On the type of integral functions

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1. Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an integral function of order ϱ and lower order λ. It is known ([1], p. 9) that $f(z)$ is of finite order if and only if,

$$\limsup_{n\to\infty} \frac{n\log n}{\log |a_n|^{-1}} = \varrho. \tag{1. 1}$$

Further SHAH ([2], p. 1047) has shown that if $\left|\frac{a_n}{a_{n+1}}\right|$ is a non-decreasing function of n for $n>n_0$, then

$$\liminf_{n\to\infty} \frac{n\log n}{\log |a_n|^{-1}} = \lambda. \tag{1. 2}$$

It is also known ([3], p. 45) that for $0<\varrho<\infty$

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{n}{e\varrho} |a_n|^{\frac{\varrho}{n}} = \begin{matrix}T\\ t\end{matrix} \tag{1. 3}$$

provided that for the inferior limit, $\left|\frac{a_n}{a_{n+1}}\right|$ forms a non-decreasing function of n for $n>n_0$. Here T, $(0\leqq T\leqq\infty)$ and t, $(0\leqq t\leqq\infty)$ are known respectively as the type and lower types of $f(z)$.

We know that a function of perfectly regular growth is of regular growth, i. e., if type and lower type are the same then the order and lower order also coincide. In this paper we obtain some results connecting the type and lower type of two or more than two integral functions and we also show that an integral function of type zero need not be of regular growth.

2. Theorem 1. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an integral function of order ϱ, $(0<\varrho<\infty)$ lower oder λ and type T, $(0<T<\infty)$. If $\left|\frac{a_n}{a_{n+1}}\right|$ is a non-decreasing function of n for $n>n_0$, then $g(z)=\sum_{n=0}^{\infty} b_n z^n$ is an integral function of the same order ϱ, lower order λ,* *and type zero if*

(2. 1) $$|a_n| = \Phi(n)\,|b_n|,$$

where $\varphi(n)$ *is monotone,*

$$\log \Phi(n) = o(n \log n) \quad \text{and} \quad \lim_{n\to\infty} \frac{\log \Phi(n)}{n} = \infty.$$

PROOF.

$$\limsup_{n\to\infty} |b_n|^{\frac{1}{n}} = \limsup_{n\to\infty} [\Phi(n)]^{-\frac{1}{n}} |a_n|^{\frac{1}{n}} = 0.$$

Hence $g(z)$ is an integral function.

Now

$$\left| \frac{b_n}{b_{n+1}} \right| = \left| \frac{\Phi(n+1)}{\Phi(n)} \cdot \frac{a_n}{a_{n+1}} \right|.$$

Since $\left| \frac{\Phi(n+1)}{\Phi(n)} \right|$ is a non decreasing function of n for $n > n_0$, $\left| \frac{b_n}{b_{n+1}} \right|$ represents a nondecreasing function of n for $n > n_0$. Let $g(z)$ be of order ϱ_1, lower order λ_1, then

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log |b_n|^{-1}}{n \log n} = \begin{cases} \dfrac{1}{\lambda_1} \\ \dfrac{1}{\varrho_1} \end{cases}.$$

But

$$\frac{\log |b_n|^{-1}}{n \log n} = \frac{\log \Phi(n)}{n \log n} + \frac{\log |a_n|^{-1}}{n \log n}.$$

Proceeding to the limits and making use of (2. 1) we get

(2. 2) $$\varrho_1 = \varrho, \quad \lambda_1 = \lambda.$$

Since $f(z)$ is of type T, for any $\varepsilon > 0$, we can find $n_0(t)$ such that

(2. 3) $$\frac{n}{e\varrho} |a_n|^{\frac{\varrho}{n}} < (T+\varepsilon) \quad \text{for} \quad n > n_0(t).$$

Now

$$\frac{n}{e\varrho} |b_n|^{\frac{\varrho}{n}} = [\Phi(n)]^{-\frac{\varrho}{n}} \cdot \frac{n}{e\varrho} |a_n|^{\frac{\varrho}{n}}.$$

Again, proceeding to the limit and making use of (2. 1) and of (2. 3), we get

$$\limsup_{n\to\infty} \frac{n}{e\varrho} |b_n|^{\frac{\varrho}{n}} = 0$$

and hence the theorem.

Here is an example on the above theorem.

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ where

$$a_n = \frac{1}{R_0 R_1 \dots R_n}, \qquad R_0=R_1=\dots=R_9=1;$$

let $n_1=10$, $n_{k+1}=(n_k)^{p+q}$ (p, q are integers, $p, q \geqq 2$) for $k=1, 2, 3 \dots$,

$$\log R_n = \delta[n \log n-(n-1)\log(n-1)], \quad \delta>0, \quad \text{for} \quad n_k<n\leqq n_k^p$$

and $\log R_n=\log R_{n_k^p}$ for $n_k^p<n\leqq n_{k+1}$, $k=1, 2, 3, \dots$. Then $f(z)$ is an integral function of order $\frac{p+q}{p\delta}$, lower order $\frac{1}{\delta}$ and type $\frac{p\delta e^{-(p+q+1)}}{(p+q)}$, while $g(z)=\sum_{n=0}^{\infty} a_n e^{-n(\log n)^\alpha} z^n$, $(0<\alpha<1)$, is an integral function of order $\frac{p+q}{p\delta}$ lower order $\frac{1}{\delta}$ and type zero.

It can be very easily seen that $\left|\frac{a_n}{a_{n+1}}\right|$ is a non-decreasing function of n.

Let $\theta(n)=\frac{\log |a_n|^{-1}}{n \log n}$.

then

$$\theta(n_k^p) \sim \frac{\sum_{n=1+n_k}^{n_k^p} \log R_n}{n_k^p \log n_k^p} \sim \delta$$

for sufficiently large k, and

$$\theta(n_{k+1}) \sim \frac{\sum_{n=n_k^p}^{n_{k+1}} \log R_n}{n_{k+1}\log n_{k+1}} \sim \frac{\{n_{k+1}-n_k^p\}\log R_{n_k^p}}{(p+q)n_{k+1}\log n_k} \sim \frac{p\delta}{p+q}.$$

From the assumptions of R_n it is easy to see that $\limsup_{n\to\infty}\theta(n)=\delta$ and $\liminf_{n\to\infty}\theta(n)=$ $=\frac{p\delta}{p+q}$. Therefore $f(z)$ is an integral function of order $\frac{p+q}{p\delta}$ and lower order $\frac{1}{\delta}$.

Let $\psi(n)=\frac{n}{e\varrho}|a_n|^{\frac{\delta}{n}}$.

Since

$$1-\delta\varrho = 1-\delta\cdot\frac{p+q}{p\delta} = -\frac{q}{p}<0$$

it follows that

$$\log\psi(n_k^p) \sim \log\frac{1}{e\varrho}+\log n_k^p-\frac{\varrho}{n_k^p}\sum_{n=0}^{n_k^p}\log R_n \sim$$

$$\sim \log\frac{1}{e\varrho}+p\log n_k-p\delta\varrho\log n_k.$$

Therefore

$$\liminf_{n\to\infty}\psi(n) = 0.$$

Similarly

$$\log \psi(n_{k+1}) \sim \log \frac{1}{e\varrho} + (p+q)\log n_k - \frac{\varrho}{n_{k+1}} \sum_{n=n_k^p+1}^{n_{k+1}} \log R_n \sim$$

$$\sim \log \frac{1}{e\varrho} + (p+q)\log n_k - \frac{\varrho\{n_{k+1} - n_k^p\}\log R_{n_k^p}}{n_{k+1}} \sim \log \frac{1}{e\varrho \cdot e^{p+q}}.$$

Therefore

$$\overline{\lim_{n\to\infty}}\, \psi(n) = \frac{p\delta e^{-(p+q+1)}}{(p+q)}.$$

It can be again very easily seen that $\frac{p\delta e^{-(p+q+1)}}{(p+q)}$ is the superior limit. Therefore $f(z)$ is an integral function of type $\frac{p\delta e^{-(p+q+1)}}{(p+q)}$. Now $g(z)$ satisfies the conditions of the theorem 1. If we move on the same lines as above we can see that order and lower order remain the same while the type becomes zero.

Theorem 2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=0}^{\infty} b_n z^n$ *be two integral functions of the same order* $\varrho (0 < \varrho < \infty)$ *and lower types* t_1, t_2 *respectively. If* $\left|\frac{a_n}{a_{n+1}}\right|$ *and* $\left|\frac{b_n}{b_{n+1}}\right|$ *form non-decreasing functions of n for* $n > n_0$, *then* $f(z) * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$ *is an integral function of lower type t such that*

$$t \geqq 2\sqrt{t_1 t_2}. \tag{2.4}$$

PROOF. We have

$$\limsup_{n\to\infty} |a_n b_n|^{\frac{1}{n}} = 0,$$

therefore $f(z) * g(z)$ is an integral function. Let this function be of order ϱ_1, then

$$\frac{1}{\varrho_1} = \liminf_{n\to\infty} \frac{\log |a_n b_n|^{-1}}{n \log n} \geqq \liminf_{n\to\infty} \frac{\log |a_n|^{-1}}{n\log n} + \liminf_{n\to\infty} \frac{\log |b_n|^{-1}}{n \log n}.$$

Hence

$$\frac{1}{\varrho_1} \geqq \frac{2}{\varrho}. \tag{2.5}$$

Since $f(z)$ and $g(z)$ are of lower types t_1, t_2, therefore for every $\varepsilon > 0$ we can find positive numbers $n_0(\varepsilon), n_0'(\varepsilon)$, such that

$$\frac{n}{e\varrho} |a_n|^{\frac{\varrho}{n}} > (t_1 - \varepsilon) \quad \text{for} \quad n > n_0(\varepsilon) \tag{2.6}$$

and

$$\frac{n}{e\varrho} |b_n|^{\frac{\varrho}{n}} > (t_2 - \varepsilon) \quad \text{for} \quad n > n_0'(\varepsilon). \tag{2.7}$$

Therefore, in view of (2. 5), (2. 6) and (2. 7)

$$\frac{n}{e\varrho_1}|a_n b_n|^{\frac{\varrho_1}{n}} > \frac{\varrho}{\varrho_1}[(t_1-\varepsilon)(t_2-\varepsilon)]^{\frac{1}{2}}$$

for

$$n > \max\{n_0(\varepsilon), n_0'(\varepsilon)\}.$$

Proceeding to limits and making use of (2. 5), we get (2. 4).

Corollary 1. *$[f(z)*g(z)]_m = \sum_{n=0}^{\infty}(a_n b_n)^m z^n$ $(m=1, 2, 3, \ldots)$ is an integral function of lower type t such that $t \geqq 2m\sqrt{t_1 t_2}$, where t_1, t_2 are the lower types of $f(z)$ and $g(z)$ respectively, and $\left|\frac{a_n}{a_{n+1}}\right|, \left|\frac{b_n}{b_{n+1}}\right|$ are non-decreasing functions of n for $n > n_0$.*

Corollary 2. *Let $f_k(z) = \sum_{n=0}^{\infty} a_{n,k} z^n$ represent m integral functions of the same order ϱ and lower types t_k, $(k=1, 2, 3, \ldots, m)$ respectively. If $\left|\frac{a_{n,k}}{a_{n+1,k}}\right|$ $(k=1, 2, \ldots)$ are non-decreasing functions of n for $n > n_0$ then $t \geqq m\,(t_1 t_2 \ldots t_m)^{\frac{1}{m}}$, where t is the lower type of*

$$f_1(z)*f_2(z)*\ldots*f_m(z) = \sum_{n=0}^{\infty} a_{n,1}a_{n,2}\ldots a_{n,m}z^n.$$

Theorem 3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an integral function of order $\varrho (0 < \varrho < \infty)$. If $|a_n|^{\varrho} \sim |b_n|^{\frac{1}{\varrho}}$, then $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is an integral function of order $\frac{1}{\varrho}$ and $f(z)*g(z) = \sum_{n=1}^{\infty} a_n b_n z^n$ is an integral function of type T such that*

$$T = \left(\varrho + \frac{1}{\varrho}\right)\sqrt{T_1 T_2}$$

where T_1 and T_2 are the types of $f(z)$ and $g(z)$ respectively.

Proof. It is obvious that $g(z)$ is of order $\frac{1}{\varrho}$. Let $f(z)*g(z)$ be of order ϱ_1, then

$$\frac{1}{\varrho_1} = \liminf_{n\to\infty}\frac{\log|a_n b_n|^{-1}}{n\log n} = \frac{1+\varrho^2}{\varrho}.$$

Now

$$\frac{n}{e\varrho_1}|a_n b_n|^{\frac{\varrho_1}{n}} \sim n\frac{(1+\varrho^2)}{e\varrho}|a_n|^{\frac{\varrho}{n}}.$$

Therefore,

$$\limsup_{n\to\infty}\frac{n}{e\varrho_1}|a_n b_n|^{\frac{\varrho_1}{n}} = (1+\varrho^2)T_1.$$

Similarly we can show that

$$\limsup_{n\to\infty} \frac{n}{e\varrho_1} |a_n b_n|^{\frac{\varrho_1}{n}} = \frac{1+\varrho^2}{\varrho^2} T_2 .$$

Therefore,

$$T = \left(\varrho + \frac{1}{\varrho}\right) \sqrt{T_1 T_2} .$$

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References

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