On the distribution of values of a class of entire functions II

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1. As stated in I., we shall deal in this paper with the distribution of values of functions of the type

(1. 1) $$f(z) = \sum_{j=1}^{n} P_j(z) e^{\omega_j z} \qquad (z = x+iy)$$

where the ω_j's are different complex numbers, $n \geqq 2$ and $P_j(z) \not\equiv 0$ polynomials of degree $\leqq p-1$; this amounts to the same as to study the distribution of zeros, since for any complex λ $(f(z)-\lambda)$ has again the form (1. 1) generally with $(n+1)$ instead of n. We are going to have the following

Theorem I. *If*

(1. 2) $$\max_{\mu \neq \nu} |\omega_\mu - \omega_\nu| \leqq M_1$$

(1. 3) $$\min_{\mu \neq \nu} |\omega_\mu - \omega_\nu| \geqq \Delta$$

then the number of zeros of $f(z)=0$ *in the square*

(1. 4) $$A \leqq x \leqq A+L, \quad B \leqq y \leqq B+L$$

cannot exceed the quantity

(1. 5) $$\psi_1(n, p, L, M_1, \Delta) \overset{\text{def}}{=} 9p^2 \log 2np + 5pn + LM_1\left(4+\frac{p}{10n}\right) + p(2n+p)\log\left(1+\frac{5n^2 p}{L\Delta}\right).$$

It seems to us somewhat remarkable that the upper bound is independent of the coefficients of the polynomials $P_j(z)$, on the position of the square (1. 4) and loosely dependent on the ω_j-exponents. Now it would be still more desirable to get rid of the dependence on Δ (if possible).

Since the zeros of $f(z)=0$ and $e^{c_1 z} f(z)=0$ are identical and n distinct points on the plane with the maximal distance M_1 can always be covered by a circle with the radius $M_1 \frac{\sqrt{2}}{2} \overset{\text{def}}{=} M$, we may suppose instead of

(1. 2) with $\omega_j + c_1 = \tilde{\omega}_j \qquad (j=1, 2, \ldots, m)$ the inequality

(1. 6) $$\max_j |\tilde{\omega}_j| \leqq M$$

and it is enough to give for the number of zeros of $f=0$ with $\tilde{\omega}_j$ instead of ω_j in the square (1. 4) under (1. 3)—(1. 6) the majorant

$$\psi(n, p, L, M, \Delta) \overset{\text{def}}{=} 9p^2 \log 2np + 5pn + + LM\left(5+\frac{p}{10n}\right) + p(2n+p)\log\left(1+\frac{5n^2p}{L\Delta}\right). \tag{1. 7}$$

(For matter of convenience we shall use ω_j instead of $\tilde{\omega}_j$.)

As remarked by Professor M. MARDEN in a conversation, introducing the polynomial

$$\Omega(x) \overset{\text{def}}{=} \prod_{\nu=1}^{n} (x-\omega_\nu) \tag{1. 8}$$

and its discriminant d, and observing that

$$|d| = \prod_{1\le\mu<\nu\le n} |\omega_\mu - \omega_\nu| = \prod_{1\le\mu<\nu\le n} |\tilde{\omega}_\mu - \tilde{\omega}_\nu|$$

we have from (1. 6)

$$\min_{\mu\ne\nu} |\omega_\mu - \omega_\nu| > \frac{|d|}{(2M)^{\binom{n}{2}-1}} = \frac{|d|}{(M_1\sqrt{2})^{\binom{n}{2}-1}}$$

and hence Δ in (1. 3) can be chosen as $|d|(M_1\sqrt{2})^{-\binom{n}{2}+1}$. Thus supposing we have already proved (1. 7) (i. e. also (1. 5)), it follows that in Theorem I. an upper bound for the number of zeros can be given by

$$\psi_2(n, p, L, M_1, |d|) \overset{\text{def}}{=} 9p^2 \log 2np + 5pn + + LM_1\left(4+\frac{p}{10n}\right) + p(2n+p)\log\left(1+\frac{5n^2p}{L|d|}(M_1\sqrt{2})^{\binom{n}{2}-1}\right). \tag{1. 9}$$

This formulation, applied to the homogeneous linear differential equation

$$F(y) \overset{\text{def}}{=} y^{(N)}(t) + a_1 y^{(N-1)}(t) + \dots + a_N y(t) = 0, \qquad (N \geqq 2) \tag{1. 10}$$

with constant coefficients, gives immediately

$$\psi_3(N, L, M_2, d) = 32N^2 \log N + 5LM_2 + 3N^2 \log\left\{1+\frac{5N^3}{L|d|}(M_2\sqrt{2})^{\binom{N}{2}-1}\right\} \tag{1. 11}$$

as an upper bound for the number of zeros of any solution of the equation (1. 10) in a square of side L, if d stands for the discriminant of $\varphi(x) = x^N + a_1x^{N-1} + \dots + a_N$ and M_2 means any upper bound for the distance of any two of the zeros of $\varphi=0$ expressible in various ways by a_j's.

We remark an immediate corollary of theorem I. In the book of BELLMAN and COOKE quoted in our first paper, in connection with the treatment of systems of linear difference-differential equations the fundamental role of the distribution of zeros of the equation

$$\det\left\{\sum_{\nu=0}^{m} (A_\nu s + B_\nu)e^{-w_\nu s}\right\} = 0 \tag{1. 12}$$

is pointed out; here A_ν and B_ν stand for $l\times l$ matrices with arbitrary complex entries. Theorem I. gives the possibility to prove at once the analogon of theorem I. for the equation (1. 12), even for complex w_v's. The role of p will be played obviously by $l+1$, that of the ω_j's by the different ω ones among the numbers

$$j_1 w_1 + j_2 w_2 + \dots + j_m w_m$$

$$j_1 + j_2 + \dots + j_m = l, \ (j_\nu \text{ nonnegative integers})$$

that of n by $\binom{l+m-1}{m-1}$. As to M_1 and Δ we have to consider the set of the non-vanishing numbers among those of the form

$$(j_1' - j_1'')w_1 + \dots + (j_m' - j_m'')w_m$$

$$\sum_{\nu=1}^{m} j_\nu' = \sum_{\nu=1}^{m} j_\nu'' = l \qquad (j_\nu', j_\nu'' \text{ nonnegative integers}), \tag{1. 13}$$

and let M_1^* resp. Δ^* stand for the absolute maximum resp. minimum of the numbers in (1. 13). Then theorem I. gives at once the

Corollary. The number of zeros of the equation (1. 12) in the square (1. 4) cannot exceed the quantity

$$\psi_1\left(\binom{l+m-1}{m-1}, l+1, L, M_1^*, \Delta^*\right)$$

with ψ_1 in (1. 5).

2. The proof will be based, as mentioned in the first paper of this series, on the following theorem, which was proved there and which we reproduce here for the reader's convenience.

If

$$\frac{\min\limits_{\mu\neq\nu} |z_\mu - z_\nu|}{\min\limits_{j} |z_j|} \geqq \delta \, (\leqq 1) \tag{2. 1}$$

and

$$\frac{\max\limits_{j} |z_j|}{\min |z_j|} \leqq D \, (\geqq 1) \tag{2. 2}$$

then for arbitrary complex $d_{\mu j}$'s, and positive integers m, n and p the inequality

$$\max_{\nu=m+1,\dots,m+np} \frac{\left|\sum\limits_{j=1}^{n} z_j^\nu \left\{\sum\limits_{\mu=0}^{p-1} d_{\mu j}\mu! \binom{\nu}{\mu}\right\}\right|}{\sum\limits_{j=1}^{n} |z_j^\nu| \sum\limits_{\mu=0}^{p-1} |d_{\mu j}|\mu! \binom{\nu}{\mu}} \geqq (m+np)^{-2p^2} \left(\frac{\delta}{8D}\right)^{2pn+\frac{p(p-1)}{2}} \tag{2. 3}$$

holds.

For the intended application we first get rid of the fact that m is an integer. If m_1 is positive then applying (2. 3) with $m=[m_1]$ we get-replacing for the time being the expression behind max by V-obviously

$$\max_{m_1 \leqq v \leqq m_1+np} V \geqq (m_1+np)^{-2p^2}\left(\frac{\delta}{8D}\right)^{2pn+\frac{p(p-1)}{2}}, \quad v \text{ integer.} \tag{2. 4}$$

Now let $a, d>0$ and the complex w_j's be such that

$$\frac{\min_{\mu \neq v}\left|e^{\frac{dw_\mu}{np}}-e^{\frac{dw_v}{np}}\right|}{\min_j\left|e^{\frac{dw_j}{np}}\right|} \geqq \delta\,(\leqq 1) \tag{2. 5}$$

and

$$e^{\frac{d}{np}\max_{\mu,v}\operatorname{Re}(w_\mu-w_v)} \leqq D\,(\geqq 1). \tag{2. 6}$$

Then (2. 4) is applicable with

$$z_j=e^{\frac{dw_j}{np}} \qquad j=1,2,\ldots,n$$

$$m_1=\frac{apn}{d};$$

hence

$$\max_{a \leqq \frac{dv}{pn} \leqq a+d} \frac{\left|\sum_{j=1}^{n} e^{w_j\frac{dv}{np}} \sum_{\mu=0}^{p-1} d_{\mu j}\,\mu!\binom{\frac{np}{d}\cdot\frac{dv}{np}}{\mu}\right|}{\sum_{j=1}^{n}\left|e^{w_j\frac{dv}{np}}\right| \sum_{\mu=0}^{p-1}|d_{\mu j}|\,\mu!\binom{\frac{np}{d}\cdot\frac{dv}{np}}{\mu}} \geqq \left(\frac{1}{np}\cdot\frac{d}{a+d}\right)^{2p^2}\left(\frac{\delta}{8D}\right)^{2np+\frac{p(p-1)}{2}}, \quad v \text{ integer}$$

and a fortiori

$$\max_{a \leqq x \leqq a+d} \frac{\left|\sum_{j=1}^{n} e^{w_jx} \sum_{\mu=0}^{p-1} d_{\mu j}\,\mu!\binom{\frac{np}{d}x}{\mu}\right|}{\sum_{j=1}^{n}|e^{w_jx}| \sum_{\mu=0}^{p-1}|d_{\mu j}|\,\mu!\binom{\frac{np}{d}x}{\mu}} \geqq \left(\frac{1}{np}\,\frac{d}{a+d}\right)^{2p^2}\left(\frac{\delta}{8D}\right)^{2pn+\frac{p(p-1)}{2}}. \tag{2. 7}$$

3. Now we can turn to the proof of our theorem. Since our upper bound for the number N of zeros is independent of the coefficients and the number of zeros of $g(z)$ and $g(z)e^{Hz}$ is identical we may suppose without loss of generality that our square is

$$0 \leqq x \leqq L, \quad -\frac{L}{2} \leqq y \leqq \frac{L}{2}. \tag{3. 1}$$

Let for $\mu \neq \nu$ be

$$K_{\mu\nu} \overset{\text{def}}{=} \omega_\mu - \omega_\nu \overset{\text{def}}{=} |\omega_\mu - \omega_\nu| e^{i\varphi_{\mu\nu}} \qquad (-\pi < \varphi_{\mu\nu} \leqq \pi). \tag{3. 2}$$

These $K_{\mu\nu}$-vectors are obviously such that with a $K_{\mu\nu}$ also $-K_{\mu\nu}$ $(=K_{\nu\mu})$ occurs; hence there is an α with $-\pi < \alpha < \pi$ such that the two sectors

$$\left|\varphi - \left(\alpha \pm \frac{\pi}{2}\right)\right| < \frac{2\pi}{n(n-1)}$$

do not contain any of our $K_{\mu\nu}$'s. In other words, for each $\varphi_{\mu\nu}$-argument one of the inequalities

$$-\frac{\pi}{2} + \frac{\pi}{n(n-1)} \leqq \varphi_{\mu\nu} - \alpha \leqq \frac{\pi}{2} - \frac{\pi}{n(n-1)} \quad \operatorname{mod}\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right), \tag{3. 3}$$

$$\frac{\pi}{2} + \frac{\pi}{n(n-1)} \leqq \varphi_{\mu\nu} - \alpha \leqq \frac{3\pi}{2} - \frac{\pi}{n(n-1)} \quad \operatorname{mod}\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right). \tag{3. 4}$$

holds. Now we choose with this α

$$w_j = \omega_j e^{-i\alpha} \qquad j = 1, 2, \ldots, n. \tag{3. 5}$$

Further wo choose

$$a = d = \frac{L}{10}. \tag{3. 6}$$

In order to make (2. 7) applicable we have first to get explicit values for δ and D in (2. 5) and (2. 6). With our choice of w_j's we have

$$\frac{\min\limits_{\mu \neq \nu} \left|e^{\frac{dw_\mu}{np}} - e^{\frac{dw_\nu}{np}}\right|}{\min\limits_{\gamma} \left|e^{\frac{dw_\gamma}{np}}\right|} \geqq \min_{\mu \neq \nu} \left|e^{\frac{d(w_\mu - w_\nu)}{np}} - 1\right| = \min_{\mu \neq \nu} \left|e^{\frac{d}{np}|\omega_\mu - \omega_\nu| e^{i(\varphi_{\mu\nu} - \alpha)}} - 1\right|.$$

In case of (3. 3) this is

$$\geqq e^{\frac{d}{np}|\omega_\mu - \omega_\nu| \cos(\varphi_{\mu\nu} - \alpha)} - 1 \geqq e^{\frac{d}{np} \Delta \sin \frac{\pi}{n(n-1)}} - 1 \geqq$$

$$\geqq \frac{d}{np} \Delta \sin \frac{\pi}{n(n-1)} \geqq \frac{2d\Delta}{n^3 p} \geqq \frac{2d\Delta}{n^3 p + 2d\Delta},$$

and in the case (3. 4)

$$\geqq 1 - e^{-\frac{d}{np} \Delta \sin \frac{\pi}{n(n-1)}} \geqq \frac{\frac{2d\Delta}{n^3 p}}{1 + \frac{2d\Delta}{n^3 p}} = \frac{2d\Delta}{n^3 p + 2d\Delta}.$$

Hence, using also (3. 6)

$$\delta = \frac{2d\Delta}{n^3 p + 2d\Delta} = \frac{L\Delta}{5n^3 p + L\Delta} \tag{3. 7}$$

can be chosen; for D in (2. 6) obviously

$$D = e^{\frac{2dM}{np}} = e^{\frac{LM}{5np}} \tag{3. 8}$$

is a permissible choice. Finally as to the coefficients $d_{\mu j}$ we choose them for $j = 1, 2, \ldots, n$ so that

$$\sum_{\mu=0}^{p-1} d_{\mu j}\, \mu! \binom{10\frac{np}{L} z}{\mu} = P_j\left(\frac{L}{2} + ze^{-i\alpha}\right) e^{\frac{L}{2}\omega_j}. \tag{3. 9}$$

Then (2. 7) assures the existence of an x_0 with

$$\frac{L}{10} \leqq x_0 \leqq \frac{L}{5} \tag{3. 10}$$

such that

$$\frac{\left|f\left(\frac{L}{2} + x_0 e^{-i\alpha}\right)\right|}{\sum_{j=1}^{n} \left|e^{\omega_j\left(\frac{L}{2} + x_0 e^{-i\alpha}\right)}\right| \sum_{\mu=0}^{p-1} |d_{\mu j}|\, \mu! \binom{10\frac{np}{L} x_0}{\mu}} \geqq \tag{3. 11}$$

$$\geqq \left(\frac{1}{2np}\right)^{2p^2} \left(\frac{L\Delta}{5n^3 p + L\Delta} \cdot \frac{1}{8} e^{-\frac{LM}{5np}}\right)^{2pn + \frac{p(p-1)}{2}}.$$

4. Now we shall complete the proof by an appropriate application of JENSEN's inequality. Writing

$$\frac{L}{2} + x_0 e^{-i\alpha} = z_0 \tag{4. 1}$$

the circle

$$|z - z_0| \leqq L\sqrt{2} \tag{4. 2}$$

certainly covers our square; hence the number N of the zeros cannot exceed

$$\max_{|z - z_0| \leqq eL\sqrt{2}} \log \left|\frac{f(z)}{f(z_0)}\right|.$$

Denoting shortly by W the denominator on the left of (3. 11) the use of (3. 11) gives the upper bound

$$2p^2 \log 2np + \left(2pn + \frac{p(p-1)}{2}\right) \left\{\frac{LM}{5np} + \right. \tag{4. 3}$$

$$\left. + \log 8 + \log\left(1 + \frac{5n^3 p}{L\Delta}\right)\right\} + \max_{|z - z_0| \leqq Le\sqrt{2}} \log \frac{|f(z)|}{W}.$$

Now let $|f(z)|$ assume its maximum on the disc (4. 2) for $z=z^*$; then we have, using (3. 9),

$$|f(z^*)| \leqq \sum_{j=1}^{n} |P_j(z^*)|\, |e^{\omega_j z^*}| \leqq \tag{4. 4}$$

$$\leqq \sum_{j=1}^{n} |e^{\omega_j z^*}| \cdot \sum_{\mu=0}^{p-1} |d_{\mu j}|\, \mu! \left| \binom{10\frac{np}{L}\left(z^* - \frac{L}{2}\right)e^{i\alpha}}{\mu} \right|.$$

But we may observe first that for $j=1, 2, \ldots, n$

$$|e^{\omega_j z^*}| \leqq |e^{\omega_j z_0}|\, e^{M|z^*-z_0|} \leqq |e^{\omega_j z_0}|\, e^{MLe\sqrt{2}}. \tag{4. 5}$$

Secondly we have (of course if $p \geqq 2$) for $\mu = 1, 2, \ldots, p-1$, using also (3. 10),

$$\left| \frac{\binom{10\frac{np}{L}\left(z^* - \frac{L}{2}\right)e^{i\alpha}}{\mu}}{\binom{10\frac{np}{L}x_0}{\mu}} \right| = \prod_{\nu=0}^{\mu-1} \left| \frac{10\frac{np}{L}\left(z^* - \frac{L}{2}\right)e^{i\alpha} - \nu}{\frac{10np}{L}x_0 - \nu} \right| \leqq \prod_{\nu=0}^{p-2} \frac{40np}{(n-1)p+2} < 80^p.$$

This, (4. 5) and (4. 4) give

$$|f(z^*)| \leqq e^{MLe\sqrt{2}} \cdot 80^p \cdot W$$

i. e.

$$\log \frac{|f(z^*)|}{W} < 4LM + 5p;$$

hence (4. 3) gives the upper bound

$$2p^2 \log 2np + LM\left(4{,}4 + \frac{p}{10n}\right) + 5p + p\left(2n + \frac{p}{2}\right)\left\{\log 8 + \log\left(1 + \frac{5n^3 p}{L\Delta}\right)\right\}.$$

Sprucing up a bit gives the upper bound

$$9p^2 \log 2np + LM\left(5 + \frac{p}{10n}\right) + 5pn + p(2n+p)\log\left(1 + \frac{5n^3 p}{L\Delta}\right). \qquad \text{Q. e. d.}$$

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