On the distribution of values of a class of entire functions I

By S. DANCS and P. TURÁN (Budapest)

1. The distribution of values of functions of type

$$f(z)=\sum_{j=1}^{n} P_j(z)e^{\omega_j z} \tag{1.1}$$

where the ω_j's are different complex numbers and the $P_j(z)$'s are polynomials of degree $\leqq p-1$ $(p\geqq 1)$ or — what amounts to the same — the distribution of their zeros is intimately connected with several important problems. Such questions occur e. g. in the study of solutions of retarded and non-retarded differential equations of finite or infinite order with constant coefficients; such problems occur in the study of developments according to their characteristical functions, dealt with in the papers of RITT, C. E. WILDER, TAMARKIN, VALIRON, GELFOND, D. G. DICKSON and others (see [1]) nothing said on the theory of mean-periodical functions. In the technical literature they are treated mainly in stability questions of control-systems; such papers are mentioned in a paper of S. SHERMAN (see [2]); see also some papers of PONTRJAGIN and MYSKIS (see [3], [4]). The vast literature on our subject can be found in the expository paper of LANGER (see [5]) and in recent books of B. LEVIN (see [6]) resp. of R. BELLMAN and K. COOKE (see [7]). See also a still more recent paper of KARLIN and SZEGŐ (see [8]) where connections with stochastic processes are indicated. Investigating the distributions of zeros of high-order derivatives of rational functions G. PÓLYA (see [9]) was lead again to the problem (1. 1). Nevertheless the following type of theorem seems not to have been observed before (see [10]).

If with complex a_j's and ω_j's

$$g(z)=\sum_{j=1}^{n} a_j e^{\omega_j z} \qquad (z=x+iy) \tag{1.2}$$

and*)

$$\max_j |\omega_j| \leqq M \tag{1.3}$$

$$\min_{\mu\neq\nu} |\omega_\mu-\omega_\nu| \geqq \Delta, \tag{1.4}$$

*) Since the roots of $g(z)$ do not change after multipyling by $e^{\alpha z}$, α complex number, (1. 3) can be replaced by

$$\max_{\mu,\nu} |\omega_\mu-\omega_\nu| \leqq M'.$$

then with arbitrary real A, B and positive L the number of zeros of $g(z)=0$ in the square

$$(1.\ 5) \qquad A \leqq x \leqq A+L, \quad B \leqq y \leqq B+L$$

(counted according to multiplicity) cannot exceed

$$(1.\ 6) \qquad 6LM + n \log\left(2 + \frac{n^3}{\Delta L}\right) + \log 2n.$$

The point of the theorem is of course that this upper bound is independent of the coefficients a_j, of the position of the square and only very loosely dependent on the configuration of the ω_j-exponents. Easy examples show that dependence upon L, M and n is indispensable; the necessity of Δ in this bound has not been established. In what follows (paper II.) we shall show that the answer is positive to the second natural question, namely that whether or not an analogous upper bound (depending also upon p) exists for the more general class (1. 1).

2. The main role in the proof of (1. 6) was played by the following theorem. If for the complex z_j-numbers with a positive δ the inequality

$$(2.\ 1) \qquad \frac{\min\limits_{\mu \neq \nu} |z_\mu - z_\nu|}{\max\limits_j |z_j|} \geqq \delta\,(\leqq 1)$$

holds, then for fixed complex b_j's and positive integers m we have the inequality

$$(2.\ 2) \qquad \max_{\nu = m+1, \ldots, m+n} \frac{\left|\sum\limits_{j=1}^{n} b_j z_j^\nu\right|}{\sum\limits_{j=1}^{n} |b_j|\,|z_j|^\nu} \geqq \frac{1}{2n}\left(\frac{\delta}{2}\right)^{n-1}$$

independently of m.

This theorem was stated and proved in the German edition of [9]; the significance in applications was realized in the Chinese one (in which the proof of (1. 6) appeared for the first time). In this paper we shall prove an analogous theorem which fits the more general situation. This runs as follows:

Theorem. *If the complex variables $z_1, \ldots, z_n$ are restricted only by*

$$(2.\ 3) \qquad \frac{\min\limits_{\mu \neq \nu} |z_\mu - z_\nu|}{\min\limits_j |z_j|} \geqq \delta\,(\leqq 1) \quad \text{and} \quad \frac{\max\limits_j |z_j|}{\min\limits_j |z_j|} \leqq D\,(\geqq 1),$$

then for all fixed complex $d_{\mu j}$, positive integers m and p the inequality

$$\max_{\nu = m+1, m+2, \ldots, m+np} \frac{\left|\sum\limits_{j=1}^{n} z_j^\nu \left\{\sum\limits_{\mu=0}^{p-1} d_{\mu j}\,\mu!\binom{\nu}{\mu}\right\}\right|}{\sum\limits_{j=1}^{n} |z_j|^\nu \sum\limits_{\mu=0}^{p-1} |d_{\mu j}|\,\mu!\binom{\nu}{\mu}} \geqq (m+np)^{-2p^2}\left(\frac{\delta}{8D}\right)^{2pn + \frac{p(p-1)}{2}}$$

holds.

The intended application to the class (1. 1) will follow in the second paper of this series and an application of theorem (1. 6) to another class of functions, which is in a certain sense much more general, in the third paper.

Another way to obtain the theorem in the second paper is via the following theorem.

If $P_j(z)$ are polynomials of degree $\leqq p-1$, $A>0$ given and m is a positive integer satisfying $p-1\leqq m+1$, then with the notations of the previous theorem there is an integer ν_0 with

$$m+1\leqq \nu_0\leqq m+np$$

so that for all w-values with $|w|\leqq A$ the inequality

$$\frac{\left|\sum_{j=1}^{n} P_j(\nu_0) z_j^{\nu_0}\right|}{\sum_{j=0}^{n} |P_j(\nu_0+w)|\,|z_j|^{\nu_0}} \geqq (m+np)^{-p^2}(m+np+A)^{-p}\left(\frac{\delta}{8D}\right)^{2pn+\frac{p(p-1)}{2}}$$

holds.

We shall not give the details of its proof.

3. Before turning to the proof of our theorem we shall make a number of observations. With the abbreviation

$$l!\binom{\mu}{l}\binom{m+1}{l}=a_{\mu l} \qquad (m \text{ positive integer}) \tag{3. 1}$$

we consider first the linear system

$$\sum_{l=0}^{\mu} a_{\mu,\mu-l}x_l=B_\mu \qquad (\mu=0, 1, \dots, r). \tag{3. 2}$$

Then we have

$$x_r=\begin{vmatrix} a_{00} & 0 & \cdots 0 & B_0 \\ a_{11} & a_{10} & 0 & B_1 \\ \vdots & \vdots & \ddots \vdots & \vdots \\ a_{r-1,r-1} & a_{r-1,r-2} & \cdots a_{r-1,0} & B_{r-1} \\ a_{r,r} & a_{r,r-1} & \cdots a_{r,1} & B_r \end{vmatrix}. \tag{3. 3}$$

Hence HADAMARD's determinant-theorem gives

$$|x_r| \leqq \left(\sum_{\nu=0}^{r}|B_\nu|^2\right)^{\frac{1}{2}}\cdot\prod_{j=0}^{r-1}(|a_{j0}|^2+|a_{j+1,1}|^2+\dots+|a_{r,r-j}|^2)^{\frac{1}{2}}.$$

Since $\binom{j+\mu}{\mu}$ increases monotonically with μ, the second factor in this product is bounded by

$$\binom{r}{j}\{1+(m+1)^2+\dots+(m+1)^{2r-2j}\}^{\frac{1}{2}}<2^r\sqrt{r+1}\,(m+1)^{r-j},$$

i. e.

$$(3.4)\qquad |x_r| \leqq \left(\sum_{\nu=0}^{r} |B_\nu|^2\right)^{\frac{1}{2}} 2^{r^2}(r+1)^{\frac{r}{2}}(m+1)^{\binom{r+1}{2}}.$$

Next let $\varphi(z)$ be analytic at the point $z=\beta(\neq 0)$ and let

$$(3.5)\qquad \beta^{\mu-m-1}\{z^{m+1}\varphi(z)\}^{(\mu)}_{z=\beta} = C_\mu = \text{prescribed} \qquad \mu=0,1,\dots,p-1.$$

What upper bound can be given to the $|\varphi^{(\mu)}(\beta)|$'s? LEIBNIZ's rule gives

$$\sum_{l=0}^{\mu}\binom{\mu}{l}\binom{m+1}{l} l!\,(\beta^{\mu-l}\varphi^{(\mu-l)}(\beta)) = C_\mu, \qquad \mu=0,1,\dots,p-1.$$

This is a linear system for $x_r=\beta^r\varphi^{(r)}(\beta)$; hence (3. 4) gives at once

$$(3.6)\qquad |\beta|^r|\varphi^{(r)}(\beta)| \leqq 2^{r^2}(r+1)^{\frac{r}{2}}(m+1)^{\binom{r+1}{2}}\left(\sum_{l=0}^{r}|C_l|^2\right)^{\frac{1}{2}}, \qquad r=0,1,\dots,p-1.$$

4. Our main concern is to get information for the ξ_h-unknowns $(h=1,2,\dots,np)$ in the linear system

$$(4.1)\qquad \sum_{h=1}^{np}\alpha_j^{h-1}\binom{m+h}{\mu}\mu!\,\xi_h = A_{\mu j}\,(=\text{prescribed})$$
$$j=1,\dots,n, \quad \mu=0,1,\dots,p-1$$

where $\alpha_1,\dots,\alpha_n$ are different fixed non-vanishing complex numbers. Introducing the polynomial

$$(4.2)\qquad \sum_{h=1}^{np}\xi_h z^{h-1} = \varphi(z)$$

with this $\varphi(z)$ the system (4. 1) can obviously be written in the form for each fixed j

$$(4.3)\qquad \alpha_j^{\mu-m-1}\{z^{m+1}\varphi(z)\}^{(\mu)}_{z=\alpha_j} = A_{\mu j} \qquad (\mu=0,1,\dots,p-1).$$

But then for each of our α_j's we have the problem (3. 5) with $C_\mu=A_{\mu j}$ and $\beta=\alpha_j$. Hence we obtained from (3. 6) the

Lemma I. *The linear system* (4. 1) *is solvable (uniquely) and the solving ξ_h-system is such that*

$$|\alpha_j|^\mu|\varphi^{(\mu)}(\alpha_j)| \leqq 2^{\mu^2}(\mu+1)^{\frac{\mu}{2}}(m+1)^{\binom{\mu+1}{2}}\left(\sum_{l=0}^{\mu}|A_{lj}|^2\right)^{\frac{1}{2}},$$
$$\mu=0,1,\dots,p-1, \quad j=1,2,\dots,n.$$

5. Now we have to represent $\varphi(z)$ in terms of the values $\varphi^{(\mu)}(\alpha_j)$. We introduce the notation

$$(5.1)\qquad \omega(z) = \prod_{j=1}^{n}(z-\alpha_j).$$

$$(5.2)\qquad l_j(z) = \frac{\omega(z)}{\omega'(\alpha_j)(z-\alpha_j)}, \qquad j=1,\dots,n$$

then we can represent $\varphi(z)$ in the form

$$\varphi(z) = \sum_{j=1}^{n} \{c_{0j} + c_{1j}(z-\alpha_j) + \dots + c_{p-1,j}(z-\alpha_j)^{p-1}\} l_j(z)^p \tag{5. 3}$$

with suitable coefficients $c_{\mu j}$. We have evidently

$$c_{0j} = \varphi(\alpha_j); \qquad j = 1, \dots, n \tag{5. 4}$$

as to the determination of the other coefficients it suffices to assume $j=1$. The determining relations are

$$\frac{1}{\mu!}\varphi^{(\mu)}(\alpha_1) = \sum_{r=1}^{\mu} \frac{1}{r!}\left(\frac{d^r}{dz^r} l_1^p(z)\right)_{z=\alpha_1} \cdot c_{\mu-r,1} + c_{\mu 1}, \qquad \mu = 0, 1, \dots, p-1. \tag{5. 5}$$

For the estimation of these coefficients $c_{\mu 1}$ we shall need the

Lemma II. *For $j = 1, 2, \dots, n$ and all non-negative integers r the inequality*

$$\left|\frac{d^r}{dz^r} l_j(z)^p\right|_{z=\alpha_j} \leqq \frac{p^r}{\delta^r}\binom{p(n-1)}{r}$$

holds, if

$$\min_{\mu \neq \nu} |\alpha_\mu - \alpha_\nu| \geqq \delta (\leqq 1). \tag{5. 6}$$

Again it suffices to take $j=1$ and $r>0$; we shall prove this by induction with respect to p. For $p=1$ we have for all positive r's

$$\left(\frac{d^r}{dz^r} l_1(z)\right)_{z=\alpha_1} = \sum \frac{1}{(\alpha_1-\alpha_{i_1})(\alpha_1-\alpha_{i_2})\dots(\alpha_1-\alpha_{i_r})}$$

where the summation is extended over all indices $(i_1, \dots, i_r)$ for which

$$2 \leqq i_1 < i_2 < \dots < i_r \leqq n.$$

Hence from (5. 6) we have

$$\left|\frac{d^r}{dz^r} l_1(z)\right|_{z=\alpha_1} \leqq \frac{1}{\delta^r}\binom{n-1}{r} \leqq \frac{p^r}{\delta^r}\binom{n-1}{r}$$

which proves our assertion for $p=1$.

Suppose now the lemma is proved for $p \leqq p_0 - 1$, $p_0 \geqq 2$ with all positive r's. LEIBNIZ's rule gives

$$\left(\frac{d^r}{dz^r} l_1(z)^{p_0}\right)_{z=\alpha_1} = \sum_{k=0}^{r}\binom{r}{k}\left\{\frac{d^k}{dz^k} l_1(z)^{p_0-1}\right\}_{z=\alpha_1} \cdot l_1^{(r-k)}(\alpha_1).$$

Therefore from the induction hypothesis

$$\left|\frac{d^r}{dz^r} l_1(z)^{p_0}\right|_{z=\alpha_1} \leqq \sum_{k=0}^{r} \binom{r}{k} \frac{1}{\delta^{r-k}} \binom{n-1}{r-k} \frac{(p_0-1)^k}{\delta^k} \binom{(p_0-1)(n-1)}{k} =$$

$$= \frac{1}{\delta^r} \sum_{k=0}^{r} \binom{r}{k} (p_0-1)^k \binom{n-1}{r-k} \binom{(p_0-1)(n-1)}{k} \leqq p_0^r \frac{1}{\delta^r} \sum_{k=0}^{r} \binom{n-1}{r-k} \binom{(p_0-1)(n-1)}{k} =$$

$$= \left(\frac{p_0}{\delta}\right)^r \text{coeffs. } z^r \text{ in } (1+z)^{n-1}(1+z)^{(p_0-1)(n-1)} = \left(\frac{p}{\delta}\right)^r \binom{p_0(n-1)}{r}$$

indeed.

From (5. 5), using the abbreviation

$$\frac{1}{r!}\left(\frac{d^r}{dz^r} l_1(z)^p\right)_{z=\alpha_1} = \gamma_r, \tag{5. 7}$$

we get for $\mu = 1, 2, \ldots, p-1$:

$$c_{\mu 1} = \begin{vmatrix} \gamma_0 & 0 & \cdots 0 & \frac{1}{0!}\varphi^{(0)}(\alpha_1) \\ \gamma_1 & \gamma_0 & \cdots 0 & \frac{1}{1!}\varphi^{(1)}(\alpha_1) \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{\mu-1} & \gamma_{\mu-2} & \cdots \gamma_0 & \frac{1}{(\mu-1)!}\varphi^{(\mu-1)}(\alpha_1) \\ \gamma_\mu & \gamma_{\mu-1} & \cdots \gamma_1 & \frac{1}{\mu!}\varphi^{(\mu)}(\alpha_1) \end{vmatrix}.$$

Hence again HADAMARD's inequality gives

$$|c_{\mu 1}| \leqq \left(\sum_{\nu=0}^{\mu} \left|\frac{1}{\nu!}\varphi^{(\nu)}(\alpha_1)\right|^2\right)^{\frac{1}{2}} \prod_{j=1}^{\mu} (|\gamma_0|^2 + |\gamma_1|^2 + \ldots + |\gamma_j|^2)^{\frac{1}{2}}. \tag{5. 8}$$

The second product is owing to (5. 7) and Lemma II.

$$< \prod_{j=1}^{\mu} \left\{\sum_{l=0}^{j} \frac{1}{l!^2} \left(\frac{p}{\delta}\right)^{2l} \binom{p(n-1)}{l}^2\right\}^{\frac{1}{2}} \leqq \left(\frac{p}{\delta}\right)^{\binom{\mu+1}{2}} \binom{p(n-1)}{\mu}^{\mu} 2^{\frac{\mu}{2}}$$

while the first factor is owing to Lemma I.

$$\leqq \left\{\sum_{\nu=0}^{\mu} \frac{1}{\nu!^2} \frac{2^{2\nu^2}(\nu+1)^\nu (m+1)^{2\binom{\nu+1}{2}}}{|\alpha_1|^{2\nu}} \sum_{l=0}^{\nu} |A_{l1}|^2\right\}^{\frac{1}{2}} \leqq$$

$$\leqq 2^{(\mu+1)^2}(m+1)^{\binom{\mu+1}{2}} \left(\sum_{l=0}^{\mu} |A_{l1}|^2\right)^{\frac{1}{2}}$$

if only

$$\min_j |\alpha_j| \geqq 1. \tag{5.9}$$

Collecting these, we get for $\mu = 0, 1, \ldots, p-1$

$$|c_{\mu 1}| \leqq \left(\frac{p}{\delta}\right)^{\binom{\mu+1}{2}} \binom{p(n-1)}{\mu}^{\mu} \cdot 2^{\frac{\mu}{2}+(\mu+1)^2} (m+1)^{\binom{\mu+1}{2}} \left(\sum_{l=0}^{\mu} |A_{l1}|^2\right)^{\frac{1}{2}}$$

and hence we have the

Lemma III. *For the* $c_{\mu j}$*-coefficients in* (5. 3) *we have*

$$S_1 \overset{\text{def}}{=} \sum_{\mu=0}^{p-1} \sum_{j=1}^{n} |c_{\mu j}|^2 \leqq$$

$$\leqq p \left(\frac{p}{\delta}\right)^{p(p-1)} \binom{p(n-1)}{p-1}^{2(p-1)} \cdot 2^{3p^2} (m+1)^{p(p-1)} \cdot \sum_{\mu=0}^{p-1} \sum_{j=1}^{n} |A_{\mu j}|^2.$$

6. What we actually need is an upper bound for $S_2 = \sum_{h=1}^{np} |\xi_h|^2$. Since from (4. 2) and (5. 3) we have

$$S_2 = \frac{1}{2\pi} \int\limits_{|z|=1} |\varphi(z)|^2 \, |dz| = \frac{1}{2\pi} \int\limits_{|z|=1} \left| \sum_{j=1}^{n} \sum_{\mu=0}^{p-1} c_{\mu j} (z-\alpha_j)^{\mu} l_j(z)^p \right|^2 |dz| \leqq \tag{6.1}$$

$$\leqq S_1 \frac{1}{2\pi} \sum_{j=1}^{n} \sum_{\mu=0}^{p-1} \int\limits_{|z|=1} |z-\alpha_j|^{2\mu} \, |l_j(z)|^{2p} \, |dz|,$$

using (5. 9) we get

$$|z-\alpha_j|^{2\mu} \leqq (1+|\alpha_j|)^{2\mu} \leqq (2|\alpha_j|)^{2p-2} \leqq (2D)^{2p-2}$$

and

$$|l_j(z)| \leqq \frac{1}{\delta^{n-1}} \prod_{\substack{\nu=1 \\ \nu \neq j}}^{n} (1+|\alpha_\nu|) \leqq \left(\frac{2D}{\delta}\right)^{n-1}$$

if only

$$\max_j |\alpha_j| \leqq D (\geqq 1). \tag{6.2}$$

Hence from these, Lemma III and (6. 1), we get the

Lemma IV. *Under suppositions* (5. 6), (5. 9) *and* (6. 2) *we have for the solution of the system* (4. 1) *the inequality*

$$\sum_{h=1}^{np} |\xi_h|^2 \leqq np^2 \left(\frac{p}{\delta}\right)^{p(p-1)} \binom{p(n-1)}{p-1}^{2(p-1)} 2^{3p^2} (m+1)^{p(p-1)} \cdot$$

$$\cdot \left(\frac{2D}{\delta}\right)^{2p(n-1)} (2D)^{2p-2} \left(\sum_{\mu=0}^{p-1} \sum_{j=1}^{n} |A_{j\mu}|^2\right).$$

7. Now we can turn to the proof of our theorem. Since the left side of the assertion is homogenous in the z_j's we may assume without loss of generality

$$(7.1)\qquad \min_j |z_j| = 1,$$

$$(7.2)\qquad \max_j |z_j| = D(\geqq 1),$$

$$(7.3)\qquad \min_{\mu\neq\nu} |z_\mu - z_\nu| \geqq \delta(\leqq 1).$$

Putting

$$(7.4)\qquad P_j(x) \overset{\text{def}}{=} \sum_{l=0}^{p-1} d_{lj}\, l! \binom{x}{l},$$

$$(7.5)\qquad f(\nu) \overset{\text{def}}{=} \sum_{j=1}^{n} P_j(\nu) z_j^\nu,$$

let ν_0 be defined as one of the indices with $m+1 \leqq \nu_0 \leqq m+np$ and

$$(7.6)\qquad \max_{\nu=m+1,\dots,m+np} |f(\nu)| = |f(\nu_0)|.$$

We consider now the system (4. 1) with the choice

$$(7.7)\qquad \alpha_j = z_j, \qquad j=1,\dots,n$$

$$A_{\mu j} = \mu!\, |z_j|^{\nu_0-m-1} \binom{\nu_0}{\mu} e^{-i\{\operatorname{arc} d_{\mu j} + (m+1)\operatorname{arc} z_j\}} \overset{\text{def}}{=} A^*_{\mu j},$$

$$\mu = 0, 1, \dots, p-1, \quad j=1,2,\dots,n.$$

Denoting the resulting ξ_h's as ξ_h^*'s, using the inequality

$$\sum_{\mu=0}^{p-1} \sum_{j=1}^{n} |A_j^*|^2 \leqq (m+np)^{2(p-1)} \cdot D^{2np} \cdot pn,$$

the application of lemma IV. yields the inequalities

$$(7.8)\qquad \sum_1^{np} |\xi_h^*|^2 \leqq n^2 p^3 \left(\frac{p}{\delta}\right)^{p(p-1)} \binom{p(n-1)}{p-1}^{2(p-1)} 2^{3p^2} (m+np)^{p^2+p-2} \cdot$$

$$\cdot \left(\frac{2D}{\delta}\right)^{2p(n-1)} (2D)^{2(n+1)p} <$$

$$< 2^{3p^2}(m+np)^{p^2+p} p^{p(p-1)+1} \left(\frac{4D}{\delta}\right)^{2p(n-1)+p\max(p-1,2n+2)} \frac{(pn)^{2(p-1)}}{(p-1)!} <$$

$$< \left(\frac{8D}{\delta}\right)^{2p(n-1)+p\max(p-1,2n+2)} (m+np)^{2(p^2+p)} \overset{\text{def}}{=} U.$$

8. Now we start from our system

$$(8.1)\qquad \sum_{h=1}^{np} \xi_k^* z_j^{h-1} \binom{m+h}{\mu} \mu! = A^*_{\mu j} \qquad \begin{matrix} j=1,\dots,n \\ \mu=0,1,\dots,p-1. \end{matrix}$$

Multiplying by z_j^{m+1} we get, using (7. 7)

$$\sum_{h=1}^{np} \xi_h^* z_j^{m+h} \binom{m+h}{\mu} \mu! = \mu! \binom{v_0}{\mu} e^{-i \operatorname{arc} d_{\mu j}} \cdot |z_j|^{v_0}.$$

Multiplying by $d_{\mu j}$ and summing with respect to μ and j and taking into account (7. 4) and (7. 5) we get

$$(8.\,2) \qquad \sum_{h=1}^{np} \xi_h^* f(m+h) = \sum_{j=1}^{n} \sum_{\mu=0}^{p-1} |d_{\mu j}| \binom{v_0}{\mu} \mu! \, |z_j|^{v_0}.$$

Hence from the definition of v_0 in (7. 6) and U in (7. 2) we get

$$\left(\sum_{j=1}^{n} \sum_{\mu=0}^{p-1} |d_{\mu j}| \binom{v_0}{\mu} \mu! \, |z_j|^{v_0} \right)^2 \leqq |f(v_0)|^2 \left(\sum_{h=1}^{np} |\xi_h^*| \right)^2 \leqq |f(v_0)|^2 \, npU.$$

But this gives

$$\max_{m+1 \leqq v \leqq m+np} \frac{|f(v)|}{\sum_{j=1}^{n} \sum_{\mu=0}^{p-1} \left| d_{\mu j} \binom{v}{\mu} \mu! \, z_j^{v} \right|} \geqq \frac{|f(v_0)|}{\sum_{j=1}^{n} \sum_{\mu=0}^{p-1} \left| d_{\mu j} \binom{v_0}{\mu} \mu! \, z_j^{v_0} \right|} \geqq \frac{1}{\sqrt{npU}}$$

from which the assertion readily follows.

References

[1] D. G. Dickson, Expansion in series of solutions of linear difference-differential and infinite order differential equations with constant coefficients, *Mem. Amer. Math. Soc.* **23** (1957), 1—72.

[2] Seymour Sherman, A note on stability calculations and time lag, *Quart. Appl. Math.* **5** (1947), 92—97.

[3] L. S. Pontrjagin, On the zeros of some elementary transcendental functions, *Izv. Akad. Nauk SSSR, Ser. Mat.* **6** (1942), 115—134.

[4] A. D. Myskis, Lineare Differentialgleichungen mit nacheilendem Argument, *Berlin*, 1955.

[5] S. R. E. Langer, On the zeros of exponential sums and integrals, *Bull. Amer. Math. Soc.* **37** (1931), 213—239.

[6] Б. Я. Левин, Распределение корней целых функций, *Москва* 1956.

[7] R. Bellmann and K. L. Cooke, Differential-difference equations, *London*, 1963.

[8] S. Karlin and G. Szegő, On certain differential-integral equations, *Math. Z.* **72** (1960), 205—228.

[9] G. Pólya, Über die Nachstellen sukzessiver Deriwierten, *Math. Z.* **12** (1922), 36—60.

[10] P. Turán, On the distribution of zeros of general exponential polynomials, *Publ. Math. Debrecen* **7** (1960), 130—136.

[11] P. Turán, Eine neue Methode der Analysis und ihre Anwendungen, *Budapest*, 1953. The Chinese edition appeared in 1956. A completely rewritten English edition will appear in the Interscience Tracts series.

(Received December 28, 1963.)