On transformations of certain hypergeometric functions of three variables

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1. Introduction

In the year 1893 LAURICELLA ([5]) conjectured the existence of ten hypergeometric functions of three variables in addition to F_A, F_B, F_C and F_D defined and studied by him. These ten functions, namely F_E, F_F, F_G, F_K, F_M, F_N, F_P, F_R, F_S and F_T, have been defined and studied by SARAN ([7]). In §§ 3 and 4 of this paper I have obtained certain hypergeometric transformations associated with the functions F_F, F_P and F_R.

It may be remarked here that the functions F_P and F_R chosen by me have not been dealt with hitherto[1]), so far as their transformations are concerned. On the other hand, no transformation into a known function has been previously obtained for F_F. PANDEY [6], however, gave the transformation

$$F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) =$$

$$= (1-x-z)^{-\beta_1}(1-y)^{-\beta_2} G_A\left(1-\gamma_1, \beta_1, \beta_2; \gamma_2; \frac{x}{x+z-1}, \frac{1}{y-1}, \frac{z}{x+z-1}\right),$$

where $\gamma_1+\gamma_2 = 1+\alpha_1$, and the new function

$$G_A(\alpha, \beta, \beta'; \gamma; x, y, z) =$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, n+p-m)(\beta, m+p)(\beta', n)}{(1, m)(1, n)(1, p)(\gamma, n+p-m)} x^m y^n z^p.$$

The four linear transformations of F_F, given by me in this paper, connect this function with either F_M or F_N. Besides, I have been able to show that, under certain conditions, the function F_F is reducible to Appell's F_4.

In the last section I have given some interesting cases of reducibility of the remaining functions F_E, F_G, F_K, F_M, F_N, F_S and F_T. Some of these reductions yield very interesting results by putting two of the variables equal. For instance, by putting $y=z$, the reduction formulae for F_S lead to well-known relations between Appell's and Gauss's hypergeometric functions.

[1]) Vide [8], p. 294, last paragraph.

2. Definitions

Following the notation of [7] the ten hypergeometric functions of there variables are defined by the equalities

$$F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) =$$

(2. 1)

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m+n+p)(\beta_1, m)(\beta_2, n+p)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n)(\gamma_3, p)} x^m y^n z^p$$

$$[r + (\sqrt{s} + \sqrt{t})^2 = 1];$$

$$F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) =$$

(2. 2)

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m+n+p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p$$

$$[(1-s)(s-t) = rs];$$

$$F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z) =$$

(2. 3)

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m+n+p)(\beta_1, m)(\beta_2, n)(\beta_3, p)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p$$

$$[r+s = 1 = r+t];$$

$$F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z) =$$

(2. 4)

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n)(\gamma_3, p)} x^m y^n z^p$$

$$[(1-r)(1-s) = t];$$

$$F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) =$$

(2. 5)

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p$$

$$[r+t = 1 = s];$$

$$F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) =$$

(2. 6)

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m)(\alpha_2, n)(\alpha_3, p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p$$

$$[(1-r)s + (1-s)t = 0];$$

$$F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_2; x, y, z) =$$

(2. 7)

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m+p)(\alpha_2, n)(\beta_1, m+n)(\beta_2, p)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p$$

$$[(st-s-t)^2 = 4rst];$$

(2. 8)
$$F_R(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) =$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m+p)(\alpha_2, n)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p$$
$$\left[s(1-\sqrt{r})^2+(1-s)t = 0\right];$$

(2. 9)
$$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z) =$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m)(\beta_2, n)(\beta_3, p)}{(1, m)(1, n)(1, p)(\gamma_1, m+n+p)} x^m y^n z^p$$
$$[r+s = rs, \ s = t];$$

and

(2. 10)
$$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, z) =$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m+n+p)} x^m y^n z^p$$
$$[r-rs+s = t];$$

where, as usual, $(\lambda, k) = \dfrac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$, and $|x|<r$, $|y|<s$ and $|z|<t$.

3. Transformations of F_F.

BAILEY ([3]) has proved the theorem that

(3. 1) $\quad F_4[\alpha, \beta; \gamma, \gamma'; z(1-Z), Z(1-z)] = {}_2F_1(\alpha, \beta; \gamma; z)\,{}_2F_1(\alpha, \beta; \gamma'; Z),$

provided that $\gamma+\gamma' = \alpha+\beta+1$, this formula being valid inside simply connected regions surrounding $z=0$, $Z=0$ for which

$$|z(1-Z)|^{1/2}+|Z(1-z)|^{1/2} < 1.$$

From definition we have

$$F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) =$$
$$= \sum_{n=0}^{\infty} \frac{(\alpha_1, n)(\beta_2, n)}{(1, n)(\gamma_2, n)} F_4(\alpha_1+n, \beta_1; \gamma_1, \gamma_2+n; x, z) y^n.$$

Therefore, using (3. 1) we get

(3. 2) $\quad F_F[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; 1+\alpha_1+\beta_1-\gamma_2, \gamma_2, \gamma_2; x(1-z), y, z(1-x)] =$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1, n)(\beta_2, n)}{(1, n)(\gamma_2, n)} {}_2F_1(\alpha_1+n, \beta_1;\ 1+\alpha_1+\beta_1-\gamma_2; x) \cdot {}_2F_1(\alpha_1+n, \beta_1; \gamma_2+n; z) y^n.$$

Applying Euler's transformation [2]

$$ {}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{-\beta}{}_2F_1\left(\gamma-\alpha, \beta; \gamma; \frac{z}{z-1}\right), \tag{3. 3}$$

we obtain

$$F_F[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; 1+\alpha_1+\beta_1-\gamma_2, \gamma_2, \gamma_2; x(1-z), y, z(1-x)] =$$

$$= (1-z)^{-\beta_1} \sum_{n=0}^{\infty} \frac{(\alpha_1, n)(\beta_2, n)}{(1, n)(\gamma_2, n)} {}_2F_1(\alpha_1+n, \beta_1; 1+\alpha_1+\beta_1-\gamma_2; x)\cdot \tag{3. 4}$$

$$\cdot {}_2F_1\left(\gamma_2-\alpha_1, \beta_1; \gamma_2+n; \frac{z}{z-1}\right) y^n.$$

On expanding Gauss's functions ${}_2F_1$ inside the summation and comparing the resulting triple series with (2. 6), we obtain the transformation

$$F_F[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; 1+\alpha_1+\beta_1-\gamma_2, \gamma_2, \gamma_2; x(1-z), y, z(1-x)] = \tag{3. 5}$$

$$= (1-z)^{-\beta_1} F_N\left(\beta_1, \gamma_2-\alpha_1, \beta_2, \alpha_1, \beta_1, \alpha_1; 1+\alpha_1+\beta_1-\gamma_2, \gamma_2, \gamma_2; x, \frac{z}{z-1}, y\right).$$

Now, using the transformation [2]

$$ {}_2F_1(\alpha, \beta; \gamma; x) = (1-x)^{-\alpha}{}_2F_1\left(\alpha, \gamma-\beta; \gamma; \frac{x}{x-1}\right), \tag{3. 6}$$

in (3. 4) and comparing the resulting triple series with (2. 6), we get the transformation

$$F_F[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; 1+\alpha_1+\beta_1-\gamma_2, \gamma_2, \gamma_2; x(1-z), y, z(1-x)] =$$

$$= (1-x)^{-\alpha_1}(1-z)^{-\beta_1} F_N\left(1+\alpha_1-\gamma_2, \gamma_2-\alpha_1, \beta_2, \alpha_1, \beta_1, \alpha_1;\right. \tag{3. 7}$$

$$\left. 1+\alpha_1+\beta_1-\gamma_2, \gamma_2, \gamma_2; \frac{x}{x-1}, \frac{z}{z-1}, \frac{-y}{x-1}\right).$$

If we use the relation [2]

$$ {}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha, \gamma-\beta; \gamma; z) \tag{3. 8}$$

in (3. 2) and compare the resulting triple series with (2. 5), we obtain the transformation

$$\begin{aligned} &F_F[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; 1+\alpha_1+\beta_1-\gamma_2, \gamma_2, \gamma_2; x(1-z), y, z(1-x)] = \\ &= (1-z)^{\gamma_2-\alpha_1-\beta_1} F_M(\beta_1, \beta_2, \beta_2, \alpha_1, \gamma_2-\alpha_1, \alpha_1; \gamma_1, \gamma_2, \gamma_2; x, z, y), \end{aligned} \tag{3. 9}$$

and if we use both (3. 6) and (3. 8) in (3. 2) and proceed as above, the transformation

(3. 10)
$$F_F[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1;\ 1+\alpha_1+\beta_1-\gamma_2, \gamma_2, \gamma_2;\ x(1-z), y, z(1-x)] =$$
$$= (1-x)^{-\alpha_1}(1-z)^{\gamma_2-\alpha_1-\beta_1} F_M\left(1+\alpha_1-\gamma_2, \beta_2, \beta_2, \alpha_1, \gamma_2-\alpha_1, \alpha_1;\ \gamma_1, \gamma_2, \gamma_2;\ \frac{x}{x-1}, z, \frac{-y}{x-1}\right)$$

is obtained, both the transformations being valid when $\gamma_2 = \beta_1+\beta_2$ and $\gamma_1+\gamma_2 = 1+\alpha_1+\beta_1$.

Now, let us write

(3. 11)
$$F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1;\ \gamma_1, \gamma_2, \gamma_2;\ x, y, z) =$$
$$= \sum_{p=0}^{\infty} \frac{(\alpha_1, p)(\beta_1, p)}{(1, p)(\gamma_2, p)} F_2(\alpha_1+p, \beta_1+p, \beta_2;\ \gamma_1, \gamma_2+p;\ x, y) z^p.$$

Using then the transformation [1]

(3. 12) $$F_2(\alpha;\ \beta, \beta';\ \gamma, \gamma';\ x, y) = (1-y)^{-\alpha} F_2\left(\alpha, \beta, \gamma'-\beta';\ \gamma, \gamma'\ \frac{x}{1-y}, \frac{y}{y-1}\right)$$

and expanding the resulting Appell's function F_2 inside the summation, we get the interesting reduction

(3. 13)
$$F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \gamma_2-\beta_1, \beta_1;\ \gamma_1, \gamma_2, \gamma_2;\ x, y, -y) =$$
$$= (1-y)^{-\alpha_1} F_4\left(\alpha_1, \beta_1;\ \gamma_1, \gamma_2;\ \frac{-x}{y-1}, \frac{y}{y-1}\right).$$

4. Transformations of F_P and F_R

On writing

(4. 1)
$$F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2;\ \gamma_1, \gamma_2, \gamma_2;\ x, y, z) =$$
$$= \sum_{p=0}^{\infty} \frac{(\alpha_1, p)(\beta_2, p)}{(1, p)(\gamma_2, p)} F_2(\beta_1, \alpha_1+p, \alpha_2;\ \gamma_1, \gamma_2+p;\ x, y) z^p,$$

and using the relation (3. 12), we obtain

(4. 2)
$$F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \gamma_2-\alpha_2;\ \gamma_1, \gamma_2, \gamma_2;\ x, y, z) =$$
$$= (1-y)^{-\beta_1} H_A\left(\alpha_1, \beta_1, \gamma_2-\alpha_2;\ \gamma_1, \gamma_2;\ \frac{x}{1-y}, \frac{-y}{1-y}, z\right),$$

where the new function

(4. 3)
$$H_A(\alpha, \beta, \beta';\ \gamma, \gamma';\ x, y, z) =$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha, m+p)(\beta, m+n)(\beta', n+p)}{(1, m)(1, n)(1, p)(\gamma, m)(\gamma', n+p)} x^m y^n z^p.$$

Similarly, on rewriting

$$(4.4)\quad \begin{aligned} &F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_2, n)(\beta_1, n)}{(1, n)(\gamma_2, n)} F_2(\alpha_1, \beta_1+n, \beta_2; \gamma_1, \gamma_2+n; x, z) y^n, \end{aligned}$$

and using (3. 12), we get the transformation

$$(4.5)\quad \begin{aligned} &F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \gamma_2-\alpha_2; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\ &= (1-z)^{-\alpha_1} H_A\left(\alpha_1, \beta_1, \alpha_2; \gamma_1, \gamma_2; \frac{x}{1-z}, y, \frac{-z}{1-z}\right). \end{aligned}$$

The equality (2. 8) can be written as

$$(4.6)\quad \begin{aligned} &F_R(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_2, n)(\beta_2, n)}{(1, n)(\gamma_2, n)} F_4(\alpha_1, \beta_1; \gamma_1, \gamma_2+n; x, z) y^n. \end{aligned}$$

Using the relation [1]

$$(4.7)\quad \begin{aligned} F_4\left(\alpha, \alpha+\frac{1}{2}; \gamma, \frac{1}{2}; x, y\right) &= \frac{1}{2}\left(1+\sqrt{y}\right)^{-2\alpha} {}_2F_1\left[\alpha, \alpha+\frac{1}{2}; \gamma; \frac{x}{(1+\sqrt{y})^2}\right] + \\ &+ \frac{1}{2}\left(1-\sqrt{y}\right)^{-2\alpha} {}_2F_1\left[\alpha, \alpha+\frac{1}{2}; \gamma; \frac{x}{(1-\sqrt{y})^2}\right], \end{aligned}$$

we obtain

$$(4.8)\quad \begin{aligned} &F_R\left(\alpha_1, \alpha_2, \alpha_1, \alpha_1+\frac{1}{2}, \beta_2, \alpha_1+\frac{1}{2}; \frac{1}{2}, \gamma_2, \gamma_2; x, y, z\right) = \\ &= \frac{1}{2}\left(1+\sqrt{x}\right)^{-2\alpha_1} F_3\left[\alpha_2, \alpha_1, \beta_2, \alpha_1+\frac{1}{2}; \gamma_2; y, \frac{z}{(1+\sqrt{x})^2}\right] + \\ &+ \frac{1}{2}\left(1-\sqrt{x}\right)^{-2\alpha_1} F_3\left[\alpha_2, \alpha_1, \beta_2, \alpha_1+\frac{1}{2}; \gamma_2; y, \frac{z}{(1-\sqrt{x})^2}\right]. \end{aligned}$$

5. Some reducible cases

On writing

$$(5.1)\quad \begin{aligned} &F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \\ &= \sum_{p=0}^{\infty} \frac{(\alpha_1, p)(\beta_2, p)}{(1, p)(\gamma_3, p)} F_2(\alpha_1+p, \beta_1, \beta_2+p; \gamma_1, \gamma_2; x, y) z^p, \end{aligned}$$

and using the relation [1]

$$F_2(\alpha, \beta, \beta'; \beta, \gamma'; x, y) = (1-x)^{-\alpha} {}_2F_1\left(\alpha, \beta'; \gamma'; \frac{y}{1-x}\right), \tag{5. 2}$$

we obtain

$$\begin{aligned} &F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \beta_1, \gamma_2, \gamma_3; x, y, z) = \\ &= (1-x)^{-\alpha_1} F_4\left(\alpha_1, \beta_2; \gamma_2, \gamma_3; \frac{y}{1-x}, \frac{z}{1-x}\right). \end{aligned} \tag{5. 3}$$

Similarly, we get

$$\begin{aligned} &F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \beta_1, \gamma_2, \gamma_2; x, y, z) = \\ &= (1-x)^{-\alpha_1} F_1\left(\alpha_1; \beta_2, \beta_3; \gamma_2, \frac{y}{1-x}, \frac{z}{1-x}\right), \end{aligned} \tag{5. 4}$$

$$\begin{aligned} &F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \beta_2, \gamma_3; x, y, z) = \\ &= (1-y)^{-\alpha_2} F_2\left(\beta_1, \alpha_1, \alpha_2; \gamma_1, \gamma_3; x, \frac{z}{1-y}\right), \end{aligned} \tag{5. 5}$$

$$\begin{aligned} &F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \alpha_1, \gamma_2, \gamma_3; x, y, z) = \\ &= (1-x)^{-\beta_1} F_2\left(\alpha_2, \beta_2, \beta_1; \gamma_2, \gamma_3; y, \frac{z}{1-x}\right), \end{aligned} \tag{5. 6}$$

$$\begin{aligned} &F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \alpha_1, \gamma_2, \gamma_2; x, y, z) = \\ &= (1-x)^{-\beta_1} F_1\left(\alpha_2, \beta_2, \beta_1; \gamma_2; y, \frac{z}{1-x}\right) \end{aligned} \tag{5. 7}$$

and

$$\begin{aligned} &F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \alpha_1, \gamma_2, \gamma_2; x, y, z) = \\ &= (1-x)^{-\beta_1} F_3\left(\alpha_2, \alpha_3, \beta_2, \beta_1; \gamma_2; y, \frac{z}{1-x}\right). \end{aligned} \tag{5. 8}$$

It may be mentioned that the transformation (5. 8) was previously given by PANDEY [6] by using Pochhammer's double loop type integral for F_N, namely

$$F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \varrho+\varrho', \varrho+\varrho'; x, y, z) = \frac{\Gamma(1-\varrho)\Gamma(1-\varrho')\Gamma(\varrho+\varrho')}{(2\pi i)^2} \cdot$$

$$\cdot \int (-t)^{\varrho-1}(t-1)^{\varrho-1} \cdot {}_2F_1(\alpha_2, \beta_2; \varrho; ty) F_2[\beta_1, \alpha_1, \alpha_3; \gamma_1, \varrho'; x, (1-t)z]\, dt,$$

where $|x|+|(1-t)z|<1$ and $|ty|<1$ along the contour.

Also, on writing

(5. 9)
$$F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\ = \sum_{n=0}^{\infty} \frac{(\alpha_2, n)(\beta_2, n)}{(1, n)(\gamma_2, n)} F_2(\beta_1, \alpha_1, \alpha_2+n; \gamma_1, \gamma_2+n; x, z) y^n,$$

and using the transformation [1]

(5. 10)
$$F_2(\alpha, \beta, \beta'; \gamma, \beta'; x, y) = (1-y)^{-\alpha} {}_2F_1\left(\alpha, \beta; \gamma; \frac{x}{1-y}\right),$$

we obtain the interesting reduction

(5. 11)
$$F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \alpha_2, \alpha_2; x, y, z) = \\ = (1-y)^{-\beta_2}(1-z)^{-\beta_1} {}_2F_1\left(\alpha_1, \beta_1; \gamma_1; \frac{x}{1-z}\right).$$

Now, writing

(5. 12)
$$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \\ = \sum_{m=0}^{\infty} \frac{(\alpha_1, m)(\beta_1, m)}{(1, m)(\gamma_1, m)} F_1(\alpha_2, \beta_2, \beta_3; \gamma_1+m; y, z) x^m,$$

and using the transformation [1]

(5. 13)
$$F_1(\alpha, \beta, \beta'; \gamma; x, y) = \\ = (1-x)^{\gamma-\alpha-\beta}(1-y)^{-\beta'} F_1\left(\gamma-\alpha, \gamma-\beta-\beta', \beta'; \gamma; x, \frac{y-x}{y-1}\right),$$

we get

(5. 14)
$$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \\ = (1-y)^{\alpha_1-\beta_2}(1-z)^{-\beta_3} F_1\left(\alpha_1, \beta_1, \beta_3; \gamma_1; x+y-xy, \frac{z-y}{z-1}\right),$$

where $\gamma_1 = \alpha_1+\alpha_2 = \beta_1+\beta_2+\beta_3$.

Two particular cases of (5.14) are

(5. 15)
$$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; -y, y, z) = \\ = (1-y)^{\alpha_1-\beta_2}(1-z)^{-\beta_3} F_1\left(\alpha_1, \beta_1, \beta_3; \gamma_1; y^2, \frac{z-y}{z-1}\right)$$

and

(5. 16)
$$F_S\left(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; \frac{y}{1-y}, y, z\right) =$$
$$= (1-y)^{\alpha_1-\beta_2}(1-z)^{-\beta_3} F_1\left(\alpha_1, \beta_1, \beta_3; \gamma_1; 2y, \frac{z-y}{z-1}\right),$$

where, as before, $\gamma_1 = \alpha_1+\alpha_2 = \beta_1+\beta_2+\beta_3$.

It is interesting to note that by putting $y=z$ in (5. 14), we get the well-known transformations [1]

$$F_3(\alpha_1, \gamma_1-\alpha_1, \beta_1, \gamma_1-\beta_1; \gamma_1; x, y) =$$
$$= (1-y)^{\alpha_1+\beta_1-\gamma_1}\, {}_2F_1[\alpha_1, \beta_1; \gamma_1; 1-(1-x)(1-y)]$$

and

$$F_1\left(\alpha_1, \beta_1, \beta_2+\beta_3; \beta_1+\beta_2+\beta_3; x, \frac{y}{y-1}\right) =$$
$$= (1-y)^{\alpha_1}\, {}_2F_1[\alpha_1, \beta_1; \beta_1+\beta_2+\beta_3; x+(1-x)y],$$

the latter being a particular case of (5. 18) below.

Lastly, writing

$$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, z) =$$

(5. 17)
$$= \sum_{m=0}^{\infty} \frac{(\alpha_1, m)(\beta_1, m)}{(1, m)(\gamma_1, m)} F_1(\alpha_2, \beta_2, \beta_1+m; \gamma_1+m; y, z)x^m =$$
$$= \sum_{n=0}^{\infty} \frac{(\alpha_2, n)(\beta_2, n)}{(1, n)(\gamma_1, n)} F_1(\beta_1, \alpha_1, \alpha_2+n; \gamma_1+n; x, z)y^n,$$

and using the relation [1]

(5. 18)
$$F_1(\alpha, \beta, \beta'; \beta+\beta'; x, y) = (1-y)^{-\alpha}\, {}_2F_1\left(\alpha, \beta; \beta+\beta'; \frac{x-y}{1-y}\right),$$

we get the reductions

(5. 19)
$$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \gamma_1-\beta_1, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, z) =$$
$$= (1-y)^{-\alpha_2} F_1\left(\beta_1, \alpha_1, \alpha_2; \gamma_1; x, \frac{z-y}{1-y}\right) =$$
$$= (1-z)^{-\alpha_2} F_3\left(\alpha_1, \alpha_2, \beta_1, \gamma_1-\beta_1; \gamma_1; x, \frac{y-z}{1-z}\right);$$

and

$$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \alpha_1+\alpha_2, \alpha_1+\alpha_2, \alpha_1+\alpha_2; x, y, z) =$$

(5. 20)

$$= (1-x)^{-\beta_1} F_1\left(\alpha_2, \beta_2, \beta_1; \alpha_1+\alpha_2; y, \frac{z-x}{1-z}\right) =$$

$$= (1-z)^{-\beta_1} F_3\left(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1+\alpha_2; \frac{x-z}{1-z} y\right).$$

On putting $y=z$ in (5. 19) and $x=z$ in (5. 20), we obtain the interesting reductions

(5. 21) $$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, y) = (1-y)^{-\alpha_2} {}_2F_1(\alpha_1, \beta_1; \gamma_1; x),$$

provided that $\gamma_1 = \beta_1+\beta_2$, and

(5. 22) $$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, x) = (1-x)^{-\beta_1} {}_2F_1(\alpha_2, \beta_2; \gamma_1; y),$$

valid when $\gamma_1 = \alpha_1+\alpha_2$.

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