \in H$. Since H/A is divisible, there is an $a \in A$ and an $h \in H$ such that $p^{i+1}g = a+p^{i+1}h$. If A is a kernel of purity, then $p^i(g+h)+z \in A \cap p^iG$ for some $z \in G[p]$ and therefore $p^ig+z \in H \cap p^iG$.

Remark. The foregoing lemmas cannot be strengthened along the lines of Theorem 7. For if the relation between H and A is as described in Lemma 1, A being a kernel of purity does not imply that H is a kernel of purity. Similarly, in Lemma 2, H being a kernel of purity does not imply that A is a kernel of purity.

Theorem 7, in a sense, reduces the problem of determining kernels of purity in primary groups to direct sums of cyclic groups. Although, as the condition (*) indicates, whether or not a subgroup H of G is a kernel of purity depends to some extent on H_t, H_t being a kernel of purity in G does not imply that H itself is. Indeed even if H is torsion free it need not be a kernel of purity. For example, if G is a non-splitting mixed group and if H is maximal in G with respect to $H \cap G_t = 0$, then H is a torsion free neat subgroup of G which is not pure.

Finally, we consider the problem of determining those groups in which all subgroups are kernels of purity, that is, those groups in which all neat subgroups are pure.

Theorem 8. *Every subgroup of the group G is a kernel of purity in G if and only if G satisfies one of the following two conditions:*

(1) *G is torsion and every p-primary component of G is either divisible or else the direct sum of at most cyclic groups of orders p^n and p^{n+1} for some positive integer n.*

(2) *$G/G_t \neq 0$ and G_t is divisible.*

PROOF. We consider first the case where G is torsion. We need, indeed, only examine the primary case. Therefore assume that G is a p-group. If G is divisible and H is a subgroup of G, then $H[p]$ is trivially dense in $G[p]$. If G is a direct sum of at most cyclic groups of orders p^n and p^{n+1} and if H is a subgroup of G, then $p^{n+2}G \cap H[p] = 0$ and $G[p] = \{H[p], (p^nG)[p]\}$; and therefore by Theorem 4, $H[p]$, and conseuqently H itself, is a kernel of purity. However if G does not satisfy (1), then for some $i \geqq 1$, $G[p]$ contains an element x of finite height $i-1$ and also a non-zero element of the form $p^{i+1}g$ for some $g \in G$. Let $H = \{x+p^ig\}$. Then

$G[p]/H[p] \nsubseteq p^i(G/H[p])$ since $x - p^{i+1}g \notin p^iG$ and moreover $p^ig + z \notin H \cap p^iG$ for lal $z \in G[p]$ since $H \cap p^iG = H[p]$.

Next consider the case where G is not torsion, that is, $G/G_t \neq 0$. If G_t is divisible, then trivially $H[p]$ is dense in $G[p]$ for all primes p and all subgroups H of G and consequently all subgroups H of G are kernels of purity in G. Suppose however that G_t is not divisible. Then for some prime p and some positive integer i, $G[p]$ contains an element x of finite p-height $i-1$. Let g be an element of infinite order in G and set $H = \{x + p^ig\}$. Then since $H[p] = 0$ and $x \notin p^iG$, $G[p]/H[p]$ is not contained in $p^i(G/H[p])$; and clearly $p^ig + z \notin H \cap p^iG$ for all $z \in G[p]$ since $H \cap p^iG = \{p^{i+1}g\}$.

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