Cohesive groups and p-adic integers

By D. W. DUBOIS[1]) (Albuquerque, N. M.)

§ 1. Introduction

The construction of indecomposable torsionfree Abelian groups of rank two, due to PONTRIAGIN ([7]; see also [6] Theorem 19 and [2] p. 151) was generalized by CORNER [8] p. 696 to obtain groups of rank $c = 2^{\aleph_0}$. In § 3 present a construction which slightly generalizes Corner's, produces indecomposable torsionfree groups of rank up to c, and uses an arbitrary set of primes. If this set is the set of all primes, then the groups obtained have a property which was first noticed by SASIADA and JAREK for pure subgroups of $I(p)$, and which we call *cohesiveness*. This very strong property is examined in § 2, where we show that every reduced cohesive group is strongly and absolutely indecomposable and give in Theorem 1 a simple computational criterion for cohesiveness. A corollary is that the quasi-isomorphism class of a cohesive group coincides with its isomorphism class.

Notation. For most terms see FUCHS [2] or KAPLANSKY [6]. Z is the additive group of the rational integers, $I(p)$ the additive group of all p-adic integers. For a group or set S, $|S|$ is the power or cardinal number of S. A subgroup K of a group G is *p-pure in G* if and only if $K \cap p^n G = p^n K$ for $n = 0, 1, 2, \ldots$; K is *pure in G* if and only if it is p-pure for every prime p. In case G is torsionfree and K is a subset of G, then the *pure (p-pure) closure of K in G* is the intersection of all pure (p-pure) subgroups containing K. A *rigid system* is a set S of torsionfree groups such that for all A and B in S, Hom (A, B) is zero if $A \neq B$, and Hom (A, A) has rank one. The height of x at the prime p is denoted by $H_p(x)$. See FUCHS [2], § 42, for discussion of heights and types in torsionfree groups.

§ 2. Cohesive groups

Definition. The torsionfree group G is *cohesive* if and only if for every nonzero pure subgroup S of G, G/S is divisible.

First we give some useful lemmas and examples.

C1. Every divisible torsionfree group is cohesive.

C2. A cohesive group is either divisible or reduced and any reduced cohesive group is indecomposable.

[1]) Research partly supported by the National Science Foundation, Grant GP 2214.

C3. A torsionfree group G is cohesive if and only if every nonzero map of G into a reduced torsionfree group is a monomorphism (zero kernel).

C4. If $G \subset H \subset G^* \subset K$, if G is cohesive, if K is torsionfree, and if G^* is the pure closure of G in K, then H is also cohesive. From this it follows readily that cohesiveness is a quasiisomorphism invariant and that every reduced cohesive group is strongly indecomposable.

C5. Every pure subgroup of a cohesive group is cohesive; thus a reduced cohesive group is absolutely indecomposable, i. e., every pure subgroup is indecomposable.

C6. $I(p)$ is cohesive. For if S is a nonzero pure subgroup of $I(p)$, then it is immediately clear that S contains a unit; this implies $I(p)/S$ is divisible.

C7. The union of a chain of cohesive subgroups of a torsionfree group is cohesive.

C8. In a cohesive group all nonzero elements have types with the same set of infinity places. For suppose that G is cohesive and that a and c are nonzero members with $H_p(a)=\infty$, $H_p(c)=0$. Let A be the pure subgroup generated by a. Then G/A is divisible, so p divides $c+A$ in G/A. This means that there is an x with $px = c+a'$, a' in A. Since every element in A has infinite height at p, p divides a' and hence p divides c. This contradicts $H_p(c)=0$. It follows that de Groot's absolutely indecomposable example G [4], § 7, is not cohesive; for in G there is a set $a_1, a_2, \ldots$ of elements with a_i having infinite height only at the prime p_i, where $a_i \to p_i$ is a one to one correspondence.

C9. If S is a rigid system, none of whose groups is divisible by p, then the subgroup $G(S, a)$ of the direct sum $T=\sum_{A\in S} A$ generated by pT and all members of the form $\sum_{i=1}^{n} a(A_i)$ where a is a fixed function on S with $a(A)$ a member of A that is not divisible by p, and $A_i \in S$, is indecomposable (see Hulanicki [5] and Fuchs [3]) but not cohesive. For the projection of $G(S, a)$ onto a fixed A' in S induces a nonzero map of G into A' that is not a monomorphism (unless S is a singleton set). But A' is a reduced torsionfree group, so by C3 above, $G(S, a)$ is not cohesive.

We shall need the following three trivial lemmas on p-adic integers.

P1. If S is torsionfree and $pS \neq S$ (where p is a prime), then there exist nonzero maps of S into $I(p)$. For let $r=$ rank of S/pS. Then

$$\begin{aligned}\operatorname{Hom}(S, I(p)) &= \operatorname{Hom}(S, \operatorname{Hom}(Z(p^\infty), Z(p^\infty)))\\ &\approx \operatorname{Hom}(S\otimes Z(p^\infty), Z(p^\infty)) \approx \operatorname{Hom}\big(\sum_r Z(p^\infty), Z(p^\infty)\big)\\ &\approx \prod_r \left(\operatorname{Hom}(Z(p^\infty), Z(p^\infty))\right) = \prod_r I(p).\end{aligned}$$

This is zero if and only if $pS=S$.

P2. Every endomorphism of $I(p)$ is a multiplication by a fixed member of $I(p)$. See Fuchs [2], p. 212.

P3. If S is p-pure in $I(p)$, then every map of S into $I(p)$ is the restriction of an endomorphism of $I(p)$ (Armstrong [1]). Fuchs' argument cited above can be generalized to prove this, as follows. Since S (assumed nonzero) is p-pure it contains a unit u. If U is the subgroup generated by u then S/U is divisible by p, while $I(p)$

has no elements of infinite height at p. It follows readily that every map of S into $I(p)$ is determined by its value at u and the assertion is proved.

Now we are ready to prove a computational criterion.

Theorem 1. *Let S be a torsionfree abelian group. The following are equivalent:* (a) *S is cohesive* (b) *for every prime p, either S/pS is zero or S is isomorphic with a p-pure subgroup of $I(p)$;* (c) *for every prime p, either S/pS is zero, or else S has no elements of infinite height at p and S/pS is a cyclic group of order p;* (d) *for every prime p, every nonzero map of S into $I(p)$ is a monomorphism.*

PROOF. (i). If (a) then (b). Let S be cohesive and suppose $pS \neq S$. By P2, there is a nonzero map $\varphi: S \to I(p)$. By C3, φ is a monomorphism. Let the image of φ be written as $p^k S'$, where S' contains a unit $u = u_0 + u_1 p + \ldots$, with $u_0 \neq 0$. Then S' is isomorphic with S. Let px belong to S', with x in $I(p)$, $x = x_0 p^t + x_1 p^{t+1} + \ldots$, and $x_0 \neq 0$. Let X be the pure subgroup of S' generated by px. Then S'/X is divisible, so the coset $u + X$ is divisible by p. Then there is a rational r with rpx in S' and $u + rpx$ divisible by p in S'. Divisibility in $I(p)$ implies $r = k/(np^{t+1})$, with k and n integers prime to p. Hence $kx = np^t(rpx)$ belongs to S'. Take integers a and b with $ak + bp = 1$, so that $x = a(kx) + b(px)$ belongs to S'. Thus S' is p-pure in $I(p)$ and the first implication is proved.

(ii). If (b) then (c). Let S satisfy condition (b) and suppose $pS \neq S$. From (b) we may assume that S is a p-pure subgroup of $I(p)$. Then S has no elements of infinite height at p since $I(p)$ has none. If $u = u_0 + u_1 p + \ldots$ is any unit in S then $u + pS$ generates S/pS. For given x in S choose m with mu_0 congruent to $x_0 \pmod p$ (m an integer). Then p divides $x - mu$. This shows (c).

(iii). If (c) then (d). Let S satisfy condition (c) and let p be a prime. If $pS = S$ then S has no nonzero maps into $I(p)$ so we suppose that S/pS is cyclic, generated by $a + pS$, and that S has no elements of infinite height at p. For every x in S a sequence x_i of members of S and a sequence n_i of integers is uniquely determined by the conditions: $x = n_0 a + px_1,\ 0 \leqq n_0 < p,\ x_i = n_i a + px_{i+1},\ 0 \leqq n_i < p,\ i = 1, 2, \ldots$. Then for every k, $x = \left(\sum_{i=0}^{k} n_i p^i\right) a + p^{k+1} x_{k+1}$ and the correspondence $x \to \sum_{i=0}^{\infty} n_i p^i$ defines a homomorphism φ from S into $I(p)$. The image of φ is p-pure in $I(p)$ since if $py = x$ then $x = 0 \cdot a + px_1$ and $x_1 \varphi = y$, and the kernel of φ is the set of all elements of infinite height at p, hence zero. In view of P2 and P3, the proof of (iii) is complete.

(iv). If (d) then (a). Suppose that S satisfies (d), that $\varphi: S \to T$ is a nonzero epimorphism, and that T is a reduced torsionfree group. Since T is reduced there is a prime p with $pT \neq T$, and so there is a nonzero homomorphism $\psi: T \to I(p)$. Then $\varphi\psi: S \to I(p)$ is not zero. By (d), $\varphi\psi$ is a monomorphism; therefore φ is also a monomorphism. This proves that S is cohesive, i. e., condition (a). The theorem is proved.

G is *quasi-isomorphic* with H if G is isomorphic with a subgroup H' of H which contains a nonzero multiple mH of H.

Corollary. *Let G be quasi-isomorphic with the cohesive group H. Then G is isomorphic with H.*

PROOF. Suppose $mH \subset H' \subset H$ with $m \neq 0$. If m is a prime then it follows immediately from Theorem 1 (c) that H' is either H or mH. An induction shows that in any case $H' = kH$ where k is a factor of m, and hence that H' is isomorphic with H.

Note that the largest possible power of a reduced cohesive group is $c = 2^{\aleph_0}$.

For an example of a p-pure subgroup of $I(p)$ that is not cohesive, let x and y be independent in $I(p)$, let q be a prime different from p, and let G be the p-pure closure in $I(p)$ of the group generated by y and all $q^{-n}x$, $n = 0, 1, 2, \ldots$. Then G is not cohesive since x has infinite height at q while y has finite height at q.

Theorem 2. *Let A be a torsionfree reduced cohesive group of rank at least two. Then $A \otimes A$ is not cohesive.*

PROOF. Let a and b be independent in A, and let F be the free subgroup of A generated by a and b. Then the natural map of $F \otimes F$ into $A \otimes A$ is a monomorphism. In $F \otimes F$ the elements $a \otimes a, a \otimes b, b \otimes a$ and $b \otimes b$ are independent generators, so $a \otimes b - b \otimes a$ is not zero in $A \otimes A$. Now choose a prime p so that $pA \neq A$ and choose g in A with $g + pA$ a generator of A/pA. This is possible by Theorem 1 (note that A is assumed reduced). Let k be a positive integer. By the construction in Theorem 1, part (iii), we can write $a = mg + p^k a'$, $b = ng + p^k b'$. Then clearly $a \otimes b - b \otimes a$ is divisible by p^k. This proves that $H_p(a \otimes b - b \otimes a) = \infty$. Next identify A with a subgroup of $I(p)$ and let φ be the homomorphism of $A \otimes A$ into $I(p)$ defined by the formula: $(x \otimes y)\varphi = xy$. Then φ is not zero and therefore $p(A \otimes A) \neq A \otimes A$, since $I(p)$ has no elements of infinite height at p. Thus it has been shown that $p(A \otimes A) \neq A \otimes A$, but not all (nonzero) elements of $A \otimes A$ have finite height at p. By Theorem 1, $A \otimes A$ is not cohesive.

Theorem 3. *If A and B are cohesive then* Hom (A, B) *is cohesive.*

PROOF. Set $H = \text{Hom}(A, B)$. If either A or B is divisible by a prime p then so is H. Suppose H not divisible by p; then neither A nor B is divisible by p, so each of A and B has, by cohesiveness, no elements of infinite height at p. If φ belongs to H and has infinite height at p in H, then the image of φ in B is divisible by p; hence φ is zero. Thus H contains no elements of infinite height at p. Let φ have height zero at p. Then Im φ is not contained in pB; choose a in A so that $a\varphi$ has height zero at p. Then a likewise has height zero at p. Now we show that $\varphi + pH$ generates H/pH. Let ψ belong to H. Since p does not divide $a\varphi$, $a\varphi + pB$ generates B/pB, whence $a\psi - ma\varphi = pb'$ $(b' \in B)$ for some integer m. For arbitrary x in A, write $x = na + px_1$. Then

$$x(\psi - m\varphi) = p(nb' + x_1\psi - mx_1\varphi).$$

Hence $\psi - m\varphi$ is divisible by p in H. By Theorem 1, H is cohesive.

§ 3. Constructions

We construct a group $G(S)$ where S is a set of functions denoted by π, with values π_p, with the following properties:

S1. $1 \leq |S| \leq c$.

S2. Each function has the same domain, denoted by $D(S)$, a nonempty set of primes.

S3. For each prime p in $D(S)$, the set S_p, where $S_p=\{\pi_p;\pi\in S\}$, is an algebraically independent subset (over the rationals) of $I(p)$; and if $\pi_p=\pi'_p$, then $\pi=\pi'$.

To construct such a set S, let L be a nonempty set of primes and for each p in L let T_p be a well-ordered algebraically independent set of p-adic integers with order type Γ, where Γ is the least ordinal of power c. For each ordinal α with $0\leqq\alpha<\Gamma$ set $\pi_\alpha(p)$ equal to the α^{th} member of T_p. Take an ordinal $\beta\leqq\Gamma$ and set $S=\{\pi_\alpha;\alpha<\beta\}$. Then S satisfies all requirements, $S_p=T_p$, $D(S)=L$, and $|S|=|\beta|$.

Let M be a linearly independent set of reals strictly between 0 and 1, with y in M and $|M|=|S|+1$. Let x be a one to one function mapping S onto $M-\{y\}$. For each π in S and p in $D(S)$ write the standard p-adic power series:

$$\pi_p = \pi_{p0}+\pi_{p1}p+\dots,$$

with $0\leqq\pi_{pi}<p$, and set $\pi_p^0=0$, $\pi_p^n=\sum_{i=0}^{n-1}\pi_{pi}$ for $n>0$. Then $G(S)$ is the group of real numbers generated by y and all $x(\pi)_n^p$, for $n=0,1,2,\dots$ and p in $D(S)$, where

$$x(\pi)_p^n = p^{-n}\big(x(\pi)+\pi_p^n y\big). \tag{1}$$

Note that $\pi_{pk}=p^{-k}(\pi_p^{k+1}-\pi_p^k)$, and that, for $k=0,1,2,\dots$,

$$px(\pi)_p^{k+1} = x(\pi)_p^k+\pi_{pk}y. \tag{2}$$

Members of $G(S)$ are simply those reals that are equal to a formula

$$a'y+\sum_\pi\sum_p\sum_{n=0}^{k(\pi,p,a)} a_{\pi pn}x(\pi)_p^n, \tag{a}$$

where the coefficients a' and $a_{\pi pn}$ are integers and almost all of them are zero; in the sums, π runs over S and p over $D(S)$. If $D(S)$ is a singleton, say $D(S)=\{p\}$, then $G(S)$ is the Fuchs example of type zero, Pontriagin's example if $p=2$. Every element b of $G(S)$ can be written (uniquely) in the form (almost all coefficients are zero)

$$b = b'y+\sum_\pi b_\pi x(\pi) \tag{3}$$

where b' and b_π are rationals.

GS1. *Let the coefficients of b in* (3) *be integers. If p belongs to $D(S)$ then p^t divides b in $G(S)$ if and only if p^t divides $b'-\sum_\pi b_\pi\pi_p^t$ in Z. If p does not belong to $D(S)$, then p^t divides b if and only if p^t divides every coefficient.*

PROOF. First let the coefficients of b be rationals, and suppose b belongs to $G(S)$. Then b is equal to a formula (a). By independence of the set M we get

$$b' = a'+\sum_\pi\sum_p\sum_{n=0}^{k(\pi,p,a)} a_{\pi pn}p^{-n}\pi_p^n,$$

$$b_\pi = \sum_p\sum_{n=0}^{k(\pi,p,a)} a_{\pi pn}p^{-n} \qquad (\pi\in S).$$

Let, for each prime q, W_q be the (logarithmic) q-adic valuation. (If r is a rational, write $r=q^t r'$ where r' is a rational with numerator and denominator prime to q, and set $W_q(r)=t$; then $W_q(r+s) \geqq \min\{W_q(r), W_q(s)\}$, $W_q(rs) = W_q(r)+W_q(s)$.) If q is a prime not in $D(S)$, then both right sides in equations above have nonnegative q-adic valuation, and so $W_q(b')\geqq 0$, $W_q(b_\pi)\geqq 0$ for all π. This proves the second part. Now suppose that q belongs to $D(S)$, let π be a member of S, set $m=k(\pi, q, a)$ and suppose that $m > -W_q(b_\pi)$, $m>0$. Equating the q-adic values of both sides in the last equation above shows that $a_{\pi qm}$ is divisible by q, say $a_{\pi qm}=qd$. If this is substituted in formula (a) and equation (2) is used, a new formula (c) is obtained with $k(\pi, q, c)<k(\pi, q, a)$ and $k(\pi', p, c)=k(\pi', p, a)$ if $(\pi', p)\neq(\pi, q)$. By induction, there exists a formula (a) for b with

$$k(\pi, p, a)\leqq \max\{0, -W_p(b_\pi)\}$$

for all π and all p in $D(S)$.

Now suppose that b has integral coefficients, and that p^t divides b in $G(S)$. Since $x(\pi)_q^0=x(\pi)=x(\pi)_p^0$, the result just obtained guaratees that for some integers a', $a_{\pi pn}$,

$$b = p^t\left(a'y + \sum_{n=0}^{t} a_{\pi pn}x(\pi)_p^n\right),$$

so that the equations expressing independence have the simpler form

$$b' = p^t(a' + \sum_\pi \sum \pi_p^n p^{-n} a_{\pi pn}), \quad b_\pi = \sum_0^t p^{t-n} a_{\pi pn}.$$

Multiply the last equation by π_p^t, sum on π, and subtract from the next to last equation. This gives

$$b' - \sum_\pi b_\pi \pi_p^t = p^t\left(a' + \sum_\pi \sum_{n=0}^{t} a_{\pi pn}p^{-n}(\pi_p^n - \pi_p^t)\right),$$

and p^n divides $\pi_p^n-\pi_p^t$ for all $n\leqq t$. Hence the "only if" part of the first assertion is proved. If, conversely, $b'-\sum_\pi b_\pi\pi_p^t = p^t c$, with integral c, then a straightforward computation shows that

$$b = p^t(cy + \sum_\pi b_\pi x(\pi)_p^t),$$

so p^t divides b in $G(S)$.

GS2. *Every nonzero element of $G(S)$ has finite height at every prime.*

PROOF. It is sufficient to prove that if b, in equation (3), has integral coefficients and infinite height at p, then $b=0$. This is immediate if p is not in $D(S)$. If p belongs to $D(S)$, then, by GS1, p^t divides $b'-\sum_\pi b_\pi\pi_p^t$ for every t. In the p-adic completion of the rationals, therefore,

$$0 = \lim(b' - \sum_\pi b_\pi \pi_p^t) = b' - \sum_\pi b_\pi \pi_p.$$

This equation and the algebraic independence of the set S_q (recall that if $\pi_p=\pi'_p$ then $\pi=\pi'$) imply $0=b'=b_\pi$ for all π, whence $b=0$.

GS3. *If p belongs to $D(S)$ then $G(S)$ is isomorphic with a p-pure subgroup of $I(p)$ lying between the subgroup generated by 1 and S_p and the pure closure (in $I(p)$) of the latter subgroup. Thus $G(S)$ is always indecomposable.*

PROOF. First we show that the coset of y generates $G(S)/pG(S)$, by showing that every generator of $G(S)$ is congruent to a natural multiple of y mod $pG(S)$, where p is any member of $D(S)$. Equation (2) shows this for generators of the form $x(\pi)_p^n$. Let q be a member of $D(S)$ different from p. In equation (2) replace p by q and subtract the result from equation (2). There results

$$q^n x(\pi)_q^n = (\pi_q^n - \pi_p^n)y + p^n x(\pi)_q^n;$$

since q is prime to p, $x(\pi)_q^n$ has the required property (for $n>0$; $x(\pi)_q^0 = x(\pi)_p^0$). Next replace a by $-y$ in the proof of part (iii) of Theorem 1 to get a map φ; since $G(S)$ has no elements of infinite height at p, by GS2, φ is a monomorphism whose image is p-pure in $I(p)$. The remaining assertion is proved by observing that $y\varphi = -1$ and that, from equation (2), $x(\pi)\varphi = \pi_p$ (see the construction of φ in part (iii) of Theorem 1).

GS4. *$G(S)$ is cohesive if and only if $D(S)$ is the set of all primes.*

PROOF. The "if" part follows from GS3 and Theorem 1 (b). Conversely, if p is not in $D(S)$ then, for arbitrary π in S, y and $x(\pi)$ are independent mod $pG(S)$, according to GS1. By Theorem 1 (c), $G(S)$ is not cohesive.

GS5. *If $D(S)$ is finite then $G(S)$ is homogeneous of type zero. If for some π in S, $\pi_{p0} = \pi_p^1 = 1$ for all p in $D(S)$, and if $D(S)$ is infinite, then $G(S)$ is not homogeneous. If for all π in S and all p in $D(S)$, $\pi_{p0} = [(\log p)^{x(\pi)}]$, then $G(S)$ is homogeneous of type zero; $[t]$ is the greatest integer $\leq t$.* (Another condition guaranteeing homogeneity is given by CORNER [8] p. 697.)

PROOF. GS1 implies the first assertion immediately, and with the hypothesis of the second, it implies that $y - x(\pi)$ is divisible by every prime in $D(S)$; since $D(S)$ is infinite, $y - x(\pi)$ has a nonzero type. But y always has type zero. To prove the third assertion we may suppose that $D(S)$ is infinite. Suppose $b = b'y + \sum_\pi b_\pi x(\pi) \neq 0$ has integral coefficients and nonzero type. Then b is divisible by infinitely many primes in $D(S)$ (only finitely many primes outside $D(S)$ can divide b). By GS1, there is an infinite set K of primes such that if p belongs to K, then p divides $b' - \sum_\pi b_\pi \pi_{p0}$. But for all large p in K,

$$|b' - \sum_\pi b_\pi \pi_{p0}| = |b' - \sum_\pi b_\pi [(\log p)^{x(\pi)}]| \leq |b'| + \sum_\pi |b_\pi| \log_p < p.$$

(Recall $0 < x(\pi) < 1$.) Hence there is an infinite subset K' of K such that if p belongs to K' then $b' - \sum_\pi b_\pi \pi_{p0} = 0$. There is a sequence p_n of primes in K' with $\lim p_n = \infty$, and, for every n,

$$b' = \sum_\pi b_\pi [(\log p_n)^{x(\pi)}].$$

Let $m = \max \{x(\pi); b_\pi \neq 0\}$ (this set is not empty since y has type zero) and suppose $x(\pi') = m$. Then the right side is dominated, for large n, by the term $b_{\pi'} [(\log p_n)^m]$ (x is a one-to-one function) so that the right side tends to infinity contrary to the above equation. The contradiction shows that $G(S)$ is homogeneous of type zero.

Theorem 4. *For every cardinal number m with $1 \leqq m \leqq c$ and every characteristic α, there exists a homogeneous cohesive group of type $[\alpha]$ and rank m and a nonhomogeneous cohesive group of rank m.*

PROOF. All that is left to prove is the assertion about type $[\alpha]$. Let $G(S)$ be cohesive (by taking $D(S)$ to be the set of all primes) and homogeneous of type zero, rank m, and let H be the group of all multiples rt, where r belongs to the unique group K of rationals in which 1 appears and has height α, and t belongs to $G(S)$. Then H is still cohesive, rank m, and is homogeneous of type $[\alpha]$.

Theorem 5. *Let W be an algebraically independent (over the rationals) subset of power c in $I(p)$ and let V be a family of power 2^c of pairwise incomparable (by set inclusion) subsets of W, each of power c. For each T in V let T' be a p-pure subgroup of $I(p)$ containing $\{1\} \cup T$ and contained in the pure closure of $\{1\} \cup T$. Then the set $V' = \{T'; T \in V\}$ is a rigid system.*

PROOF. Let T and U be members of V and consider a nonzero map, necessarily a multiplication m_h, from T' to U'. Let t_1 be an arbitrary member of T and let t_2 be either 1 or a member of T different from t_1. There exist u_i in U and rationals a, b, a_i, b_i with

$$t_1 h = a + \Sigma a_i u_i$$

$$t_2 h = b + \Sigma b_i u_i$$

$$t_1/t_2 = (a + \Sigma a_i u_i)/(b + \Sigma b_i u_i).$$

The last equation and the algebraic independence of W imply that t_1 belongs to U. Thus $T \subset U$ so, by hypothesis, $T = U$. The algebraic independence of T now shows that h is rational. This completes the proof.

From this and GS3 we get:

Corollary. *A rigid system $\{G(S(T)); T \in V\}$ is obtained if V satisfies the hypothesis of Theorem 5 and, for each T in V, $(S(T))_p = T$.*

References

[1] J. ARMSTRONG, On *p*-pure subgroups of the *p*-adic integers, *Topics in Abelian Groups*, Chicago, 1963.
[2] L. FUCHS, Abelian groups, *Budapest*, 1958.
[3] L. FUCHS, On a directly indecomposable group of power greater than continuum, *Acta Math. Sci. Hung.* **8** (1957), 453–454.
[4] J. DE GROOT, Indecomposable abelian groups, *Indagationes Math.* **19** (1957), 137–145.
[5] A. HULANICKI, Note on a paper of de Groot, *Indagationes Math.* **20** (1958), 114.
[6] I. KAPLANSKY, Infinite Abelian groups, *Ann Arbor*, 1954.
[7] L. PONTRIAGIN, Theory of topological commutative groups, *Ann. of Math.* **35** (1934), 361–388.
[8] A. L. S. CORNER, Every countable reduced torsionfree ring is an endomorphism ring, *Proc. London Math. Soc.* (3) **13** (1963), 687–710.

(Received March 14, 1964.)