On proximity functions and symmetrical topogenous structures

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To Professor A. Rapcsák on h s 50th birthday

I.

It is the aim of this note to point out how topogenous orders, more exactly symmetrical topogenous structures, can be put to use in studying proximity functions and equivalent concepts. Our terminology and notations will be those of [1] and [2]

Definition 1. A symmetrical topogenous structure on a set E is a relation $<$ defined on the set of all subsets of E and satisfying the following axioms [1]):

(O1) $0<0,\ E<E$;

(O2) $A<B$ implies $A\subset B$;

(O3) $A\subset A'<B'\subset B$ implies $A<B$;

(S) $A<B$ implies $E-B<E-A$;

(Q″) the formulae $A<B$ and $A'<B'$ imply[2]) $A\cup A'<B\cup B'$;

(7. 9) if $A<B$, there exists a set C such that $A<C<B$.

The set of all symmetrical topogenous strctures on E possesses a natural partial order, namely the one induced by set-theoretical inclusion in $\mathfrak{P}(E)\times\mathfrak{P}(E)$: $<$ is "smaller" than $<_1$ if $<\subset<_1$, i. e. if $A<B$ implies $A<_1 B$.

Definition 2. A proximity function on a set E is a mapping α from the set of all subsets of E into the set of all filters on E satisfying the following conditions[3]) for $A, B, C\subset E$:

[1]) See [2], pp. 7., 9., 12., and 59. In [2] it is not the relation $<$ itself, but the one-element set $T=\{<\}$ which is called a symmetrical topogenous structure.

[2]) (S) and (Q'') together yield:

(Q') The formulae $A<B$ and $A'<B'$ imply

$$A\cap A'<B\cap B'.$$

Similarly, (S) and (Q') together imply (Q''). Thus, in the above definition (Q'') can be replaced by (Q').

[3]) See [1], p. 6. Thus, contrarily to the terminology of BOURBAKI, we do not exclude the improper filter, i. e. the set of all subsets of the set considered. Filters different from this will be called proper filters.

(A1) If $B\in\alpha(A)$ then $B\supset A$;

(A2) If $A\subset B$ then $\alpha(A)\supset\alpha(B)$;

(A3) If $B\in\alpha(A)$ then $E-A\in\alpha(E-B)$;

(A4) For any $B\in\alpha(A)$ there exists a C such that $B\in\alpha(C)$ and $C\in\alpha(A)$.

The set of all proximity functions on a set E possesses a natural partial order, defined by the condition "$\alpha(A)\subset\alpha'(A)$ for all $A\subset E$", which will be denoted by $\alpha\subset\alpha'$. (See [1], p. 7.)

The two concepts just defined are in fact equivalent as is shown by the following

Theorem 1. (1) *If $<$ is a symmetrical topogenous structure on E then the function $\alpha_<$ defined on the subsets of E by*

$$\alpha_<(A)=\{X|A<X\}$$

is a proximity function on E.

(2) *If α is a proximity function on E then the relation $<_\alpha$ defined for subsets of E by*

$$A<_\alpha B\Leftrightarrow B\in\alpha(A)$$

is a symmetrical topogenous structure on E.

(3) *The mappings $<\to\alpha_<$ and $\alpha\to<_\alpha$ are one-to-one correspondences, inverse to each other, between the sets of all symmetrical topogenous structures and all proximity functions on E which preserve the respective partial orders.*

PROOF. (1) For any $A\subset E$,

$$\alpha_<(A)=\{X|A<X\}$$

is a filter. Indeed, $A<X\subset Y$ implies $A<Y$ by (O3), while by virtue of (Q′) $A<X_1$ and $A<X_2$ imply $A<X_1\cap X_2$.

Moreover, this filter $\alpha_<(A)$ has the properties (A1)—(A4):

If $B\in\alpha_<(A)$, i. e. if $A<B$, then $A\subset B$ by (O2), i. e. (A1) holds.

Let now be $A\subset B$ and $C\in\alpha_<(B)$, i. e. $B<C$. Then $A\subset B<C$ implies $A<C$ by (O3), i. e. $C\in\alpha_<(A)$. This proves (A2).

As to (A3), $B\in\alpha_<(A)$ i. e. $A<B$ implies $E-B<E-A$ i. e. $E-A\in\alpha_<(E-B)$.

Finally, if $B\in\alpha_<(A)$ i. e. if $A<B$ then by (7. 9) there is C such that $A<C<B$ i. e. such that $B\in\alpha_<(C)$ and $C\in\alpha_<(A)$. Thus (A4) holds.

(2) We clearly have $E\in\alpha(E)$ i. e. $E<_\alpha E$; $O<_\alpha O$ i. e. $O\in\alpha(O)$ follows then by (A3). This establishes (O1). $A<_\alpha B$ i. e. $B\in\alpha(A)$ implies $B\supset A$ by (A1), i. e. (O2) holds.

Let now be $A\subset A'<_\alpha B'\subset B$. This means that

$$B\supset B'\in\alpha(A')\subset\alpha(A),$$

and consequently $B\in\alpha(A)$ i. e. $A<_\alpha B$ (by (A2) and by the fact that the α's are filters). So we have (O3).

$<_\alpha$ is also symmetrical. We indeed have

$$A<_\alpha B\Leftrightarrow B\in\alpha(A)$$

and

$$E-B<_\alpha E-A\Leftrightarrow E-A\in\alpha(E-B).$$

Thus the validity of (S) follows from (A3).

In order to establish (Q″) we first remark[4]) that for any $A, B \subset E$,

$$\alpha(A \cup B) = \alpha(A) \cap \alpha(B).$$

Now, from $A <_\alpha B$ and $A' <_\alpha B'$ i. e. from $B \in \alpha(A)$ and $B' \in \alpha(A')$ we infer $B \cup B' \in \alpha(A) \cap \alpha(A') = \alpha(A \cup A')$, i. e. $A \cup A' <_\alpha B \cup B'$.

Finally, (7. 9) follows from (A4): If $A <_\alpha B$ i. e. if $B \in \alpha(A)$ then by (A4) there exists a set C such that $B \in \alpha(C)$ and $C \in \alpha(A)$, i. e. such that

$$C <_\alpha B \text{ and } A <_\alpha C.$$

(3) Let $< \to \alpha_<$; $\alpha \to <_\alpha$. If $\alpha = \alpha_<$ then $<_\alpha = <$.
As a matter of fact, $A <_\alpha B$ means that $B \in \alpha(A)$, i. e. for $\alpha = \alpha_<$ we get

$$A <_\alpha B \Leftrightarrow B \in \alpha_<(A) \Leftrightarrow A < B.$$

On the other hand, let $\alpha \to <_\alpha$; $< \to \alpha_<$. If $< = <_\alpha$ then $\alpha_< = \alpha$.
As a matter of fact,

$$\alpha_<(A) = \{X | A < X\},$$

and for $< = <_\alpha$ we accordingly get

$$\alpha_<(A) = \{X | A <_\alpha X\} = \{X | X \in \alpha(A)\} = \alpha(A).$$

Finally, let $\alpha \subset \alpha_1$, i. e. $\alpha(A) \subset \alpha_1(A)$ for any $A \subset E$. Then $<_\alpha \subset <_{\alpha_1}$. — As a matter of fact, $A <_\alpha B$ i. e. $B \in \alpha(A)$ implies $B \in \alpha_1(A)$ i. e. $A <_{\alpha_1} B$. On the other hand, if $< \subset <_1$ then $\alpha_< \subset \alpha_{<_1}$. Indeed, $B \in \alpha_<(A)$ i. e. $A < B$ implies $A <_1 B$ i. e. $B \in \alpha_{<_1}(A)$.

This completes the proof of Theorem 1.

Besides proximity functions, B. BANASCHEWSKI and J. M. MARANDA introduce in [1] several equivalent concepts: Proximity relations, regular kernel operators and regular classes of filters. By Theorem 1. each of these concepts is equivalent to that of symmetrical topogenous structure. For proximity relations this result is due to Á. CSÁSZÁR [5]).

Here we are now going to state and to prove in a direct way the theorem linking together symmetrical topogenous structures and regular kernel operators. For this purpose we need the following

Definition 3.[6]) A regular kernel operator on the set $\Phi(E)$ of all filters on a set E is a mapping γ of $\Phi(E)$ into $\Phi(E)$ satisfying the following conditions for $\mathfrak{A}, \mathfrak{B} \in \Phi(E)$:

(K1) If $\mathfrak{A} \subset \mathfrak{B}$ then $\gamma\mathfrak{A} \subset \gamma\mathfrak{B}$;

(K2) $\gamma\mathfrak{A} \subset \mathfrak{A}$;

(K3) $\gamma(\gamma\mathfrak{A}) = \gamma\mathfrak{A}$;

(K4) $\gamma(\mathfrak{A} \cap \mathfrak{B}) = \gamma\mathfrak{A} \cap \gamma\mathfrak{B}$;

(K5) If $\gamma\mathfrak{A}$ and $\mathfrak{B}$ are incompatible then $\gamma\mathfrak{A}$ and $\gamma\mathfrak{B}$ are incompatible.[7])

[4]) See proposition 4. on p. 6. in [1].

[5]) See [2], p. 65., (7.26). The definition of proximity relation given in [1] differs slightly from that given in [2], but the two definitions can be proved to be equivalent.

[6]) [1], p. 10.

[7]) Two filters $\mathfrak{A}$ and $\mathfrak{B}$ are said to be incompatible (written: $\mathfrak{A}\Delta\mathfrak{B}$) if together they generate the improper filter, i. e. if $A \cap B = 0$ for some $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$.

For a given set E the set of all regular kernel operators $\Phi(E)$ possesses a natural partial ordering $\gamma \subset \gamma'$ which is defined by the condition:

$$\gamma\mathfrak{A} \subset \gamma'\mathfrak{A} \text{ for all } \mathfrak{A} \in \Phi(E).$$

The symmetrical topogenous structures on E and the regular kernel operators on $\Phi(E)$ correspond to each other as described by the following

Theorem 2. (1) *If $<$ is a symmetrical topogenous structure on E then the mapping $\gamma_<$ defined by*

$$\gamma_< \mathfrak{A} = \bigcup_{A \in \mathfrak{A}} \{X \mid A < X\}$$

for $\mathfrak{A} \in \Phi(E)$ is a regular kernel operator on $\Phi(E)$.

(2) *If γ is a regular kernel operator on $\Phi(E)$ then the relation $<_\gamma$ defined for subsets of E by*

$$A <_\gamma B \Leftrightarrow B \in \gamma[A]$$

is a symmetrical topogenous structure on E.

(3) *The mappings $< \to \gamma_<$ and $\gamma \to <_\gamma$ are one-to-one correspondences, inverse to each other, between the sets of all symmetrical topogenous structures on E and all regular kernel operators on $\Phi(E)$ which preserve the respective partial orders.*

Proof. (1) If $\mathfrak{A} \subset \mathfrak{B}$ then $A < X$ $(A \in \mathfrak{A})$ implies $A < X$ $(A \in \mathfrak{B})$, i. e. $\gamma_< \mathfrak{A} \subset \gamma_< \mathfrak{B}$. This proves (K1).

$A < X$ implies $A \subset X$ and this together with $A \in \mathfrak{A}$ yields $X \in \mathfrak{A}$. Thus $\gamma_< \mathfrak{A} \subset \mathfrak{A}$, i. e. (K2) holds.

By (K2) $\gamma_<(\gamma_< \mathfrak{A}) \subset \gamma_< \mathfrak{A}$. If, on the other hand, $X \in \gamma_< \mathfrak{A}$ i. e. $A < X$ for some $A \in \mathfrak{A}$ then by (7. 9) there is a subset C of E such that $A < C < X$. This means however that $C \in \gamma_< \mathfrak{A}$ and $X \in \gamma_<(\gamma_< \mathfrak{A})$, i. e. that $\gamma_< \mathfrak{A} \subset \gamma_<(\gamma_< \mathfrak{A})$. So we have (K3).

Making use of (K1) we easily get

$$\gamma_<(\mathfrak{A} \cap \mathfrak{B}) \subset \gamma_< \mathfrak{A} \cap \gamma_< \mathfrak{B}.$$

If, on the other hand, $X \in \gamma_< \mathfrak{A} \cap \gamma_< \mathfrak{B}$, then

$$A < X \, (A \in \mathfrak{A}) \quad \text{and} \quad B < X \, (B \in \mathfrak{B}).$$

From this we infer by (Q″) that

$$A \cup B < X \quad (A \cup B \in \mathfrak{A} \cap \mathfrak{B}),$$

i. e. that $X \in \gamma(\mathfrak{A} \cap \mathfrak{B})$. This proves (K4).

Finally, if $\gamma\mathfrak{A} \Delta \mathfrak{B}$, i. e. if $X \cap B = 0$ for some $X > A$ $(A \in \mathfrak{A})$ and $B \in \mathfrak{B}$, then by (7. 9) there is a $C \subset E$ such that $A < C < X$. By (S) we have $E - X < E - C < E - A$. Now $B \subset E - X < E - C$ implies $B < E - C$. Thus $E - C \in \gamma_< \mathfrak{B}$, and this, together with $C \in \gamma_< \mathfrak{A}$ yields $\gamma_< \mathfrak{A} \Delta \gamma_< \mathfrak{B}$, i. e. we have (K5).

(2) Let $A <_\gamma B$ i. e. $B \in \gamma[A]$. By (K2) we have $B \in [A]$, i. e. $A \subset B$. This proves (O2).

In order to establish (S) we first point out that by virtue of (K2) and of (K5) $\gamma\mathfrak{A}\Delta\mathfrak{B}$ if and only if $\gamma\mathfrak{A}\Delta\gamma\mathfrak{B}$.

Let now again be $A <_\gamma B$, i. e. $B \in \gamma[A]$. Then we get the following relations each of which implies the next one:

$$\gamma[A]\Delta[E-B],$$
$$\gamma[A]\Delta\gamma[E-B],$$
$$[A]\Delta\gamma[E-B],$$
$$E-A\in\gamma[E-B],^{8})$$
$$E-B<_\gamma E-A.$$

Thus (S) holds.

As to (O1), clearly $E\in\gamma[E]$ i. e. $E<_\gamma E$, and $O<_\gamma O$ then follows by (S).

In order to show the validity of (O3) we first remark that $A\subset B$ implies $[B]\subset\subset[A]$ and this in turn implies $\gamma[B]\subset\gamma[A]$. Let now be $A\subset A'<_\gamma B'\subset B$. We have $B'\in\gamma[A']$ and consequently $B\in\gamma[A']\subset\gamma[A]$, i. e. $B\in\gamma[A]$, or equivalently $A<_\gamma B$.

Let now be $A<_\gamma B$ and $A'<_\gamma B'$ i. e. $B\in\gamma[A]$ and $B'\in\gamma[A']$. By the implication $A\subset B\Rightarrow\gamma[B]\subset\gamma[A]$ just established $\gamma[A]\cup\gamma[A']\subset\gamma[A\cap A']$. Thus we have $B, B'\in\gamma[A\cap A']$ and consequently $B\cap B'\in\gamma[A\cap A']$, i. e. $A\cap A'<_\gamma B\cap B'$. Thus (Q′) holds.

If $A<_\gamma B$ i. e. $B\in\gamma[A]$, then $\gamma[A]\Delta[E-B]$ and by (K5) $\gamma[A]\Delta\gamma[E-B]$, i. e. there exist disjoint $X\in\gamma[A]$ and $Y\in\gamma[E-B]$. Thus, by the definition of $<_\gamma$, $E-B<_\gamma Y$ and consequently $E-Y<_\gamma B$.

At the same time $E-Y\supset X\in\gamma[A]$ implies $E-Y\in\gamma[A]$, i. e. $A<_\gamma E-Y$. This establishes (7. 9.)

(3) Consider the mappings $<\to\gamma_<$ and $\gamma\to<_\gamma$. If $\gamma=\gamma_<$, then $<_\gamma=<$. Indeed, $A<_\gamma B$ means that $B\in\gamma[A]$. For $\gamma=\gamma_<$ this condition can be written in any of the following equivalent forms:

$$B\in\gamma_<[A];$$
$$X<B \text{ for some } X\in[A];$$
$$A\subset X<B \text{ for some } X;$$
$$A<B.$$

Thus we have in fact $<_\gamma=<$.

On the other hand, let $\gamma\to<_\gamma$ and $<\to\gamma_<$. If $<=<_\gamma$, then $\gamma_<=\gamma$.

As a matter of fact, $X\in\gamma_<\mathfrak{A}$ means that $A<X$ for some $A\in\mathfrak{A}$. For $<=<_\gamma$ this condition can be written in any of the following equivalent forms:

$$A<_\gamma X \text{ for some } A\in\mathfrak{A};$$
$$X\in\gamma[A] \text{ for some } A\in\mathfrak{A}.$$

[8]) $[A]\Delta\gamma[E-B]$ means that $A\cap X=0$ i. e. that $X\subset E-A$ for some $X\in\gamma[E-B]$, and then $E-A\in\gamma[E-B]$ since $\gamma[E-B]$ is a filter.

Thus we get

$$(*)\qquad \gamma_{<}\mathfrak{A} = \bigcup_{A\in\mathfrak{A}} \gamma[A].$$

From this formula it is now possible to derive [9]) $\gamma_{<} = \gamma$.

Finally, it is clear from the definitions of $\gamma_{<}$ and $<_{\gamma}$ respectively that $< \subset <_1$ implies $\gamma_{<} \subset \gamma_{<_1}$ and that $\gamma \subset \gamma_1$ implies $<_{\gamma} \subset <_{\gamma_1}$. This completes the proof of Theorem 2.

Let us now enter upon a brief discussion of the equivalence existing between symmetrical topogenous structures and so called regular classes of filters. We start with the following [10])

Definition 4. A set Ψ of filters on E is called a regular class of filters if it satisfies the following conditions:

(R1) For any family $\{\mathfrak{A}_i | i\in I\} \subset \Psi$,

$$\sum \{\mathfrak{A}_i | i \in I\} \in \Psi.$$

(R2) If $\mathfrak{A}, \mathfrak{B} \in \Psi$, then $\mathfrak{A} \cap \mathfrak{B} \in \Psi$.

(R3) If $\mathfrak{A} \in \Psi$ is incompatible with $\mathfrak{B} \in \Phi(E)$ then there exists a $\mathfrak{C} \subset \mathfrak{B}$ in Ψ also incompatible with $\mathfrak{A}$.

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[9]) For a proof see [1], p. 11., (3). — For completeness sake, we are going to repeat this proof here. For this purpose we need the following

Lemma. *If the filter $\mathfrak{A}$ is properly contained in the filter $\mathfrak{B}$ then there is an ultrafilter $\mathfrak{U}$ containing $\mathfrak{A}$ and incompatible with $\mathfrak{B}$.*

Proof of the Lemma. By hypothesis there is a set H satisfying $H \in \mathfrak{B}$ and $H \notin \mathfrak{A}$. Now $H \notin \mathfrak{A}$ clearly implies $(E-H) \cap A \neq 0$ for $A \in \mathfrak{A}$. (Of course, $\mathfrak{A}$ is a proper filter, since it is properly contained in another filter.) Thus there is a proper filter $\mathfrak{F}$ satisfying $\mathfrak{A} \subset \mathfrak{F}$ and $E-H \in \mathfrak{F}$. If $\mathfrak{U}$ is an ultrafilter containing $\mathfrak{F}$ then $\mathfrak{U}$ fulfils the requirements of the lemma.

Let us now give a

Proof of the equality $\gamma_{<} = \gamma$. (K1) clearly implies

$$\bigcup_{A\in\mathfrak{A}} \gamma[A] \subset \gamma\mathfrak{A},$$

i. e. by $(*)$, $\gamma_{<}\mathfrak{A} \subset \gamma\mathfrak{A}$.

From this by conditions (K1) and (K3) valid for $\gamma_{<}$ one obtains $\gamma_{<}(\gamma_{<}\mathfrak{A}) \subset \gamma_{<}(\gamma\mathfrak{A})$ and $\gamma_{<}\mathfrak{A} \subset \subset \gamma_{<}(\gamma\mathfrak{A})$. At the same time $\gamma\mathfrak{A} \subset \mathfrak{A}$ implies $\gamma_{<}(\gamma\mathfrak{A}) \subset \gamma_{<}\mathfrak{A}$, and so one has $\gamma_{<}\mathfrak{A} = \gamma_{<}(\gamma\mathfrak{A})$.

Similarly, from

$$\gamma_{<}\mathfrak{A} = \gamma_{<}(\gamma_{<}\mathfrak{A}) \subset \gamma(\gamma_{<}\mathfrak{A}) \subset \gamma_{<}\mathfrak{A}$$

it is seen that $\gamma_{<}\mathfrak{A} = \gamma(\gamma_{<}\mathfrak{A})$.

Now suppose that for some $\mathfrak{A} \in \Phi(E)$ the filter $\gamma_{<}\mathfrak{A}$ is properly contained in $\gamma\mathfrak{A}$. Then there exists an ultrafilter $\mathfrak{U} \supset \gamma_{<}\mathfrak{A}$ incompatible with $\gamma\mathfrak{A}$, and this implies by (K5) that $\gamma\mathfrak{U}$ and $\gamma\mathfrak{A}$ are also incompatible. If $X \in \gamma\mathfrak{A}$ has void intersection with some set belonging to $\gamma\mathfrak{U}$, one furthermore has $[X] \Delta \gamma\mathfrak{U}$ and consequently $\gamma[X] \Delta \gamma\mathfrak{U}$.

However, $\mathfrak{U} \supset \gamma_{<}\mathfrak{A}$ gives

$$\gamma\mathfrak{U} \supset \gamma(\gamma_{<}\mathfrak{A}) = \gamma_{<}\mathfrak{A} = \gamma_{<}(\gamma\mathfrak{A}),$$

and by formula $(*)$ $X \in \gamma\mathfrak{A}$ implies $\gamma[X] \subset \gamma_{<}(\gamma\mathfrak{A})$, hence $\gamma[X] \subset \gamma\mathfrak{U}$ which is a contradiction since $\gamma\mathfrak{U}$, being contained in the ultrafilter $\mathfrak{U}$, is a proper filter. It follows that $\gamma_{<}\mathfrak{A} = \gamma\mathfrak{A}$ for $\mathfrak{A} \in \Phi(E)$, i. e. that $\gamma_{<} = \gamma$.

This result, combined with $(*)$ yields

$$\gamma\mathfrak{A} = \bigcup_{A\in\mathfrak{A}} \gamma[\mathfrak{A}]$$

for any filter $\mathfrak{A}$ on E and any regular kernel operator γ on $\Phi(E)$.

[10]) See [1], Proposition 9.

As is shown in [1] (Proposition 9.), there exists an order-preserving one-to-one correspondence between the sets of all proximity functions and all regular classes of filters on a set E. Since, on the other hand, we have by Theorem 1. an order-preserving one-to-one correspondence between symmetrical topogenous structures and proximity functions on E, a correspondence of the same kind exists between symmetrical topogenous structures and regular classes of filters on E. More explicitly this statement can be formulated as follows:

Theorem 3. (1) *If $<$ is a symmetrical topogenous structure on E then the set $\Psi_<$ of those filters $\mathfrak{A}$ on E which satisfy the condition*

$$\mathfrak{A} = \bigcup_{A \in \mathfrak{A}} \{X \mid A < X\}$$

is a regular class of filters. [11])

(2) *If Ψ is a regular class of filters on E then the relation $<_\Psi$ defined by*

$$A <_\Psi B \leftrightarrow B \in \sum \{\mathfrak{A} \mid [A] \supset \mathfrak{A} \in \Psi\}$$

yields a symmetrical topogenous structure $<_\Psi$.

(3) *The mappings $< \to \Psi_<$ and $\Psi \to <_\Psi$ are one-to-one correspondences, inverse to each other, between the sets of all regular classes of filters and all symmetrical topogenous structures on E, which preserve the respective partial orders.* [12])

PROOF. This theorem is an immediate consequence of Theorem 1. and of Proposition 9. in [1]. — A direct proof, modelled on the proof of Proposition 9. in [1] is also feasible.

II.

We are going now to establish a few results concerning (symmetrical) topogenous structures. Our starting point will be the following

Definition 5. A topogenous structure [13]) $<$ defined on the set E is said to be compatible with the mapping f of the set E into the set E', if $f(x) = f(y)$ implies

$$\{x\} < X \Leftrightarrow \{y\} < X.$$

Proposition 1. *If the topogenous structure $<$ on E is compatible with the mapping f of E into E', then $A < X$ implies $f^{-1}(f(A)) < X$ for any A, $X \subset E$.*

PROOF. By (O3) $A < X$ implies $\{a\} < X$ for $a \in A$. Let now be $x \in f^{-1}(f(A))$. Then $f(x) = f(a_0)$ for a suitable $a_0 \in A$ and $\{a_0\} < X$ implies $\{x\} < X$. Thus we see that $A < X$ implies

(I) $\{x\} < X$ for $x \in f^{-1}(f(A))$.

[11]) In the German version of [2] these filters are called „round" (runde Filter), a terminology due to KOWALSKY.

[12]) I. e. $< \subset <_1$ if and only if $\Psi_< \subset \Psi_{<_1}$ for the corresponding regular classes of filters.

[13]) The definition of a topogenous structure is obtained from Definition 1. by omitting (S) and by adding (Q'). — The compatibility defined here is in fact equivalent to that introduced in [2] on p. 106. — Let us also note right here that Proposition 2. below is but a special case of Theorem (9.29) in [2]. It will be stated and proved only in order to make the presentation self-contained as far as possible.

Now (I) in turn implies $f^{-1}(f(A))\subset X$. This yields the implication:

(II) $A<X<H\Rightarrow f^{-1}(f(A))<H$.

By (7. 9) $A<X$ implies the existence of a set K such that $A<K<X$, and so by (II) we get $f^{-1}(f(A))<X$. (If, on the other hand, $f^{-1}(f(A))<X$, then by $A\subset f^{-1}(f(A))$ we of course get $A<X$.)

Proposition 2. *Let a mapping f of the set E onto the set E' be given. If $<$ is a symmetrical topogenous structure defined on E and compatible with f, then the relation $<'$ on E' defined by*

$$A'<'B' \quad \textit{means} \quad \textit{that} \quad f^{-1}(A')<f^{-1}(B')$$

is a symmetrical topogenous structure on E' and $<=f^{-1}(<')$.

Proof. One easily sees that $<'$ is a symmetrical topogenous structure. As a matter of fact, if $A'<'B'$, i. e. if $f^{-1}(A')<f^{-1}(B')$, then $f^{-1}(A')<C<f^{-1}(B')$ for a suitable $C\subset E$. — By (O3) and by Prpoosition 1. we now get

$$f^{-1}(A')<f^{-1}(f(C))<f^{-1}(B'),$$

i. e.

$$A'<'f(C)<'B'.$$

This shows that $<'$ satisfies Condition (7. 9) of Definition 1. As to the remaing conditions of that definition, it has been pointed out in [2] (see (6. 30) and (6. 34)) that $<'$ satisfies them.

We put now for the time being $<^*=f^{-1}(<')$. Then $A<^*B$ means that $f(A)<'E'-f(E-B)$, i. e. that

$$f^{-1}(f(A))<f^{-1}(E'-f(E-B)),$$

or equivalently

$$f^{-1}(f(A))<E-f^{-1}(f(E-B)).$$

From this we infer by (O3) $A<E-f^{-1}(f(E-B))$, and with the help of axiom (S) and of (O3) we get

$$f^{-1}(f(E-B))<E-A\Rightarrow E-B<E-A\Rightarrow A<B.$$

Thus we have proved that $A<^*B$ implies $A<B$. By going through the steps of the above proof in reverse order[14]), we obtain a proof of the reverse implication. Thus $<=<^*=f^{-1}(<')$. This completes the proof of the proposition.

Let us now denote by $\Pi(E)$ the set of all proximity functions defined on a set E. The one-to-one correspondence existing between proximity functions and symmetrical topogenous structures can be put to use when investigating the relation between the sets $\Pi(E)$ and $\Pi(E')$ for two sets E and E' which is determined by a given mapping f of E into E'.

As a matter of fact, let $\alpha'\in\Pi(E')$. The corresponding symmetrical topogenous structure $<_{\alpha'}=<'$ is defined by

$$A'<'B' \text{ means that } B'\in\alpha'(A').$$

[14]) In doing so, we have to use Proposition 1. instead of (03) on the appropriate places.

By the mapping f of E into E' there corresponds to $<'$ defined on E' a relation $f^{-1}(<')=<$ defined on E as follows (see [2], (6. 1)):

$$A<B \text{ means that } f(A)<'E'-f(E-B),$$

or equivalently

$$A<B \Leftrightarrow E'-f(E-B)\in\alpha'[f(A)].$$

This relation $<$ is in fact a symmetrical topogenous structure defined on E (see [2], p. 100.) and so there is on E a corresponding proximity function $\alpha_< = \alpha$ defined by

$$\alpha(A) = \alpha_<(A) = \{X|A<X\},$$

i. e. by

$$\alpha(A) = \{X|E'-f(E-X)\in\alpha'[f(A)]\}.$$

This proximity function α is said to be the inverse image[15]) with respect to the mapping f of the proximity function α':

$$\alpha=f^{-1}(\alpha').$$

(With the help of the inverse of the corresponding symmetrical topogenous structure one can also define the inverse of a regular kernel operator or of a regular class of filters.)

An important property of the inverse of a proximity function is expressed by the following

Proposition 3. *Let f be a mapping of E into E', $\alpha'\in\Pi(E')$ and $\alpha=f^{-1}(\alpha')$. Then for any $A\subset E$ the filter $\alpha(A)$ is the one generated by $f^{-1}\{\alpha'[f(A)]\}$.*

Proof. For $X\in\alpha(A)$ we have

$$X\supset f^{-1}[E'-f(E-X)]\in f^{-1}\{\alpha'[f(A)]\}.$$

On the other hand, if $H=f^{-1}(H')$ where $H'\in\alpha'[f(A)]$, then $f(E-H)\cap H'=0$. Thus

$$E'-f(E-H)\supset H'\in\alpha'[f(A)]$$

and so $H\in\alpha(A)$.

The validity of the following propositions is immediately clear:

Proposition 4. *The correspondence between a proximity function and its inverse with respect to a mapping f is order preserving, i. e. $\alpha_1'\subset\alpha_2'$ on E' implies $f^{-1}(\alpha_1')\subset$ $\subset f^{-1}(\alpha_2')$ on E.*

Proposition 5. *The proximity functions α on E which are inverse images with respect to a mapping f are compatible with the equivalence relation*

$$R_f = \{(x,y)|f(x)=f(y);\ x,y\in E\}$$

in the sense that $f(x)=f(y)$ implies $\alpha(\{x\})=\alpha(\{y\})$.

Proposition 6. *In case f is a mapping of E onto E' the relation between the proximity functions α' on E' and their inverses $\alpha=f^{-1}(\alpha')$ on E is one-to-one and it maps $\Pi(E')$ onto the set of all $\alpha\in\Pi(E)$ which are compatible with R_f.*

[15]) In [1] our $f^{-1}(\alpha')$ is denoted by $(f^*\alpha')$.

PROOF. That f^{-1} maps $\Pi(E')$ one-to-one into $\Pi(E)$ is a direct consequence of the fact that for a mapping which is *onto* the relation between the corresponding symmetrical topogenous structures is one-to-one. (See [2], (6. 5).) We have already pointed out that the inverse image of a proximity function with respect to a mapping f is always compatible with R_f. There remains still to be shown that for any $\alpha \in \Pi(E)$ compatible with R_f there exists an $\alpha' \in \Pi(E')$ with $f^{-1}(\alpha')=\alpha$, this is however an immediate consequence of Proposition 2.

Propositions 3—6. together express the fundamental result of BANASCHEWSKI and MARANDA characterizing the inverse of a proximity function.

Finally, let it be mentioned that in two short communications (see [3]) M. HACQUE introduces concepts generalizing some of those considered by BANASCHEWSKI and MARANDA. — I am indebted to professor Á. CSÁSZÁR for having pointed out this to me.

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