Asymptotic distribution of zeros of exponential sums

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1. Introduction

This study is concerned with the distribution in the complex plane of the zeros of an exponential sum of the form

$$f(z) = \sum_{j=1}^{n} A_j z^{m_j}[1+\varepsilon_j(z)]e^{\omega_j z} \tag{1}$$

where $n>1$; the A_j and ω_j are complex numbers such that $A_j \neq 0$ and the ω_j are distinct; the m_j are non-negative integers; the functions ε_j are analytic for $|z| \geqq r_0 > 0$ with $\lim_{z\to\infty} \varepsilon_j(z) = 0$. For $r > r_0$, let $n(r)$ denote the number of zeros of f having modulus in the interval $(r_0, r]$, each zero being counted according to its multiplicity. Let L be the perimeter of the convex hull Q of the set $\{\overline{\omega}_j\}_{j=1}^n$, where L is to be taken as twice the length of Q if Q reduces to a line segment. If f is entire, Q is the indicator diagram of f. The most important special case of (1) is

$$f(z) = \sum_{j=1}^{n} P_j(z)e^{\omega_j z}, \tag{2}$$

where the P_j are polynomials that are not identically zero. In this case we may let $r_0 = 0$.

Professor TURÁN has pointed out that PÓLYA [7, p. 594] stated that for (2)

$$n(r) = Lr/(2\pi) + O(1), \tag{3}$$

while indicating that no formal proof had been given up to that time. The purpose of this paper is to show that (3) holds for functions of the form (1).

In [6], Pólya established (3) with $O(1)$ replaced by $O(\log r)$ for the case (2) and has since then sketched an unpublished proof of (3) for this case, the proof being based on the work of SCHWENGELER [8]. Schwengeler himself showed that (3) followed for the special case of (2) which occurs when f is asymptotically an exponential sum involving exactly two ω_j's in the logarithmic strips (to be described) that contain the zeros of f. In the case that the P_j are constants and the ω_j are real, (3) follows from the work of C. E. WILDER [11, pp. 420—422]. A proof of (3) when the m_j are zero and the coefficients are asymptotically constant is sketched by TAMARKIN in [9]; the proof is based on his work in [10]. LEVIN [5, p. 297] shows that (3) holds for a general class of functions which, while not including all functions of the form (2), does include such functions for which the P_j are constants. Roughly, Levin's functions are limits in an appropriate topology of functions of the form $\sum_{j=1}^{\infty} A_j \exp(\omega_j z)$ where $\sum_{j=1}^{\infty} |A_j| < \infty$ and the ω_j are bounded.

As shown by LANGER in [4], the study of (1) can be reduced to the study of (2) where the P_j are constants and the ω_j are real. The result of Wilder that implies (3) in this special case was used by the author in [2] to obtain a similar result concerning the distribution of zeros of (1). Here, we apply that result to obtain (3) for functions of the form (1). We state the result as a theorem.

Theorem. *If f is an exponential sum of the form* (1), *then* $n(r) = Lr/(2\pi) + O(1)$.

The studies mentioned above are all concerned with the distribution of zeros of large modulus, and the techniques employed are similar. Using quite different techniques, DANCS and TURÁN [1] have obtained an upper bound on the number of zeros of (2) in a square of side S, the bound depending upon the maximum degree of the polynomial coefficients, n, S, $\max|\omega_i - \omega_j|$, and $\min|\omega_i - \omega_j|$ where $i \neq j$. In particular, the bound is independent of the coefficients in the polynomials, and a simple translation argument shows that the result holds for *all* squares of side S in the plane.

2. Preliminaries

Let f be given by (1). It has been shown (for example, see [3, p. 15]) that all but a finite number of the zeros of f of large modulus greater than r_0 are contained in a finite number (at most n) of strips of bounded width with boundary curves asymptotically logarithmic. Each strip tends to infinity in a direction that is asymptotically logarithmic to an exterior normal to Q. Several strips may be associated with each such normal direction. There exists a $K > 0$, depending only on f, such that these strips may be given in the form

$$V_p = \{z; \mathfrak{I}(ze^{-i\Phi_p}) \geqq 0, \ |\mathscr{R}(ze^{-i\Phi_p}) + \mu_p \log|z|| \leqq K\},$$

$p = 1, \ldots, m \leqq n$; where the Φ_p are in $[-\pi/2, 3\pi/2)$, and the μ_p are real. The Φ_p are arguments of differences of consecutive vertices of Q and are not necessarily mutually distinct. The subsets of the V_p containing z's of sufficiently large modulus are individually connected and mutually disjoint with respect to p. These facts can easily be verified using the simplified notation of the next section.

For each $p = 1, \ldots, m$ let

$$F_p(z) = \mathfrak{I}(ze^{-i\Phi_p}) + \mu_p \arg z, \quad \text{where} \quad \arg z \in (\Phi_p, \Phi_p + \pi),$$

$$E_p(z) = \mathscr{R}(ze^{-i\Phi_p}) + \mu_p \log|z|.$$

For each pair of real numbers (α, s) with $s > 0$, let

$$R_p(\alpha, s) = \{z; \alpha \leqq F_p(z) \leqq \alpha + s, \ |E_p(z)| \leqq K\}.$$

We will examine such sets more closely in the next section. If T is a set in the plane, $N(T)$ will denote the number of zeros of f, counted according to their multiplicities, in T. It follows from Theorem 2 of [2], that there is an $\alpha_0 > 0$ and numbers $L_p > 0$ with $\sum_{p=1}^{n} L_p = L$ such that whenever $\alpha \geqq \alpha_0$ and $s > 1$,

$$|N(R_p(\alpha, s)) - sL_p/(2\pi)| < n. \tag{4}$$

The sum of the L_p associated with a single exterior normal to Q is the length of the corresponding side of Q. The theorem actually provides a better bound which depends on p, but this fact will not be of use to us here.

3. Proof of the theorem

Initially we will examine sets V_p and $R_p(\alpha, s)$ for a fixed p using a change of variable. Let $z' = x' + iy' = z \exp(-i\Phi_p)$, and then suppress the primes in the notation. V_p and $R_p(\alpha, s)$ then have the forms

$$V_p = \{z;\, y \geqq 0,\, |x + \mu_p \log|z|| \leqq K\}$$

$$R_p(\alpha, s) = \{z;\, \alpha \leqq y + \mu_p(\Phi_p + \arg z) \leqq \alpha + s,\, |x + \mu_p \log|z|| \leqq K\},$$

where $\arg z$ is in $(0, \pi)$. It is easy to verify [3, p. 28] that $|y/x| \to \infty$ and $\arg z \to \pi/2$ as $z \to \infty$ in V_p. From these observations we note that the curves $x + \mu_p \log|z| = \pm K$ are asymptotic to the curves $x + \mu_p \log y = \pm K$ and that the curve $y + \mu_p(\Phi_p + \arg z) = \gamma$ is approximated by $y + \mu_p(\Phi_p + \pi/2) = \gamma$ if γ is large. The set $R_p(\alpha, s)$ is then approximately a rectangle of dimensions $2K$ by s if α is large. If α is large, $R_p(\alpha, s) \subset V_p$.

We assert that there exists an $r_1 > 0$ such that if $\varrho > r_1$ and z is in V_p, then $|z| > \varrho$ when $y + \mu_p(\arg z - \pi/2) = \varrho + 1$ and $|z| < \varrho$ when $y + \mu_p(\arg z - \pi/2) = \varrho - 1$. In the respective cases,

$$|z| = [\mu_p(\pi/2 - \arg z) + \varrho \pm 1][1 + (x/y)^2]^{1/2}.$$

Since $|x/y| \to 0$ and $\arg z \to \pi/2$ as $z \to \infty$ in V_p, the assertion follows. Returning to our original notation and letting $v_p = \mu_p(\Phi_p + \pi/2)$, if $\varrho > r_1$ and z is in V_p, then $|z| > \varrho$ when $F_p(z) - v_p = \varrho + 1$ and $|z| < \varrho$ when $F_p(z) - v_p = \varrho - 1$. Since there are only finitely many p, r_1 may be chosen independently of p.

Choose r_2 sufficiently large so that $r_2 > r_0$, $r_2 > r_1$, and the subsets of the V_p composed of points of modulus greater than r_2 are individually connected and mutually disjoint with respect to p. Choose α_1 so that $\alpha_1 \geqq \alpha_0$ and $\alpha_1 > r_2 + v_p + 1$ for all p. By the choice of r_1, it follows that each point of $R_p(\alpha_1, s)$ is of modulus greater than r_2. For if $F_p(z) = \gamma \geqq \alpha_1$, then with $\varrho = \gamma - v_p - 1 > r_1$, it follows that $|z| > \gamma - v_p - 1 > r_2$. By the choice of r_2, such $R_p(\alpha, s)$ are disjoint with respect to p.

Choose an r_3 so that $r_3 > \alpha_1 - v_p + 1$ for all p. For $r > r_3$, define $C_p(r)$ to be the set of all z in V_p for which $F_p(z) \geqq \alpha_1$ and $|z| \leqq r$. Then, except for a bounded number of zeros of f, each zero of f with modulus in $(r_0, r]$ is in precisely one $C_p(r)$. For $r > r_3$, let $R' = R_p(\alpha_1, r + v_p - \alpha_1 - 1)$ and $R'' = R_p(\alpha_1, r + v_p - \alpha_1 + 1)$. Then for $r > r_3$, we assert that $R' \subset C_p(r) \subset R''$. For if z is in R', then $F_p(z) = \gamma$ where γ is in $[\alpha_1, r + v_p - 1]$. With $\varrho = \gamma - v_p + 1 > \alpha_1 - v_p + 1 > r_1$, it follows that $|z| < \gamma - v_p + 1 \leqq r$, and z is in $C_p(r)$. To establish the second inclusion it suffices to show that if $F_p(z) = \gamma > r + \nu_p + 1$, then $|z| > r$. With $\varrho = \gamma - \nu_p - 1 > r_1$, it follows that if $\gamma > r + \nu_p + 1$, then $|z| > \gamma - \nu_p - 1 > r$. As a result,

$$N(R') \leqq N(C_p(r)) \leqq N(R''). \tag{5}$$

Applying (4) with $\alpha = r + \nu_p - 1$ and $s = 2$ when $r > r_3$, we have

$$|N(R_p(r + \nu_p - 1, 2)) - L_p/\pi| < n.$$

Hence $N(R_p(r + \nu_p - 1, 2))$ is bounded uniformly in r, and a fortiori the number q of zeros on its boundary along $F_p(z) = r + \nu_p - 1$ is similarly bounded. Since $N(R') + N(R_p(r + \nu_p - 1, 2)) - q = N(R'')$, it follows that $N(R') = N(R'') + O(1)$.

From (5) it then follows that

$$N(C_p(r)) = N(R'') + O(1). \tag{6}$$

Applying (4) with $\alpha = \alpha_1$ and $s = r + \nu_p - \alpha_1 + 1$ when $r > r_3$,

$$|N(R'') - (r + \nu_p - \alpha_1 + 1) L_p/(2\pi)| < n$$

and

$$N(R'') = r L_p/(2\pi) + O(1).$$

Combining this with (6) and then summing over p, we obtain

$$N(C_p(r)) = r L_p/(2\pi) + O(1),$$

and

$$n(r) = r L/(2\pi) + O(1).$$

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References

[1] S. Dancs and P. Turán, On the distribution of values of a class of entire functions, I and II, in this journal, 257—272.

[2] D. G. Dickson, Zeros of exponential sums, *Proc. Amer. Math. Soc.* (forthcoming).

[3] D. G. Dickson, Expansions in series of solutions of linear difference-differential and infinite order differential equations with constant coefficients, *Mem. Amer. Math. Soc.* **23** (1957), 1—72.

[4] R. E. Langer, On the zeros of exponential sums and integrals, *Bull. Amer. Math. Soc.* **37** (1931), 213—239.

[5] B. J. Levin (Lewin), Nullstellenverteilung ganzer Funktionen, *Berlin*, 1962.

[6] G. Pólya, Geometrisches über die Verteilung der Nullstellen gewisser ganzer transzendenter Funktionen *Sitzungsb. der Bayerischen Akad.* (1920), 285—290.

[7] G. Pólya, Untersuchungen über Lücken und Singularitäten von Potenzreihen, *Math. Z.* **29** (1929), 549—640.

[8] E. Schwengeler, Geometrisches über die Verteilung der Nullstellen spezieller ganzer Funktionen, *Dissertation, Zurich*, 1925.

[9] J. D. Tamarkin, The zeros of certain integral functions, *J. London Math. Soc.* **2** (1927), 66—69.

[10] J. D. Tamarkin, Some general problems of the theory of ordinary linear differential equations and expansions of an arbitrary function in series of fundamental functions, *Math. Z.* **27** (1928), 1—54.

[11] C. E. Wilder, Expansion problems of ordinary linear differential equations with auxiliary conditions at more than two points, *Trans. Amer. Math. Soc.* **18** (1917), 415—442.

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