On unitary and Π-unitary rings

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We call an element a of a ring A right (left) unitary if and only if for each a in A, there exists an element $x=x(a)$ of A such that $a=ax$ (or $a=xa$) and a ring is right (left) unitary if and only if each element of A is right (left) unitary. Every right (left) valuation ring is right (left) unitary but not conversely. Further, an element a of a ring is said to be Π-unitary if and only if there exists a positive integer $n=n(a)$ and an element $x=x(a)$ in the center such that $a^n=a^nx$. A ring all of whose elements are Π-unitary is termed as Π-unitary.

We have obtained a few results about such rings and also prove a theorem on the existence of a unity. The arguments are very straight forward and do not depend much on the structure theorems. We shall refer to a subring C of a ring A (cf. [2]) defined as follows:

$$C=\{c\,|\,c\in A;\ ce=ec=c\}$$

where e is some fixed element of A.

1. We begin with

Theorem 1. *In a right (left) unitary ring A if to each $a\in A$, there corresponds a positive integer $n=n(a)$ such that $a^n\in C$, then A is either nil or it has an identity element.*

PROOF. Let $a\in A$. Then $a^{n(a)}\in C$. In particular, $e^n\in C$. It follows then, $e^ne=e^n$, viz., $e^{n+1}=e^n$. In general $e^{n+k}=e^n$ $(k=0, 1, 2, \ldots)$. This implies that $w=e^n$ is an idempotent. It is easy to verify that $wAw=C$. In case $w=0$, $C=0$. Therefore A is nil. Let $w\neq 0$. Consider the Peirce decomposition relative to w

$$A = wAw+wA(1-w)+(1-w)Aw+(1-w)A(1-w).$$

If $a\in(1-w)\,Aw$, then $(w+a)^2 = w+a$. So that $(w+a)^n = (w+a)$, for all positive integer n. By our hypothesis, this implies that $w+a\in C = wAw$. But since $w\in wAw$, $a\in wAw$. Thus we have obtained that $a\in(1-w)Aw$ implies $a\in wAw$ too. This requires that $a=0$. Hence $(1-w)Aw=0$. Similarly $wA(1-w)=0$. Thus w is then central.

The Peirce-decomposition then reduces to $A = wA+A(1-W)$. Set $B = A(1-w)$. If $x\in B$, then $x^n\in B\cap wA$ for some $n>0$. Thus $x^n=0$. B is then a nil ideal.

Next, A being a right (left) unitary ring, $x\in xA$ (or $x\in Ax$) for all x. In particular for every x in B, $x\in xA$ (or $x\in Ax$). So that $x\in x(Aw+B)$ or $x\in(Aw+B)x$, for each $x \in B$. But since, for each $x \in B$, wx or xw is zero, it follows that $x \in xB$ (or $x \in Bx$) for all x in B.

Therefore, for every x in B, there exists a y in B such that $x = xy$ (or $x = yx$). This implies $x = xy^n$ (or $x = y^n x$), for every integer $n > 0$.

Since B is a nil ideal, it follows that $x = 0$ for all x in B. So that $B = 0$. Hence $A = Aw = wA$ and it is immediate that $A = Ae = eA$. This completes the proof.

2. Following exactly Brown and McCoy [1] for obtaining the maximal regular ideal of a ring we can show that the join of all right (left) unitary ideals is a right (left) unitary ideal and hence there exists a unique maximal right (left) unitary ideal $M = M(A)$ in A. The properties of $M(A)$: (i) $M(A/M(A)) = 0$ and (ii) If B is an ideal of A, then $M(B) \subseteq B \cap M(A)$, are also easy to verify. We shall prove the corresponding results for the maximal Π-unitary ideal in a ring.

We first prove the following

Lemma. *If to each element a of a ring A there corresponds a positive integer $n = n(a)$ and a central element $x = x(a)$ of A such that $a^n - a^n x$ is Π-unitary then a is Π-unitary.*

Proof. By hypothesis there exists a positive integer m and a central element y of R such that

$$(a^n - a^n x)^m y = (a^n - a^n y)^m$$

so that

$$a^{mn}(1-x)^m y = a^{mn}(1-x)^m$$

where 1 is formal. But then

$$a^{mn}(1-x)^m(1-y) = 0.$$

So we get

$$a^{mn}(1-z) = 0,$$

where z is obviously in the center. Thus a is Π-unitary and this completes the proof.

We shall indicate by (a) the principal ideal in B denerated by a.

We now prove the following theorem.

Theorem 2. *If M is the set of all elements a of R such that (a) is Π-unitary then M is an ideal in R.*

Proof. If $z \in M$ and $t \in R$, then $zt \in M$, since $(zt) \subset (z)$. Similarly, $tz \in M$. If $z, w \in M$ and $a \in (z - w)$, then $a = u - v$ for some u in (z) and v in (w). Since (z) is Π-unitary. $u^{n(u)} = u^{n(u)} r$ for some element r of R. Then

$$a^n - a^n r = (u-v)^n - (u-v)^n r = \textit{terms each of which has } v \textit{ as one of its factors.}$$

Since $v \in (w)$, we obtain that $a^n - a^n r \in (w)$ and is therefore Π-unitary. The above lemma now implies that a is Π-unitary and hence $z - w \in M$. This completes the proof.

Theorem 2. 1. *If R is any ring, $M(R/M(R)) = 0$.*

Proof. Let $\bar{a}$ denote the residue class modulo $M(R)$ which contains the element a of R. If $\bar{b} \in M(R/M(R))$ and $\bar{a} \in (\bar{b})$, then $a \in (b)$. Since $(\bar{b})$ is a Π-unitary ideal in $R/M(R)$, $\bar{a}$ is Π-unitary. For

$$\bar{a}^n = \bar{a}^n \bar{x}, \text{ then } a^n - a^n x \text{ is in } M(R).$$

Therefore, $a^n - a^n x$ is Π-unitary and hence a is Π-unitary. But a is an arbitrary element of (b). It follows then (b) is a Π-unitary ideal.

Hence $b \in M(R)$ and therefore $\bar{b} = 0$, completing the proof.

The case $R = M(R)$. Let R be finite Π-unitary ring having $a_1, a_2, \ldots, a_n$ as its elements. For each a_i we have a positive integer n_i and a central element b_i such that

$$a_i^{n_i} b_i = a_i^{n_i} \quad \text{i. e.} \quad a_i^{n_i}(1 - b_i) = 0,$$

where 1 is formal.

Let us define

$$c = 1 - (1 - b_1)(1 - b_1) \ldots (1 - b_n).$$

Then

$$\begin{aligned} a_i^{n_i} c &= a_i^{n_i} - (1 - b_1) \ldots (1 - b_n) a_i^{n_i} \\ &= a_i^{n_i} - (1 - b_1)(1 - b_2) \ldots (1 - b_n)(1 - b_i) a_i^{n_i} \\ &= a_i^{n_i}, \text{ for } b_i\text{'s are all central and } a_i^{n_i}(1 - b_i) = 0. \end{aligned}$$

Evidently c which is a product of central elements is also central.

We have, thus, proved the following

Theorem 2. 2. *A finite Π-unitary ring R has a central element c such that $a^{n(a)} c = a^{n(a)}$, for each a in R.*

Corollary. *A finite non-nil Π-unitary ring has a nil ideal modulo which it has a unity element.*

Finally we prove:

Theorem 3. 1. *If for every x in a non-nil ring A there exists a positive integer $n = n(x)$ such that $x^n \in C$ and if every proper ideal is contained in a modular right (left) ideal then A has a unity element.*

Proof. As in theorem 1. 1 we have a nil ideal B in A such that

$$A = Aw + B.$$

The ideal Aw cannot be zero, for otherwise A is nil. If Aw were different from A, it can by hypothesis be imbedded in a modular right (left) ideal and therefore by Zorn Lemma it can be imbedded in a modular maximal right (left) ideal F. If R is the Jacobson radical then the nil ideal.

$$\begin{aligned} B &\subseteq R \\ &\subseteq F \end{aligned}$$

and thus we are led to a contradiction

$$A = Aw + B \subseteqq F \subset A.$$

Hence $A = Aw = wA$ and it is immediate that $e = u$.

This completes the proof of the theorem.

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References

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[2] S. K. Jain, On existence of identity, *Math. Student*, to appear.

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