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Some fixed point theorems for product spaces

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Abstract. Let $(X, \mathcal{F}, T)$ be a complete probabilistic metric space, Y be a space having the fixed point property and $f : X \times Y \to X \times Y$ be a continuous mapping. In this paper, we prove some fixed point theorems for f which satisfies some conditions in the first variable. The results presented in this paper generalize and develop the fixed point theory on product spaces and probabilistic metric spaces.

1. Introduction

It is well known that the theory of probabilistic metric spaces is a new frontier branch between probability theory and functional analysis and has an important practical back-ground, which contains the ordinary metric spaces as a special case. The fixed point theory in probabilistic metric spaces has been extensively developed in the last thirty years [1]–[6], [8], [11], [15], [17], [19], [20]. On the other hand, the existence of fixed points in product spaces has been studied by some authors [7], [9], [10], [13], [14], [16]. Let $(X, \mathcal{F}, T)$ be a $\mathcal{T}$-complete probabilistic metric space and Y be a space having the fixed point property, $f : X \times Y \to X \times Y$ be a continuous mapping.

In this paper, we prove some fixed point theorems for f which satisfies some conditions in the first variable. Our main results generalize and develope the fixed point theory on product spaces and probabilistic metric spaces.

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2. Preliminaries

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$.

Definition 2.1. A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a *distribution function* if it is nondecreasing and left-continuous with $\inf F(t) = 0$ and $\sup F(t) = 1$.

In what follows we always denote by $\mathcal{D}$ the set of all distribution functions and by H the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.2. A *probabilistic metric space* (briefly, PM-space) is an ordered pair $(X, \mathcal{F})$, where X is a nonempty set and $\mathcal{F}$ is a mapping from $X \times X$ into $\mathcal{D}$. We shall denote the distribution function $\mathcal{F}(x, y)$ by $F_{x,y}$ and $F_{x,y}(t)$ will represent the value of $F_{x,y}$ at $t \in \mathbb{R}$. The function $F_{x,y}$ is assumed to satisfy the following conditions:

(PM-1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$,

(PM-2) $F_{x,y}(0) = 0$,

(PM-3) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in \mathbb{R}$,

(PM-4) if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$, then $F_{x,z}(t_1 + t_2) = 1$.

Definition 2.3. A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if it is satisfied the following conditions:

(T-1) $T(a, 1) = a$,

(T-2) $T(a, b) = T(b, a)$,

(T-3) $T(c, d) \geq T(a, b)$ for $c \geq a$ and $d \geq b$,

(T-4) $T(T(a, b), c) = T(a, T(b, c))$.

Definition 2.4. A *Menger PM-space* is a triplet $(X, \mathcal{F}, T)$, where $(X, \mathcal{F})$ is a PM-space and T is a t-norm satisfying the following triangle inequality:

$$F_{x,z}(t_1 + t_2) \geq T(F_{x,y}(t_1), F_{y,z}(t_2))$$

for all $x, y, z \in X$ and $t_1, t_2 \geq 0$.

Schweizer, Sklar and Thorp [18] have proved that if $(X, \mathcal{F}, T)$ is a Menger PM-space and the t-norm T satisfies $\sup_{t<1} T(t, t) = 1$, then $(X, \mathcal{U})$ is a Hausdorff uniform space with the uniformity $\mathcal{U}$ induced by the family of subsets

$$U_{\epsilon,\lambda} = \{(x, y) \in X \times X : F_{x,y}(\epsilon) > 1 - \lambda\}, \quad \epsilon, \lambda > 0,$$

and therefore, the $(X, \mathcal{F}, T)$ is a Hausdorff space in the topology $\mathcal{T}$ induced by the family of neighborhoods:

$$\{U_p(\epsilon, \lambda) : p \in X, \epsilon > 0, \lambda > 0\},$$

where

$$U_p(\epsilon, \lambda) = U_{\epsilon,\lambda}[p] = \{x \in X : F_{x,p}(\epsilon) > 1 - \lambda\}.$$

Furthermore, the uniformity $\mathcal{U}$ is metrizable.

Let X, Y be two topological spaces and $f : X \times Y \to X \times Y$ be a mapping. In what follows $p_1 : X \times Y \to X$ will denote the first projection mapping defined by $p_1(x, y) = x$ and, for any $(x, y) \in X \times Y$,

$$(p_1 f)^0(x, y) = x, \quad (p_1 f)^n = p_1 f((p_1 f)^{n-1}(x, y), y), \quad n = 1, 2, \ldots.$$

Definition 2.5. Let $(X, \mathcal{U})$ be a Hausdorff uniform space, Y be a topological space and $f : X \times X \to X \times Y$ be a continuous mapping. f is called to have the *property C.U. at* x_0 in X if, for any y in Y, there exists a neighbourhood $V(y)$ of y such that for any $U \in \mathcal{U}$, there exists a positive integer $N = N(y, U)$ such that

$$((p_1 f)^n(x_0, b), (p_1 f)^m(x_0, b)) \in U$$

for all $b \in V(y)$ and $n, m \geq N$.

Lemma 2.1. [9] *Let $(X, \mathcal{U})$ be a complete Hausdorff uniform space and Y be a space having the fixed point property. If the mapping $f : X \times Y \to X \times Y$ has the property C.U. at x_0 in X, then f has a fixed point in $X \times Y$.*

The following lemma can be obtained from Theorem 24 in [12] immediately.

Lemma 2.2. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space with $\sup_{t<1} T(t, t) = 1$. Then $(X, \mathcal{F}, T)$ is $\mathcal{T}$-complete if and only if $(X, \mathcal{U})$ is $\mathcal{U}$- complete, where $\mathcal{U}$ is the uniformity induced by the family of subsets $U_{\epsilon,\lambda}$.*

3. Main Results

Now, we are ready to give our main theorems.

Theorem 3.1. *Let $(X, \mathcal{F}, T)$ be a $\mathcal{T}$-complete Menger PM-space with $\sup_{t<1} T(t,t) = 1$, Y be a space having the fixed point property and $f : X \times Y \to X \times Y$ be a continuous mapping. Suppose that there exists a point x_0 in X such that for any y in Y, there exists a neighbourhood $V(y)$ of y such that*

(1) for any $\lambda > 0$, there exists $t_\lambda > 0$ such that

$$\inf_{\{z \in (p_1 f)^i (x_0, b)\}_{i=1}^{\infty}} F_{x_0,z}(t) > 1 - \lambda \tag{3.1}$$

for all $b \in V(y)$ and $t \geq t_\lambda$,

(2) for any $b \in V(y), x, z \in \{(p_1 f)^i (x_0, b)\}_{i=1}^{\infty}$ and $t > 0$, the following condition holds

$$\begin{aligned} F_{p_1 f(x,b), p_1 f(z,b)}(t) \geq \min\{&F_{x,z}(\Phi_y(t)), F_{x, p_1 f(x,b)}(\Phi_y(t)), \\ &F_{x, p_1 f(z,b)}(\Phi_y(t))\}, \end{aligned} \tag{3.2}$$

where $\Phi_y : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function such that $\lim_{i\to\infty} \Phi_y^i(t) = +\infty$ for all $t > 0$ and Φ_y^i represents the i-th iteration of Φ_y.

Then f has a fixed point in $X \times Y$.

Proof. For all b in $V(y)$, let

$$s_i = s_i(b) = (p_1 f)^i (x_0, b), \quad i = 0, 1, 2, \ldots. \tag{3.3}$$

It follows from (3.2) and (3.3) that for any non-negative integers i, k, $b \in V(y)$ and $t > 0$,

$$\begin{aligned} F_{s_i, s_{i+k}}(t) &= F_{p_1 f(s_{i-1}, b), p_1 f(s_{i+k-1}, b)}(t) \\ &\geq \min\{F_{s_{i-1}, s_{i+k-1}}(\Phi_y(t)), F_{s_{i-1}, s_i}(\Phi_y(t)), \\ &\qquad F_{s_{i-1}, s_{i+k}}(\Phi_y(t))\} \\ &\geq \cdots \\ &\geq \inf_{z \in \{s_j\}_{j=0}^{\infty}} F_{x_0, z}(\Phi_y^i(t)). \end{aligned} \tag{3.4}$$

Let $(X, \mathcal{U})$ be a Hausdorff uniform space with the uniformity $\mathcal{U}$ induced by the family of subsets $U_{\epsilon,\lambda}$. For any $U \in \mathcal{U}$, there exists $\epsilon, \lambda > 0$ such that

$U_{\epsilon,\lambda} \subset U$. Since $\lim_{i\to\infty} \Phi_y^i(t) = +\infty$ for all $t > 0$, there exists a positive integer $N = N(y, U)$ such that

$$\Phi_y^i(\epsilon) < t_\lambda, \quad i \geq N. \tag{3.5}$$

From (3.1), (3.3), (3.4) and (3.5), we have

$$\begin{aligned} F_{s_i,s_{i+k}}(\epsilon) &\geq \inf_{z\in\{s_j\}_{j=0}^\infty} F_{x_0,z}(\Phi_y^i(\epsilon)) \\ &\geq \inf_{z\in\{s_j\}_{j=0}^\infty} F_{x_0,z}(t_\lambda) \\ &> 1-\lambda \end{aligned}$$

for all $i \geq N$, $b \in V(y)$ and $k = 0, 1, 2, \ldots,$ This implies that for any positive integers $n, m \geq N$,

$$((p_1 f)^n(x_0, b), (p_1 f)^m(x_0, b)) = (s_n, s_m) \in U_{\epsilon,\lambda} \subset U$$

for all $b \in V(y)$. Hence f has the property C.U. at x_0 in X. It follows from Lemmas 2.1 and 2.2 that f has a fixed point in $X \times Y$. This completes the proof.

Theorem 3.2. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space with $\sup_{t<1} T(t, t) = 1$, Y be a space having the fixed point property and $f : X \times Y \to X \times Y$ be a continuous mapping. Suppose that there exists a point x_0 in X such that for any y in Y, there exists a neighbourhood $V(y)$ of y such that for any $b \in V(y)$, $x, z \in \{(p_1 f)^i(x_0, b)\}_{i=0}^\infty$ and $t > 0$, the following implication holds:*

$$\min_{u,v\in\{x,z,p_1f(x,b),p_1f(z,b)\}} F_{u,v}(t) > 1-t \tag{3.6}$$

implies

$$F_{p_1f(x,b),p_1f(z,b)}(\varphi_y(t)) > 1 - \varphi_y(t),$$

where $\varphi_y : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing and semicontinuous function from the right and $\varphi_y(t) < t$ for all $t > 0$. Then f has a fixed point in $X \times Y$.

Proof. For any b in $V(y)$, let

$$s_i = s_i(b) = (p_1 f)^i(x_0, b), \quad i = 0, 1, 2, \ldots. \tag{3.7}$$

Since $1 + t > 1$ for all $t > 0$, we have that $F_{x,y}(1 + t) > 1 - (1 + t)$ for all x, y in X. It follows from (3.6) and (3.7) that

$$\min_{u,v\in\{s_i,s_{i+k},p_1f(s_i,b),p_1f(s_{i+k},b)\}} F_{u,v}(1+t) > 1-(1+t)$$

implies

$$F_{s_{i+1},s_{i+k+1}}(\varphi_y(1+t)) > 1 - \varphi_y(1+t)$$

all $i, k \in \{0, 1, 2, \cdots\}$, b in $V(y)$ and $t > 0$, which means that

$$\min_{u,v\in\{s_i,s_{i+k},p_1f(s_i,b),p_1f(s_{i+k},b)\}} F_{u,v}(\varphi_y(1+t)) > 1 - \varphi_y(1+t)$$

for all $i \in \{1, 2, \dots\}$, $k \in \{0, 1, 2, \dots\}$, $b \in V(y)$ and $t > 0$. Continuing this procedure, we have

$$\min_{u,v\in\{s_i,s_{i+k},p_1f(s_i,b),p_1f(s_{i+k},b)\}} F_{u,v}(\varphi_y^{n-1}(1+t)) > 1 - \varphi_y^{n-1}(1+t)$$

for all $i \in \{n-1, n, n+1, \dots\}$, $k \in \{0, 1, 2, \dots\}$, $b \in V(y)$ and $t > 0$, which implies that

$$F_{s_i,s_{i+k}}(\varphi_y^n(1+t)) > 1 - \varphi_y^n(1+t) \tag{3.8}$$

for all $i \in \{n, n+1, \dots\}$, $k \in \{0, 1, 2, \dots\}$, $b \in V(y)$ and $t > 0$.

Let $(X,\mathcal{U})$ be a Hausdorff uniform space with the uniformity $\mathcal{U}$ induced by the family of subsets $U_{\epsilon,\lambda}$. For any $U \in \mathcal{U}$, there exist $\epsilon, \lambda > 0$ such that $U_{\epsilon,\lambda} \subset U$. Since $\varphi_y : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing and semi-continuous function form the right and $\varphi_y(t) < t$ for all $t > 0$, we have $\lim_{i\to\infty} \varphi_y^i(t) = 0$ for all $t > 0$. Hence, there exists a positive integer $N = N(y, U)$ such that

$$\varphi_y^n(1+\epsilon) < \min\{\epsilon, \lambda\} \tag{3.9}$$

for all $n \geq N$. It follows from (3.8) and (3.9) that for all $n \geq N$ and b in $V(y)$,

$$\begin{aligned} F_{s_i,s_{i+k}}(\epsilon) &\geq F_{s_i,s_{i+k}}(\varphi_y^n(1+\epsilon)) \\ &> 1 - \varphi_y^n(1+\epsilon) \\ &> 1 - \lambda \end{aligned}$$

for $i = N, N+1, \dots$ and $k = 0, 1, 2, \dots$, which implies that for any positive integers $n, m \geq N$,

$$((p_1f)^n(x_0, b), (p_1f)^m(x_0, b)) = (s_n, s_m) \in U_{\epsilon,\lambda} \subset U$$

for all $b \in V(y)$. Therefore, f has the property C.U. at x_0 in X. It follows from Lemmas 2.1 and 2.2 that f has a fixed point in $X \times Y$. This completes the proof.

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