Applications of analytic number theory to the study of type sets of torsionfree Abelian groups, I

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We consider the problem of finding necessary and sufficient conditions that a set T of types be the type set $T(G)=\{Tx; 0 \neq x \in G\}$ of a (torsionfree Abelian) group G, where Tx is the type of x in G. In this first paper, *only the rank two case will be considered.* This restricted problem is studied in the memoir of BEAUMONT and PIERCE [1] (hereafter referred to as B-P) in section 7. They show that $T(G)$ is a τ_0-set (see Definitions below), if G has rank 2, and that if T is a finite τ_0-set containing τ_0 then T is the type set of a rank two group. We give a stronger necessary condition in Theorem 1 and a weaker sufficient condition in Theorem 2, but no necessary and sufficient condition has been found. Example 1 shows that our necessary condition is stronger than that of B-P while Example 2 shows that our sufficient condition is not necessary. Problems 2a and 2b, B-P page 40, are solved in the negative; Koehler's Theorem [5], which gives an affirmative solution to question 2c, is generalized. The questions are: 2a. Is every infinite τ_0-set the type set of a rank two group? This is answered by Example 1. 2b. If $T(G)$ is infinite does there exist a completely anisotropic group (see Definitions) B with $T(B)=$ $=T(G)$? We answer this, only under the additional hypotheses that G and B are both rank two groups, in Example 2. 2c. If T is a finite τ_0-set containing τ_0 and τ_0 is non-nil, does there exist a quotient-divisible group G of rank two with $T(G)=T$? See the definitions preceding Corollary 3 below; Corollary 3 solves this problem.

Both of our theorems use the arithmetical functions $\gamma(n)$ and $\pi(n)$, where $\gamma(n)$ is the number of ordered pairs (a, b) of integers with $a \geqq 0$, a prime to b (*coprime* pairs) and $\max\{a, |b|\} \leqq n$; while $\pi(n)$ is the number of primes $\leqq n$. In [4] estimates of these functions are given: $\gamma(n)=4\Phi(n)$ is approximately $12n^2/\pi^2$; $\pi(n)$ is approximately $n/\log n$ (the prime number theorem); the last gives also that the n-th prime p_n is approximately $n \log n$. All approximations are, of course, for large n, and the coarsest (order of magnitude) estimates are sufficient. In the sufficiency theorem (Theorem 2) and in Example 2, we use a simple group construction based on the same idea as our groups $G(S)$ in [2], similar to B-P, Theorem 7.9, and KOEHLER [5], [6].

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For a discussion of characteristics, heights and types see FUCHS [3], (§ 42) and B-P (§§ 3, 7). We shall denote types by the lower case Greek letter τ and characteristics by the lower case Greek letter α. The height of x at p is denoted by $H_p x$. We shall deal only with torsionfree Abelian groups of rank two, i. e., additive subgroups, containing two lineary independent elements, of a two dimensional vector space over the field of rational numbers. Every set or sequence of types introduced will be assumed countable. The type containing the characteristic α is denoted by $[\alpha]$.

Definitions. Let G be a group of rank two, τ_0 a type, T a set of types and $S: \tau_1, \tau_2, \ldots$ a sequence of types. We say that *T is a τ_0-set* if and only if $\tau' \neq \tau''$ implies $\tau' \cap \tau'' = \tau_0$, for all τ', τ'' in T, and that *S is a τ_0-sequence* if $0 < i < j$ implies that $\tau_i \cap \tau_j = \tau_0$. Note that the only type that can be repeated in a τ_0-sequence is τ_0, which need not appear. We say that *S is a type sequence of G* provided that for independent elements x and y in G the set C of all coprime ordered pairs (a, b) of integers with $0 \leqq a$ can be indexed so that $T(a_n x + b_n y) = \tau_n$ for all n. Note that $T(G) = \{\tau_i; i = 1, 2, \ldots\}$ if S is a type sequence of G. In the last definition above it is immaterial whether we insert "all" or "some" between the words "for" and "independent." A group G is *completely anisotropic* provided independent elements have distinct types. The last definition and the Lemma following are due to B-P, and KOEHLER [6, Theorem 1. 6].

Lemma. *Let G be a rank two group with x and y independent in G, let $\alpha_0(p) = \min\{H_p x, H_p y\}$ for every prime p, and let (a, b) and (c, d) be coprime pairs in C with $H_p(ax + by) > \alpha_0(p)$ for some prime p. Then:* 1. *$T(G)$ is an $[\alpha_0]$-set and every type sequence of G is an $[\alpha_0]$-sequence.* 2. *If x' and y' have, in G, equal type greater than $[\alpha_0]$ then x' and y' are dependent.* 3. *$H_p(cx + dy) > \alpha_0(p)$ if and only if p divides the determinant $ad - bc$.* 4. *If $H_p(ax + by) = \infty > \alpha_0(p)$ then $H_p(cx + dy) = \alpha_0(p) + H_p(ad - bc)$, where the last height is computed in the additive group Z of all integers.*

PROOF. If G is torsionfree and each of M and M' is a maximal independent subset, then g. l. b. $\{Tx; x \in M\} \leqq$ g. l. b. $\{Tx; x \in M'\}$ is immediate; both 1 and 2 are easy consequences. Though more tedious, the other parts are equally trivial; their proof is omitted.

Let S be an $[\alpha_0]$ sequence $\tau_1, \tau_2, \ldots$, and let α_i belong to τ_i, $i = 1, 2, \ldots$. An attempt to select the α_i so that $\alpha_i \cap \alpha_j = \alpha_0$ for all $i \neq j$ may be foiled by virtue of the infinity places. Consider the proposition

$$D(i, j, p): \alpha_0(p) < \alpha_i(p) < \alpha_j(p) = \infty.$$

If $D(i, j, p)$, for some i, holds for infinitely many pairs j, p, then the above attempt will be foiled. For a term α_i of S and a subsequence S' of S, we say that *α_i is a snarl of S'* provided there is a subsequence S'' of S', say $S'': \alpha_{n(1)}, \alpha_{n(2)}, \ldots$, such that for every k there is a prime p_k with $D(i, n(k), p_k)$. (We assume here that $n(1) < n(2) < \ldots$.) A term α_i is a *snarl* provided it is a snarl of some subsequence, and a subsequence with no snarls is called *free*. For an $[\alpha_0]$-set T, a type $[\alpha]$ in *T is a snarl of the subset T'* of T if T' contains an infinite subset T'' such that for every $[\alpha'']$ in T'', there is a prime p with

$$\alpha_0(p) < \alpha(p) < \alpha''(p) = \infty;$$

while a subset T' of T is *free* if it has no snarls. The hypothesis of Theorem 7. 10 of B-P is equivalent to the assumption that the type set in question is free. Finite sets are always free.

Now let the set C of coprime ordered pairs with non-negative first member be well-ordered so that if $\max\{a, |b|\}$ is less than $\max\{c, |d|\}$, then (a, b) precedes (c, d). This well-ordered set will be called the *standard list* (of coprime pairs). For the analytic number theory estimates in the following paragraph see the introductory paragraph, noting that $\gamma(n)=|I_n|$ where I_n is defined below.

Now let x and y be linearly independent elements of the rank two group G. Take the characteristic α_0 so that for all p, $\alpha_0(p)=\min\{H_p x, H_p y\}$. Let (a_i, b_i) be the i-th member of the standard list of ordered coprime pairs and set $\alpha_i=$ height of $a_i x+b_i y$. Then S: $[\alpha_1]$, $[\alpha_2]$, ... is a type sequence of G and is an $[\alpha_0]$-sequence. We define sets

$$K=\{j;\ \text{for all } i<j \text{ and all } p, \text{ not } D(i, j, p)\},$$

$$I_n=\{j;\ \max\{a_j, |b_j|\}\leqq n\}.$$

For each j in I_n-K there is a prime q_j and an index $i<j$ such that $D(i, j, q_j)$. By Lemmas 1 and 3, respectively, $q_j=q_k$ implies that $j=k$ and that q_j divides $|a_i b_j - a_j b_i| \leqq 2n^2$. Hence the power $|I_n-K|$ is no greater than $\pi(2n^2)$, where $\pi(n)$ is the number of primes not exceeding n; hence $|I_n\cap K|$ is at least $|I_n|-\pi(2n^2)$. For all large n, therefore, $|I_n\cap K|$ is approximately $(12n^2/\pi^2)-(n^2/\log 2n^2)$, which tends to infinity with n. Hence K is infinite. If the set $F=\{[\alpha_i];\ i\in K\}$ is finite, then for almost all j in K, $[\alpha_j]=[\alpha_0]$. It is easy to see that K and F are free. We have proved:

Theorem 1. *The type sequence of a rank two torsionfree Abelian group has an infinite free subsequence. If the group is completely anisotropic then its type set contains an infinite free subset.*

Next we shall prove a sufficiency condition. Let S: $\tau_1, \tau_2, \ldots$ be a τ_0-sequence and *suppose that S is free*. By permuting indices if necessary we may assume that if $\tau_0(p)<\tau_n(p)=\infty$ then $p\geqq p_n=n$-th prime. Now choose characteristics α_n in τ_n so that for every p, $\alpha_0(p)=\min\{\alpha_1(p), \alpha_2(p)\}$; $\alpha_n(p)>\alpha_0(p)$ implies $p\geqq p_n$; $\alpha_n(p)=\alpha_0(p)$ for all n except one at most, while $\alpha_n(p)\geqq\alpha_0(p)$ holds for all n. In case $[\alpha_0]$ appears several times in the sequence we the take characteristic α_0 every time. Now let (a_n, b_n) be the n-th coprime pair in the standard list and select independent real numbers x and y with $0<x<y<1$. For each prime p we construct a pair u_p, v_p of p-adic integers; there are exactly four possibilities for p:

1. $\alpha_n(p)=\alpha_0(p)$ for all n.
   a) $\alpha_0(p)=\infty$. Set $u_p=v_p=0$.
   b) $\alpha_0(p)<\infty$. Take u_p and v_p linearly independent with

$$u_p = p^{\alpha_0(p)}\big([(\log p)^x]+\ldots\big)$$

$$v_p = p^{\alpha_0(p)}\big([(\log p)^y]+\ldots\big),$$

where, for example, $u_p = u_{p_0}+u_{p_1}p+\ldots, 0\leqq u_{p_i}<p$, is the usual power series representation, and $[t]$ is the greatest integer $\leqq t$.

2. $\alpha_n(p) > \alpha_0(p)$. Recall that this holds for only one value of n.

a) $\alpha_n(p) = \infty$. Choose u_p and v_p so that the minimum of their heights (in the p-adic integers) is $\alpha_0(p)$ and so that, moreover, $a_n u_p + b_n v_p = 0$.

b) $\alpha_n(p) < \infty$. Make the same height requirements as in 2a, make $H_p(a_n u_p + b_n v_p) = \alpha_n(p)$, and make u_p and v_p linearly independent.

Let G be the set of all rational combinations $ax+by$ such that for all p, au_p+bv_p is a p-adic integer (it will be a p-adic number in any case). Then G is a rank two group and $H_p(ax+by) = H_p(au_p+bv_p)$, where $ax+by$ belongs to G, and the heights are computed in G and the group of p-adic integers respectively. This construction is very similar to [2], and as in that paper we argue that for almost all p in $E = \{p;$ for all n, $\alpha_n(p) = \alpha_0(p)\}$, $H_p(ax+by) = \alpha_0(p)$, where $ax+by$ is an arbitrary member of G. We shall show that $[\alpha_k]$ is the type of $a_k x + b_k y$ in G. The infinity places are disposed of easily. If $\alpha_k(p) > \alpha_0(p)$ then our construction makes $H_p(a_k x + b_k y) = \alpha_k(p)$. Hence suppose $q(1), q(2), \ldots$ to be an infinite sequence of distinct primes such that for every i,

$$\alpha_k(q(i)) = \alpha_0(q(i)) < H_{q(i)}(a_k x + b_k y),$$

where k is a fixed index. By a previous remark we may assume $q(i)$ not in E. This means that there is an $n(i)$ with $\alpha_{n(i)}(q(i)) > \alpha_0(q(i))$ and the left member of this inequality is therefore the height at $q(i)$ of $a_{n(i)}x + b_{n(i)}y$. By Lemma 3, $q(i)$ divides $\det(k, n(i)) = a_k b_{n(i)} - a_{n(i)} b_k$. For all large values of $n(i)$ we have, by [4],

$$0 < |\det(k, n(i))| < \mathrm{Bn}(i) < B' n(i) \log n(i) < p_{n(i)} \leqq q(i),$$

where B and B' are constants independent of $n(i)$. This contradicts the divisibility of the determinant by $q(i)$. It follows that $\{n(i); i = 1, 2, \ldots\}$ is finite; for some j there is an infinite set I such that for all i in I, $n(i) = j$. Divisibility of $\det(k, j)$ by every $q(i)$ implies that $\det(k, j) = 0$, so $j = k$. But now if i belongs to I, then $\alpha_k(q(i)) = \alpha_0(q(i))$, $\alpha_0(q(i)) < \alpha_{n(i)}(q(i))$, $n(i) = j = k$. This is another contradiction. We have proved:

Theorem 2. *Every free τ_0-sequence is a type sequence of some torsionfree Abelian group of rank two.* (Cf. Theorem 7.10 of B-P.)

Corollary 1. If T is a finite τ_0-set containing τ_0 then T is the type set of a rank two group. (This is Theorem 7.9 of B-P.)

Corollary 2. If T is an infinite τ_0-set whose members are infinite only where τ_0 is infinite, then T is the type set of some rank two group. (Cf. Corollary 7. 11 of B-P.)

A type $[\alpha]$ is *non-nil* provided for almost all primes p, $\alpha(p) = 0$ or $\alpha(p) = \infty$. A torsionfree group G is *quotient-divisible* if there is a free subgroup F in G such that G/F is a divisible torsion group. These definitions are due to B-P.

The group G constructed in Theorem 2 is quotient-divisible, if $\tau_0 = [\alpha_0]$ is non-nil; take F to be the free group generated by x and y.

Corollary 3. A free τ_0-sequence with τ_0 non-nil is a type sequence of a quotient-divisible group of rank two. (Cf. B-P, page 40, and also [5].)

Example 1. Partition the set of all primes by infinite sets $P_1, P_2, \ldots$. Let α_1 have value one on P_1, zero elsewhere; for $n = 2, 3, \ldots$, let α_n be infinity at the

n-th member of each P_i for all $i<n$, one on P_n, and zero elsewhere. Then the set U of all $[\alpha_n]$ is a τ_0-set, where τ_0 is the zero type. Theorem 1 shows that if T is any subset of U then T is not the type set of any completely anisotropic group of rank two. This gives a strong negative answer to question 2a, page 40, of B-P.

Example 2. Let α_{2n} be the characteisticr that is everywhere zero, $n=0, 1, \ldots$; let α_1 be one at every prime; let α_{2n+1}, for $n \geqq 1$, be zero everywhere except at the n-th prime p_n, and $\alpha_{2n+1}(p_n)=\infty$. Let S be the $[\alpha_0]$-sequence $[\alpha_1], [\alpha_2], \ldots$, and let $T=[\{\alpha_i\}; i=1, 2, \ldots\}$. Now $[\alpha_1]$ is a snarl of every infinite subset of T so by Theorem 1, T is not the type set of a completely anisotropic group of rank two. Professor Koehler has pointed out that this example is nearly identical with one in his thesis.

But *S is the type sequence of the rank two group G*, which we now define. Let $c_1=(1, 0)$, $c_2=(0, 1)$; for every $n=1, 2, \ldots$ set c_{2n+1} equal to $(1, -p_n)$ and c_{2n+2} equal to the first coprime pair in the standard list that has not previously been selected. Note that there is no conflict in these directions for constructing the c_n and that we get a one-to-one correspondence between the set of all coprime pairs (a, b) with $0 \leqq a$ and the characteristics α_n by associating c_n with α_n. Set $c_n=(a_n, b_n)$. Let x and y be independent real numbers and let G be the set of all reals of the form $ax+by$ where a and b are rational numbers and $ap+b$, for every p, is a p-adic integer. Then G is a rank two group. It is easy to verify that the heights of $x = a_1x+$ $+b_1y$ and $y = a_2x+b_2y$ are α_1 and α_2 respectively, while a_nx+b_ny has infinite height at p if and only if $n = 2k+1$, $k \geqq 1$ and $p=p_k$. Let $k>2$ and suppose that every prime in the set K divides a_kx+b_ky. Since every prime in K also divides a_1x+b_1y, we can apply Lemma 3 to deduce that the determinant $a_1b_k-a_kb_1$ is divisible by every prime in K, while this determinant is not zero. Hence K is finite, and $[\alpha_k]$ is the type of a_kx+b_ky. Noting that T is the type *set* of G we see that question 2b of B-P (page 40) has a negative answer. Also, neither S nor T is free, so the condition of Theorem 2 is not necessary.

References

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