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Note on an inequality for regular functions

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Abstract. We determine the best possible constant in an inequality for regular functions due to R. N. Mera.

Inspired by a theorem of G. Pólya and G. Szegö which states that an entire function of exponential type which is bounded on the real axis is bounded along every horizontal line, R. N. Mera [2] published in 1989 the following result.

Proposition. *Let $p > 1$ be a real number. If the function f is regular in $G = \{re^{it} \mid r \geq 0,\, 0 \leq t \leq \pi/(2p)\}$ and satisfies*

$$|f(x)| \leq M_1 \quad \text{for all} \quad x \geq 0 \tag{1}$$

and

$$|f(z)| \leq M_2 \exp(\tau |z|^p) \quad \text{for all} \quad z \in G, \tag{2}$$

where M_1, M_2 and τ are positive constants, then we have for each $y > 0$:

$$|f(x+iy)| \leq \max(M_1, M_2) \exp(\tau \alpha(p) y x^{p-1}), \tag{3}$$

valid for all $x \geq y \cot(\pi/(2p))$, where

$$\alpha(p) = \begin{cases} p\left(1 - \frac{1}{p}\right)^{1-p}, & \text{if } p \notin \mathbb{N}, \\ p, & \text{if } p \in \mathbb{N}. \end{cases}$$

Inequality (3) extends a result given in [1, p. 82], where the Proposition is stated for the special case $p = 1$. For related theorems we refer to [1], [2], and the references therein.

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It is natural to ask whether in inequality (3) the constant $\alpha(p)$ can be replaced by a smaller number. It is the aim of this note to show that inequality (3) can be sharpened for all real numbers $p > 1$ which are not an integer. In particular we determine the best possible constant for all $p \geq 2$. We need the following

Lemma. *Let $p > 1$ be a real number. If*

$$\beta(p) = \max_{0 \leq x \leq \pi/(2p)} \frac{\sin(px)(\cos(x))^{1-p}}{\sin(x)},$$

then

$$\beta(p) = \begin{cases} (\cos(\pi/(2p)))^{1-p}/\sin(\pi/(2p)), & \text{if } 1 < p < 2, \\ p, & \text{if } p \geq 2. \end{cases} \tag{4}$$

Proof. Let

$$f_p(x) = \frac{\sin(px)(\cos(x))^{1-p}}{\sin(x)}. \tag{5}$$

We establish:
If $1 < p < 2$, then $x \mapsto f_p(x)$ is increasing on $[0,\ \pi/(2p)]$, and, if $p \geq 2$, then $x \mapsto f_p(x)$ is decreasing on $[0,\ \pi/(2p)]$. Let

$$F_p(x) = \log(f_p(x)) = \log(\sin(px)) + (1-p)\log(\cos(x)) - \log(\sin(x));$$

then we get

$$\frac{dF_p(x)}{dx} = p\cot(px) + (p-1)\tan(x) - \cot(x) = g_x(p), \text{ say.}$$

Let $x \in (0, \pi/2)$; then we obtain for all $p \in (0, \pi/x)$:

$$\frac{d^2 g_x(p)}{dp^2} = 2x(\sin(px))^{-3}[px\cos(px) - \sin(px)] \leq 0,$$

which implies that $p \mapsto g_x(p)$ is concave on $(0, \pi/x)$. Since $g_x(1) = g_x(2) = 0$, we conclude that

$$g_x(p) \geq 0 \quad \text{for all} \quad p \in [1, 2],$$

and

$$g_x(p) \leq 0 \quad \text{for all} \quad p \in [2, \pi/x).$$

This leads to

$$\frac{dF_p(x)}{dx} \geq 0, \quad \text{if} \quad 1 \leq p \leq 2,$$

and

$$\frac{dF_p(x)}{dx} \le 0, \quad \text{if} \quad 2 \le p < \pi/x. \qquad \square$$

We are now in a position to prove a refinement of inequality (3).

Theorem. *If the assumptions of the Proposition are fulfilled, then we have for each $y > 0$:*

$$|f(x+iy)| \le \max(M_1, M_2)\exp(\tau\beta(p)yx^{p-1}), \tag{6}$$

valid for all $x \ge y\cot(\pi/(2p))$, where $\beta(p)$ is given by (4)*. If $p \ge 2$, then the constant $\beta(p) = p$ is best possible.*

It is shown in [2] that

$$f_p(x) \le \alpha(p) \quad \text{for all} \quad x \in [0, \pi/(2p)], \tag{7}$$

where f_p is given by (5). From (7) we obtain

$$\beta(p) = \max_{0 \le x \le \pi/(2p)} f_p(x) \le \alpha(p).$$

The proof of inequality (6) is the same as the one for (3), with the exception that instead of (7) the sharper inequality

$$f_p(x) \le \beta(p) \quad \text{for all} \quad x \in [0, \pi/(2p)]$$

is used. For completeness the short proof is given here.

Proof of Theorem. We define

$$F(z) = f(z)\exp(i\tau z^p),$$

where z^p denotes the function which attains positive values for positive z. From (1) we conclude

$$|F(x)| = |f(x)|\,|\exp(i\tau x^p)| = |f(x)| \le M_1 \quad \text{for } x \ge 0,$$

and (2) implies

$$\begin{aligned} |F\left(re^{i\pi/(2p)}\right)| &= |f\left(re^{i\pi/(2p)}\right)|\exp(-\tau r^p) \\ &\le M_2\exp(\tau r^p)\exp(-\tau r^p) = M_2 \end{aligned}$$

and

$$|F(z)| \le M_2\exp(2\tau|z|^p) \quad \text{for} \quad z \in G.$$

Applying the theorem of Phragmén–Lindelöf we obtain for $z \in G$:

$$|F(z)| \leq \max(M_1, M_2) = M, \text{ say.} \tag{8}$$

From (8) we get for $z = re^{it} \in G$:

$$|f(z)| \leq M|\exp(-i\tau z^p)| = M \exp(\tau r^p \sin(pt)). \tag{9}$$

Since $z = x + iy = r\cos(t) + ir\sin(t)$ implies

$$r^p \sin(t) = (\cos(t))^{1-p} y x^{p-1}, \tag{10}$$

we conclude from (9), (10) and the Lemma:

$$\begin{aligned} |f(x+iy)| &\leq M \exp\left(\tau \frac{\sin(pt)(\cos(t))^{1-p}}{\sin(t)} y x^{p-1}\right) \\ &\leq M \exp\left(\tau \beta(p) y x^{p-1}\right), \end{aligned}$$

valid for all $x + iy \in G$, that is, $y \geq 0$ and $x \geq y \cot(\pi/(2p))$. It remains to show that the constant $\beta(p) = p$ is best possible if $p \geq 2$. Let $p > 1$; we assume that inequality (6) remains valid if we replace $\beta(p)$ by $\gamma(p) \in (0, p)$. We set $f(z) = \sin(z^p)$; then we obtain

$$|f(x)| \leq 1 \quad \text{for all} \quad x \geq 0,$$

and

$$|f(z)| \leq \exp|z|^p \quad \text{for all} \quad z \in G.$$

Hence, we have $M_1 = M_2 = \tau = 1$. Let $y > 0$ be fixed; putting

$$z^p = (x+iy)^p = u_p(x,y) + iv_p(x,y)$$

we get for all sufficiently large x:

$$\begin{aligned} |f(z)|^2 &= (\sin(u_p(x,y)))^2 + (\sinh(v_p(x,y)))^2 \\ &\leq \exp(2\gamma(p) y x^{p-1}) \end{aligned}$$

and

$$\begin{aligned} &|\exp(v_p(x,y) - \gamma(p) y x^{p-1}) - \exp(-v_p(x,y) - \gamma(p) y x^{p-1})| \\ &\qquad = 2|\sinh(v_p(x,y))| \exp(-\gamma(p) y x^{p-1}) \leq 2. \end{aligned} \tag{11}$$

Since

$$\frac{v_p(x,y)}{y x^{p-1}} = p + \sum_{k=1}^{\infty} \binom{p}{2k+1} (-1)^k \left(\frac{y}{x}\right)^{2k}$$

we obtain

$$\lim_{x\to\infty} \frac{v_p(x,y)}{yx^{p-1}} = p,$$

which implies, since $p - \gamma(p) > 0$,

$$\lim_{x\to\infty} (v_p(x,y) - \gamma(p)yx^{p-1}) = \infty.$$

This contradicts inequality (11). Hence, if $p \geq 2$, then inequality (6) is in general not true if $\beta(p)$ is replaced by a smaller number than p. $\square$

Remark. From the proof we conclude that if $p \in (1,2)$, then the best possible constant $\beta^*(p)$ in inequality (6) satisfies $p \leq \beta^*(p) \leq (\cos(\pi/(2p)))^{1-p}/\sin(\pi/(2p))$. It remains an open problem to determine the exact value of $\beta^*(p)$ for $p \in (1,2)$.

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References

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