Boolean valuation in commutative groups*)

By P. S. REMA (Madras)

1. Introduction

This paper deals with the theory of Boolean metrization developed in [6] in the particular case of commutative groups. We introduce the notions of Boolean valuations and Boolean norms in (commutative) groups and obtain a necessary and sufficient condition for a Boolean valuation to be a Boolean norm. In terms of the Boolean valuation $\mathcal{V}$ we then introduce a notion of metrizability called $(B, \mathcal{V}, M)$-metrizability and show with the help of an embedding theorem (cf. (3. 9)) that the Hausdorff $(B, \mathcal{V}, M)$-metrizable topological groups are precisely the Hausdorff s-groups (i. e. groups having a system of subgroup neighbourhoods of zero). We also obtain some sufficient condition for the $(B, \mathcal{V}, M)$-metrizability of non-Hausdorff topological groups. By defining an "invariant" Boolean-metric in an obvious fashion on a group we show that the invariant Boolean-metrics are precisely those which are determined by Boolean norms. Finally we prove that under certain conditions the existence of a Boolean metric on a group G implies the existence of an invariant Boolean-metric on G. The paper ends with a brief indication of similar considerations for commutative rings and vector spaces.

2. Preliminaries and basic results

Here we shall briefly recall some definitions and results from [6] which will be made use of in the sequel.

Let B be a Boolean algebra and S any set. Then a mapping of the product set $S \times S$ in B is said to be a Boolean-metric of S in B or a B-metric on S if it satisfies the following conditions:

(1) $d(a, b)=0 \Leftrightarrow a=b \ (a, b \in S)$

(2) $d(a, b)=d(b, a)$ and

(3) $d(a, b) \leqq d(a, c) \vee d(c, b)$ (for $a, b, c \in S$) where $\vee$ denotes the lattice sum in B.

Let P be any dual ideal of B. Then the subset $A_p, p \in P$ of $S \times S$ is defined by $A_p=[(x, y), x, y \in S / d(x, y) \leqq p]$.

*) Forms a part of the author's doctorate dissertation, University of Madras (1963).

We have

(2. 1): Let P be any dual ideal of the Boolean algebra B. Then the subsets A_p, $(p \in P)$ form a base for a uniformity and define a uniform*) structure A_P on S.

The uniformities A_P are called the *Boolean-uniformities* (or more particularly the (B, d, P)-uniformities) on S, and the topologies defined on S by these uniformities are called the *Boolean-topologies* (or (B, d, P)-topologies) on S.

(2. 2): Let A_P be any Boolean uniformity on S. Then the fundamental neighbourhoods $A_p(a) = [x \in S/(x, a) \in A_p]$ are both open and closed in (S, T_P).

(2. 3): A topological space (S, T) is said to be *(B, d, M)-metrizable* if there exists a Boolean algebra B and a dual ideal M of B such that (1) S admits a B-metric d into B and (2) the (B, d, M)-topology on S is equivalent to T.

Let $(S, \mathfrak{U})$ be a uniform space. Then the surrounding U_α in a base for $\mathfrak{U}$ is said to be *idempotent* if $U_\alpha \circ U_\alpha = U_\alpha$; also we recall that U_α is called *symmetric* if $U_\alpha = U_\alpha^{-1}$. A base $[U_\alpha]$ of the uniformity $\mathfrak{U}$ is called idempotent (symmetric) if each U_α is idempotent (symmetric).

Then we have

(2. 4): Let (S, T) be a Hausdorff topological space which is uniformisable by means of a uniformity $\mathfrak{U}$ which has a symmetric and idempotent base $[U_\alpha]$, $(\alpha \in \Lambda)$. Then (S, T) can be uniformly imbedded (with respect to $\mathfrak{U}$) as a dense subspace of a projective limit of discrete spaces $S_\alpha = S/\theta_\alpha$, $(\alpha \in \Lambda)$ (where θ_α is the equivalence $x\theta_\alpha y \Leftrightarrow (x, y) \in U_\alpha$).

(2. 5): A Hausdorff topological space is (B, d, M)-metrizable if and only if it has a uniformity $\mathfrak{U}$ having a symmetric and idempotent base.

3. Boolean valuations

We shall first define the notion of a Boolean valuation and $(B, \mathcal{V}, M)$-metrizability in a commutative group. Here G will always denote a commutative group and B a Boolean algebra.

Definition. A mapping $\mathcal{V}$ of G into a Boolean algebra B is said to be a *group Boolean valuation* (or B-valuation) on G if it satisfies the following conditions:

(1) $\mathcal{V}(x) = 0_B$ (the zero element of B) if and only if $x = 0$ (the zero element of G),

(2) $\mathcal{V}(x+z) \leqq \mathcal{V}(x) \vee \mathcal{V}(z)$ (where $\vee$ is the lattice sum in B).

Definition. A B-valuation $\mathcal{V}$ on G is called a *B-norm* if $d(x, y) = \mathcal{V}(x-y)$ defines a B-metric on G and d is said to be the B-metric *determined by* $\mathcal{V}$.

(3. 1) Proposition. *The necessary and sufficient condition for the B-valuation $\mathcal{V}$ on G to be a B-norm is that $\mathcal{V}(x) = \mathcal{V}(-x)$ for all $x \in G$.*

Proof. Suppose $\mathcal{V}$ determines a B-metric on G. Then $d(x, y) = \mathcal{V}(x-y)$ is a B-metric. Hence $d(x, y) = d(y, x)$ i. e. $\mathcal{V}(x-y) = \mathcal{V}(y-x)$ for all $x, y \in G$. In particular putting $y = 0$ we have $\mathcal{V}(x) = \mathcal{V}(-x)$ for all $x \in G$. Thus the condition is necessary.

Conversely suppose the condition is satisfied. Then $d(x, y) = 0_B \Leftrightarrow \mathcal{V}(x-y) = 0_B \Leftrightarrow x - y = 0 \Leftrightarrow x = y$. Again $d(x, y) = \mathcal{V}(x-y) = \mathcal{V}(-(x-y))$ (by the given

*) For uniformities the notation of Kelley [3] is used.

condition) $= \mathcal{V}(y-x) = d(y, x)$. Further $d(x, y) = \mathcal{V}(x-y) = \mathcal{V}(x-z+z-y) \leqq \mathcal{V}(x-z) \vee \mathcal{V}(z-y) = d(x, z) \vee d(z, y)$. Thus d is a B-metric and therefore the condition is also sufficient.

(3. 2) Proposition. *Let G be a group with a B-metric d and let M be any dual ideal of B. Then the group operations of G are uniformly continuous with respect to the (B, d, M)-uniformity A_M (cf. (2. 1)) if and only if the following condition $(*)$ is satisfied:*

$(*)$ Given $m \in M$ there exists an element $n \in M$ such that $d(x, a) \leqq n$, $d(y, b) \leqq n \Rightarrow d(x-y, a-b) \leqq m$.

Proof. Suppose the group operations of G are uniformly continuous with respect to the uniformity A_M. Then given $m \in M$ there exists $n \in M$ such that $A_n(x) - A_n(y) \subseteqq A_m(x-y)$ whatever be $x, y \in G$. Now $d(x, a) \leqq n$, $d(y, b) \leqq n$ implies $(x, a) \in A_n$, $(y, b) \in A_n$. Therefore $a \in A_n(x)$, $b \in A_n(y)$. Hence $a-b \in A_n(x) - A_n(y) \subseteqq A_m(x-y)$ i. e. $d(x-y, a-b) \leqq m$. Thus the condition is satisfied.

Conversely suppose the condition $(*)$ holds good. Let $m \in M$. Then by $(*)$ there exists $n \in M$ such that $d(x, a) \leqq n$, $d(y, b) \leqq n \Rightarrow d(x-y, a-b) \leqq m$. Now $z \in A_n(x) - A_n(y) \Rightarrow z = a-b$, $a \in A_n(x)$, $b \in A_n(y)$. Hence $d(a, x) \leqq n$, $d(b, y) \leqq n$. Therefore $d(x-y, a-b) \leqq m$ by $(*)$ i. e. $z = a-b \in A_m(x-y)$. Therefore $A_n(x) - A_n(y) \subseteqq A_m(x-y)$. This is true whatever be $x, y \in G$. Therefore the group operations of G are uniformly continuous.

Corollary. *If d is the B-metric determined by a B-norm $\mathcal{V}$ on G, then the group operations of G are uniformly continuous with respect to any (B, d, M)-uniformity A_M on G.*

Proof. To prove this it suffices to show that the condition $(*)$ defined in (3. 2) is satisfied. Given $m \in M$ let $d(x, a) = \mathcal{V}(x-a) \leqq m$ and $d(y, b) = \mathcal{V}(y-b) \leqq m$. Then $d(x-y, a-b) = \mathcal{V}(x-y-a+b) = \mathcal{V}(x-a+b-y) \leqq \mathcal{V}(x-a) \vee \mathcal{V}(b-y) = \mathcal{V}(x-a) \vee \mathcal{V}(y-b) = d(x, a) \vee d(y, b) \leqq m \vee m = m$. Thus $(*)$ is satisfied and hence the result.

Now we shall prove:

(3. 3) Proposition. *Let $\mathcal{V}$ be a B-norm on G determining the B-metric d on G, and let M be any dual ideal of B. Then the uniformity A_M is separated if and only if $\mathcal{V}(G) \cap C(M) = 0_B$ (where $C(M)$ is the cut-complement of M).*

Proof. Suppose A_M is a separated uniform structure. Then $\bigcap_{m \in M} A_M = \Delta$. If $x \in \mathcal{V}(G) \cap C(M)$ then $x = \mathcal{V}(g)$, $d(0, g) = \mathcal{V}(g) \leqq$ each $m \in M$. Hence $(0, g) \in A_m$ for each $m \in M$. Thus $(0, g) \in \bigcap_{m \in M} A_M = \Delta$ i. e. $g = 0$. Therefore $x = \mathcal{V}(g) = 0_B$. Thus $\mathcal{V}(G) \cap C(M) = 0_B$.

Conversely let $\mathcal{V}(G) \cap C(M) = 0_B$. If $(x, y) \in \bigcap_{m \in M} A_m$ then $\mathcal{V}(x-y) = d(x, y) \leqq m$ for each $m \in M$. Hence $\mathcal{V}(x-y) \in \mathcal{V}(G) \cap C(M) = 0_B$. Therefore $x-y = 0$ i. e. $x = y$. Thus $\bigcap_{m \in M} A_m = \Delta$ i. e. A_M is separated.

(3. 4) Lemma. *If G_λ, $(\lambda \in \Lambda)$ are a system of groups admitting B_λ-valuations $\mathcal{V}_\lambda$'s respectively then the direct product $G = \prod_{\lambda \in \Lambda} G_\lambda$ admits a B-valuation $\mathcal{V}$ into the product Boolean algebra $B = \prod_{\lambda \in \Lambda} B_\lambda$.*

PROOF. Let $g = (g_\lambda)$ be any element of G. Define $\mathcal{V}(g) = (\mathcal{V}_\lambda(g_\lambda))$, $(\lambda \in \Lambda)$. Then $\mathcal{V}$ is a mapping of G in B. Further $\mathcal{V}(g) = 0_B \Leftrightarrow \mathcal{V}_\lambda(g_\lambda) = 0_{B\lambda}$ for each $\lambda \Leftrightarrow g_\lambda = 0_\lambda$ for each $\lambda \in \Lambda \Leftrightarrow g = 0$. For any two elements $g = (g_\lambda)$, $h = (h_\lambda)$ in G $\mathcal{V}(g+h) = (\mathcal{V}_\lambda(g_\lambda + h_\lambda)) \leqq (\mathcal{V}_\lambda(g_\lambda) \vee \mathcal{V}_\lambda(h_\lambda)) = (\mathcal{V}_\lambda(g_\lambda)) \vee (\mathcal{V}_\lambda(h_\lambda)) = \mathcal{V}(g) \vee \mathcal{V}(h)$. Therefore $\mathcal{V}$ is a B-valuation on G.

Corollary. *If each $\mathcal{V}_\lambda$ is a B_λ-norm then $\mathcal{V}$ is a B-norm.*

PROOF. Let $g = (g_\lambda) \in G$. Then $\mathcal{V}(g) = (\mathcal{V}_\lambda(g_\lambda)) = (\mathcal{V}_\lambda(-g_\lambda))$ (as each $\mathcal{V}_\lambda$ is a B-norm) $= \mathcal{V}(-g)$. Thus $\mathcal{V}$ is a B-norm on G.

Definition. The B-valuation $\mathcal{V}$ of (3. 4) is called the *direct product* of the B_λ-valuations $\mathcal{V}_\lambda$.

Definition. A topological group (G, T) is said to be *$(B, \mathcal{V}, M)$-metrizable* if there exists a Boolean algebra B and a dual ideal M of B such that (1) G admits a B-norm and (2) the (B, d, M)-topology on G is equivalent to T where d is the B-metric determined by $\mathcal{V}$.

Now any group G with the discrete topology is $(B, \mathcal{V}, M)$-metrizable; for, let B be the two element Boolean algebra $(0_B, 1_B)$ and let $M = B$. Define $\mathcal{V}(x) = 0_B$ if $x = 0$ and $\mathcal{V}(x) = 1_B$ if $x \neq 0$. Then $\mathcal{V}$ is obviously a B-norm on G and the (B, d, M)-topology on G is discrete, d being the B-metric determined by $\mathcal{V}$. It should also be remarked that any subgroup of a $(B, \mathcal{V}, M)$-metrizable topological group is $(B, \mathcal{V}, M)$-metrizable.

The proof of the following lemma can easily be verified.

(3. 5) Lemma. *Let B_λ, $(\lambda \in \Lambda)$, be a set of Boolean algebras and for each λ let M_λ be a dual ideal of B_λ. Let M_1 be the set of all elements $x = (x_\lambda)$, $(\lambda \in \Lambda)$ of the direct product $B = \prod_{\lambda \in \Lambda} B_\lambda$ such that $x_\lambda = 1_\lambda$ for all but possibly a finite number of places $\lambda = \lambda_i$ $(i = 1, \ldots, n)$ and for these λ_i, x_{λ_i} is an arbitrary element of M_{λ_i}. Then*

(1) M_1 is a dual ideal of B.

(2) If each M_λ is non-principal then so is M_1 and

(3) If $C(M_\lambda) = 0_\lambda$ for each $\lambda \in \Lambda$ then $C(M_1) = 0$ (where $C(M_\lambda) =$ the cut-complement of M_λ).

Now we shall prove

(3. 6) Proposition. *The direct product G (with the product topology) of any family (G_λ, T_λ), $(\lambda \in \Lambda)$,* of *$(B_\lambda, \mathcal{V}_\lambda, M_\lambda)$-metrizable topological groups is $(B, \mathcal{V}, M_1)$-metrizable (where B is the direct product of the B_λ's, $\mathcal{V}$ is the direct product of the $\mathcal{V}_\lambda$'s and M_1 is the dual ideal of B corresponding to the dual ideals M_λ, $(\lambda \in \Lambda)$, as defined in (3. 5)).*

PROOF. Let d be the B-metric determined by the B-norm $\mathcal{V}$ on G and for any element $m_1 \in M_1$ let

$$A_{m_1}(0) = [x \in G / \mathcal{V}(x) = d(x, 0) \leqq m_1]. \tag{I}$$

Then the subsets $[A_{m_1}(0)]$, $(m_1 \in M)$ form a nuclear base of G under the (B, d, M_1)-topology for G. We call this system of nuclei as system (I).

Any fundamental nucleus of G in the Cartesian product topology of G is of the form

$$\text{(II)} \qquad U(0) = \prod_{\lambda \in \Lambda} U_\lambda(0_\lambda)$$

where $U_\lambda(0_\lambda) = G_\lambda$, for $\lambda \neq \lambda_i$, $(i = 1, \ldots, n)$ and $U_{\lambda_i}(0_{\lambda_i}) = A_{m_{\lambda_i}}(0_{\lambda_i})$, $(i = 1, \ldots, n)$ where $m_{\lambda_i} \in M_{\lambda_i}$ and $A_{m_{\lambda_i}}(0_{\lambda_i}) = [x_{\lambda_i} \in G_{\lambda_i} / \mathcal{V}_{\lambda_i}(x_{\lambda_i}) \leqq m_{\lambda_i}]$. We call this system of nuclei of G as system (II).

Since G is a topological group, to prove that these two topologies are equivalent it suffices to prove their equivalence at zero.

Now given a nucleus $U(0)$ of G in system (II) let $U(0)$ be defined as in (II). Let $m_1 = (m_{1\lambda})$ such that $m_{1\lambda} = 1_\lambda$ for $\lambda \neq \lambda_i$, $(i = 1, \ldots, n)$ and $m_{1\lambda_i} = m_{\lambda_i}$ $(i = 1, \ldots, n)$. Then $m_1 \in M_1$. We will show that $A_{m_1}(0) \subseteqq U(0)$. If $x = (x_\lambda) \in A_{m_1}(0)$ then $\mathcal{V}(x) \leqq m_1$. Hence $\mathcal{V}_\lambda(x_\lambda) \leqq m_{1\lambda}$ for each $\lambda \in \Lambda$. In particular $\mathcal{V}_{\lambda_i}(x_{\lambda_i}) \leqq m_{1\lambda_i} = m_{\lambda_i}$ for $i = 1, \ldots, n$. Hence $x_{\lambda_i} \in A_{m_{\lambda_i}}(0_{\lambda_i}) = U_{\lambda_i}(0_{\lambda_i})$, $(i = 1, \ldots, n)$. Therefore $x \in U(0)$. Thus $A_{m_1}(0) \subseteqq U(0)$.

Next suppose a basic nucleus $A_{m_1}(0)$ of G in system (I) is given. Let $A_{m_1}(0)$ be defined as in (I). Then $m_1 = (m_{1\lambda}) \in M_1$. Hence $m_{1\lambda} = 1_\lambda$ for $\lambda \neq \lambda_i$, $(i = 1, \ldots, n)$ and $m_{\lambda_i} \in M_{\lambda_i}$, $(i = 1, \ldots, n)$. Define $U(0) = \prod_{\lambda \in \Lambda} U_\lambda(0_\lambda)$ where $U_\lambda(0_\lambda) = G_\lambda$ for $\lambda \neq \lambda_i$, $(i = 1, \ldots, n)$ and $U_{\lambda_i}(0_{\lambda_i}) = A_{m_{1\lambda_i}}(0_{\lambda_i})$, $(i = 1, \ldots, n)$. We will show that $U(0) \subseteqq A_{m_1}(0)$. If $x = (x_\lambda) \in U(0)$, then $x_{\lambda_i} \in A_{m_{\lambda_i}}(0_{\lambda_i})$, $(i = 1, \ldots, n)$. Hence $\mathcal{V}_{\lambda_i}(x_{\lambda_i}) \leqq m_{1\lambda_i}$ $(i = 1, \ldots, n)$. Hence $\mathcal{V}(x) = (\mathcal{V}_\lambda(x_\lambda)) \leqq m_1$. Therefore $x \in A_{m_1}(0)$. Thus $U(0) \subseteqq A_{m_1}(0)$. Therefore the two topologies are equivalent.

Corollary. *The projective limit of $(B_\lambda, \mathcal{V}_\lambda, M_\lambda)$-metrizable topological groups (G_λ, T_λ) is $(B, \mathcal{V}, M_1)$-metrizable.*

Definition. A topological group (G, T) is said to be a s-group if it has a nuclear base (i. e. a base of neighbourhoods of 0) consisting of subgroups.

We have:

(3. 7) Proposition. *Any $(B, \mathcal{V}, M)$-metrizable topological group (G, T) is a s-group.*

PROOF. Let d be the B-metric determined by the B-norm $\mathcal{V}$ on G. Since (G, T) is (B, d, M)-metrizable the subsets $A_m(0) = [y/d(y, 0) \leqq m]$, $(m \in M)$, form a nuclear base for (G, T). We will show that each $A_m(0)$ is a subgroup of G. If $x, y \in A_m(0)$ then $\mathcal{V}(x) = d(x, 0) \leqq m$, $\mathcal{V}(y) = d(y, 0) \leqq m$. Hence $d(x - y, 0) = \mathcal{V}(x - y) \leqq \mathcal{V}(x) \vee \mathcal{V}(-y) = \mathcal{V}(x) \vee \mathcal{V}(y) \leqq m \vee m = m$. Hence $x - y \in A_m(0)$. Thus $A_m(0)$ is a subgroup of G. This is true for each $m \in M$. Therefore (G, T) is a s-group.

(3. 8) Lemma. *If (G, T) is a s-group and H is any subgroup of G then G/H is a s-group with respect to the quotient topology.*

PROOF. Since (G, T) is a s-group it has a nuclear base consisting of subgroups $[N_\lambda]$, $(\lambda \in \Lambda)$. Let f be the natural homomorphism $G \to G/H$. Then the subsets $[f(N_\lambda)]$, $(\lambda \in \Lambda)$ form a nuclear base for G/H in the quotient topology (cf. [5]). Since f is a homomorphism and each N_λ is a subgroup of G it follows that $f(N_\lambda)$ is a subgroup of G/H for each $\lambda \in \Lambda$. Hence G/H is a s-group.

In a s-group (G, T) the base for the usual (two-sided) translation uniformity $\mathfrak{U}$ are the subsets $U_\lambda = [(x, y)/x \equiv y(N_\lambda)]$ where $[N_\lambda]$, $(\lambda \in \Lambda)$ is a nuclear base of subgroups for (G, T). By the uniformity of s-group (G, T) we shall mean $\mathfrak{U}$ and by the completion of (G, T) we shall mean the completion of $(G, \mathfrak{U})$. It should be observed that if $[N_\lambda]$, $[N_{1\mu}]$ are two nuclear bases of subgroups for (G, T) then $(G, \mathfrak{U})$, $(G, \mathfrak{U})$ are unimorphic ($\mathfrak{U}$, $\mathfrak{U}_1$ being the corresponding uniformities).

(3. 9) Proposition. *Any Hausdorff s-group can be uniformly imbedded as a dense subgroup of a projective limit of discrete groups.*

Proof. Let (G, T) be a Hausdorff s-group and let $[N_\lambda]$, $(\lambda \in \Lambda)$ be a nuclear base of subgroups of (G, T). Now $U_\lambda = [(x, y)/x \equiv y \pmod{N_\lambda}]$. Each N_λ being a subgroup of G defines a congruence of G. Hence every U_λ is symmetric and idempotent. Thus (G, T) is a Hausdorff topological group which is uniformisable by means of a uniformity having a symmetric and idempotent base. Hence by (2. 4) (G, T) can be uniformly embedded as a dense subspace of the projective limit I of discrete factor spaces G/N_λ, φ^μ_λ, (where for $\lambda < \mu$, $\varphi^\mu_\lambda[x]_\mu \to [x]_\lambda$, $x \in G$, and $[x]_\mu$ denotes the residue class containing x with respect to the congruence determined by N_μ). The mappings φ^λ_μ can easily be verified to be group homomorphisms. Therefore the projective limit I is a group. Further it can easily be seen that G can be embedded as a subgroup of I and this completes the proof.

As a consequence of (3. 9) we have

Corollary i. *The completion of a $(B, \mathfrak{P}, M)$-metrizable Hausdorff topological group (G, T) is a projective limit of discrete groups.*

Proof. Now by (3. 7) (G, T) is a s-group, and being also Hausdorff by (3. 9) it can be uniformly imbedded as a dense subgroup of a projective limit of discrete groups. Since any discrete group is complete it follows that the direct product P of discrete groups is complete. The projective limit I being a closed subgroup of P is therefore complete. Since (G, T) is unimorphic to a dense subgroup of I it follows that I is the completion of (G, T).

Corollary ii. *The completion of a $(B, \mathfrak{P}, M)$-metrizable Hausdorff topological group is $(B', \mathfrak{P}', M')$-metrizable.*

Proof. From corollary i to (3. 9) it follows that the completion of a $(B, \mathfrak{P}, M)$-metrizable topological group (G, T) is the projective limit I of discrete factor groups G_λ, $(\lambda \in \Lambda)$. Now each G_λ being discrete is $(B_\lambda, \mathfrak{P}_\lambda, M_\lambda)$-metrizable. Therefore I, being their projective limit is by the corollary to (3. 6) $(B', \mathfrak{P}', M')$-metrizable.

Now using (3. 9) we shall prove the converse of (3. 7) for Hausdorff topological groups. We have:

(3. 10) Proposition. *Any Hausdorff s-group (G, T) is $(B, \mathfrak{P}, M)$-metrizable.*

Proof. Let $[N_\lambda]$, $(\lambda \in \Lambda)$ be a nuclear base of subgroups of (G, T). Then by (3. 9) (G, T) is isomorphic and unimorphic to a dense subgroup of the projective limit I of the discrete factor groups $G_\lambda = G/N_\lambda$, $(\lambda \in \Lambda)$. Each G_λ, being discrete is $(B_\lambda, \mathfrak{P}_\lambda, M_\lambda)$-metrizable. I, being their projective limit is by the corollary to (3. 6) $(B, \mathfrak{P}, M)$-metrizable. (G, T) being a sugbroup of I is therefore $(B, \mathfrak{P}, M)$-metrizable.

Combining (3. 7) and (3. 10) we have:

(3. 11) Proposition. *The $(B, \mathfrak{P}, M)$-metrizable Hausdorff topological groups are precisely the Hausdorff s-groups.*

Thus (3. 11) characterizes the Hausdorff s-groups in terms of Boolean valuations. Now it has been proved in [8] that any zero dimensional locally compact Hausdorff topological group (G, T) contains arbitrarily small open subgroups. Thus (G, T) is a s-group and is therefore by (3. 11) $(B, \mathfrak{P}, M)$-metrizable. Therefore the zero-dimensional locally compact Hausdorff topological groups belong to the class of $(B, \mathfrak{P}, M)$-metrizable topological groups.

In (3. 11) we have shown that any Hausdorff s-group is $(B, \mathfrak{P}, M)$-metrizable. The question naturally arises as to whether this result can be extended to the non-Hausdorff topological groups also. Even though we do not know the complete answer to this question, yet we shall answer it in the affirmative for a particular class of topological groups viz. those s-groups with a nuclear base of subgroups whose intersection is a direct summand.

We recall that H is a direct summand of G if there exists a subgroup K of G such that $G = H + K$ and $H \cap K = 0$ (denoted by $G = H \dotplus K$. In this case K is isomorphic to G/H and H is isomorphic to G/K. It is also well known that the direct product $H \times K$ is isomorphic to G (the isomorphism being given by the mapping $f: (h, k) \to h + k$).

(3. 12) Lemma. *Let (G, T) be a s-group with a nuclear base $[N_\lambda]$, $(\lambda \in \Lambda)$ of subgroups such that $\bigcap_{\lambda \in \Lambda} N_\lambda = N$ is a direct summand of G, $G = N \dotplus K$. Then (G, T) is the direct product of N and K both algebraically and topologically.*

Proof. It suffices to prove that the mapping $f: (n, k) \to n + k$ of $N \times K \to N \dotplus K$ is a homeomorphism.

Since $N = \bigcap_{\lambda \in \Lambda} N_\lambda$, N has the coarsest (indiscrete) topology. Therefore any basic neighbourhood of (n, k) (in $G' = N \times K$) is of the form $(N, (N_\lambda \cap K)(k))$, $((N_\lambda \cap K)(k) =$ the residue class containing k in K with respect to the congruence determined by $N_\lambda \cap K$). Suppose a neighbourhood $N_\lambda(n+k)$ of $n+k$ is given. Then consider the neighbourhood $(N, (N_\lambda \cap K)(k))$ of (n, k) in G'. If $y \in f(N, (N_\lambda \cap K)(k))$ then $y = f((n_1, k_1)) = n_1 + k_1, n_1 \in N, k_1 \equiv k \pmod{N_\lambda \cap K}$ i. e. $k_1 - k \in N_\lambda \cap K$. Since $N \subseteq N_\lambda$ and $n_1, n \in N$ we have $k_1 - k + n_1 - n \in N_\lambda$ i. e. $y = n_1 + k_1 \in N_\lambda(n+k)$. Hence $f(N, (N_\lambda \cap K)(k)) \subseteq N_\lambda(n+k)$. Therefore f is continuous.

Conversely given $y = n + k \in G$, let the neighbourhood $(N, (N_\lambda \cap K)(k))$ of $(n, k) = f^{-1}(y) \in G'$ be given. Consider the neighbourhood $N_\lambda(n+k)$ of $n+k (=y)$. If $(n_1, k_1) = Y \in f^{-1}(N_\lambda(n+k))$ then $f(Y) = n_1 + k_1 \in N_\lambda(n+k)$. Hence $n_1 - n + k_1 - k \in N_\lambda$. Since $N \subseteq N_\lambda$ it follows that $n_1 - n \in N_\lambda$. Hence $k_1 - k \in N_\lambda$ i. e. $k_1 - k \in N_\lambda \cap K$. Hence $k_1 \in (N_\lambda \cap K)(k)$. Therefore $Y = (n_1, k_1) \in (N, (N_\lambda \cap K)(k))$, i. e. $f^{-1}(N_\lambda(n+k)) \subseteq (N, (N_\lambda \cap K)(k))$. Thus f^{-1} is also continuous and therefore f is a homeomorphism.

(3. 13) Lemma. *Let (G, T) be a s-group with a nuclear base $[N_\lambda]$, $(\lambda \in \Lambda)$ of subgroups such that $\bigcap_{\lambda \in \Lambda} N_\lambda = N$ is a direct summand of G. i. e. $G = N \dotplus K$. Then K is isomorphic and homeomorphic to G/N.*

PROOF. The mapping f of K on G/N defined by $f(k)=[k]=\varphi(k)$ (where $[k]$ is the residue class containing k with respect to the congruence determined by N and φ is the natural homeomorphism $G\to G/N$) can easily be verified to be an algebraic isomorphism. We will now show that it is a homeomorphism. Now any basic neighbourhood of $f(k)$ in G/N is of the form $(\varphi(N_\lambda))(f(k))$, $(\lambda\in\Lambda)$. Suppose a neighbourhood $(\varphi(N_\lambda))(f(k))$ of $f(k)$ in G/N is given. If $Y\in f((K\cap N_\lambda)(k))$, then $Y=f(z)$, $z\in K$, $z-k\in K\cap N_\lambda$. Hence $\varphi(z)-\varphi(k)=\varphi(z-k)\in\varphi(N_\lambda)$ i. e. $Y=f(z)=\varphi(z)\in\varphi(N_\lambda)(\varphi(k))=\varphi(N_\lambda)(f(k))$. Therefore $f((K\cap N_\lambda)(k))\subseteqq\varphi(N_\lambda)(f(k))$. Thus f is continuous.

Conversely suppose the neighbourhood $(K\cap N_\lambda)(k)$ of k is given. If $y\in f^{-1}(\varphi(N_\lambda)(f(k)))$, $(y\in K)$, then $\varphi(y-k)=\varphi(y)-\varphi(k)=f(y)-f(k)\in\varphi(N_\lambda)$. Hence $y-k-n_\lambda\in N$ for some $n_\lambda\in N_\lambda$. Since $N\subseteqq N_\lambda$ we have $y-k\in N_\lambda$ and $y-k\in K$. Hence $y\in(N_\lambda\cap K)(k)$. Therefore $f^{-1}(\varphi(N_\lambda))(f(k))\subseteqq(N_\lambda\cap K)(k)$. Thus f^{-1} is also continuous i. e. f is a homeomorphism.

Corollary. *Let (G, T) be a s-group with a nuclear base $[N_\lambda]$, $(\lambda\in\Lambda)$, of subgroups such that $N=\bigcap_{\lambda\in\Lambda}N_\lambda$ is a direct summand of G. Then (G, T) is isomorphic and homeomorphic to the direct product of N and G/N.*

PROOF. This is an immediate consequence of (3. 12) and (3. 13).

It should be observed that if (G, T) is a topological group with the coarsest topology then it is $(B, \mathcal{V}, M)$-metrizable; for let B be the two-element Boolean algebra $(0_B, 1_B)$ and let M be the dual ideal of B consisting of a single element 1_B. Let $\mathcal{V}$ be defined as follows: $\mathcal{V}(x)=0_B$ if $x=0$, $\mathcal{V}(x)=1_B$ if $x\neq 0$. Then $\mathcal{V}$ is obviously a B-norm on G. Further $A_{1_B}(0)=G$ is the only (basic) nucleus of G in the $(B, \mathcal{V}, M)$-topology of G. Therefore it is the coarsest topology on G. Now we will prove:

(3. 14) Proposition. *Let (G, T) be a s-group with a nuclear base $[N_\lambda]$, $(\lambda\in\Lambda)$ of subgroups such that $N=\bigcap_{\lambda\in\Lambda}N_\lambda$ is a direct summand. Then (G, T) is $(B, \mathcal{V}, M)$-metrizable.*

PROOF. Now by the corollary to (3. 13) (G, T) is isomorphic and homeomorphic to $N\times G/N$. Now N has the coarsest topology and is therefore $(B_2, \mathcal{V}_2, M_2)$-metrizable. Further N is closed in (G, T) as each N_λ is closed. Hence G/N is Hausdorff. Further G/N is a s-group. Therefore from (3. 10) it follows that G/N is $(B_3, \mathcal{V}_3, M_3)$-metrizable. (G, T) being the direct product of N and G/N is therefore by (3. 6) $(B, \mathcal{V}, M)$-metrizable.

It is known that any (real)metrizable topological group (Hausdorff) is metrizable by means of an invariant metric (cf. Birkhoff-Kakutani metrization theorem [4]). We shall now study a similar situation in the case of B-metrics.

Definition. A B-metric d on a group G is said to be *invariant* if $d(x, y)=d(x+a, y+a)$ for all $x, y, a\in G$.

(3. 15) Proposition. *A necessary and sufficient condition for a B-metric d on a group G to be invariant is that d is determined by a B-norm.*

PROOF. Suppose d is a B-metric determined by a B-norm. Then $d(x, y)=\mathcal{V}(x-y)$. Now $d(x+a, y+a)=\mathcal{V}(x+a-y-a)=\mathcal{V}(x-y)=d(x, y)$. This is true for all $x, y, a \in G$ and therefore d is invariant. Therefore the condition is sufficient.

Conversely suppose d is an invariant B-metric on G. Define $\mathcal{V}(x) = d(x, 0)$. Now $\mathcal{V}(x) = 0_B \Leftrightarrow d(x, 0) = 0_B \Leftrightarrow x = 0$. Again $\mathcal{V}(x+z) = d(x+z, 0) = d(0, x+z) \leqq d(0, x) \vee d(x, x+z) = d(0, x) \vee d(x-x, x+z-x) = d(0, x) \vee \vee d(0, z) = \mathcal{V}(x) \vee \mathcal{V}(z)$. Finally $\mathcal{V}(-x) = d(-x, 0) = d(-x+x, 0+x) = d(0, x) = \mathcal{V}(x)$. Thus $\mathcal{V}$ is a B-norm on G. Further $d(x, y) = d(x-y, y-y) = d(x-y, 0) = \mathcal{V}(x-y)$ for all $x, y \in G$. Thus d is the B-metric determined by the B-norm $\mathcal{V}$. Thus the condition is also necessary and this proves the result.

(3. 16) Lemma. *Let (G, T) be a topological group such that*

(1) *(G, T) is uniformisable with a uniformity $\mathfrak{U}$ having a symmetric and idempotent base and*

(2) *The group translations of G form an equicontinuous family with respect to $\mathfrak{U}$.*

Then (G, T) is a s-group.

Proof. Let $[U_\alpha]$, $(\alpha \in \Lambda)$, be a symmetric and idempotent base for $\mathfrak{U}$. For each $\alpha \in \Lambda$ define $V_\alpha = [(x, y) \in U_\alpha / (x+a, y+a) \in U_\alpha$ for each $a \in G]$. Then $V_\alpha \subseteqq U_\alpha$. Since the translations form an equicontinuous family with respect to $\mathfrak{U}$, given U_α, $(\alpha \in \Lambda)$, there exists a U_β, $(\beta \in \Lambda)$ such that $(x, y) \in U_\beta \Rightarrow (x+a, y+a) \in U_\alpha$ for all $a \in G$. In particular for $a = 0$ we have $(x, y) \in U_\beta \Rightarrow (x, y) \in U_\alpha$ i. e. $U_\beta \subseteqq U_\alpha$. Further $U_\beta \subseteqq V_\alpha$ (from definition of V_α). Therefore we have

$$\text{(I)} \qquad U_\beta \subseteqq V_\alpha \subseteqq U_\alpha.$$

From (I) it follows that the subsets V_α, $(\alpha \in \Lambda)$ of $G \times G$ form a base for the uniformity $\mathfrak{U}$. We will now show that each $V_\alpha(0)$ is an open subgroup of G.

Since $U_\alpha = U_\alpha^{-1}$ it follows that $V_\alpha = V_\alpha^{-1}$. Now if $x, y \in V_\alpha(0)$ then $(x, 0) \in V_\alpha$, $(y, 0) \in V_\alpha$. Hence $(0, y) \in V_\alpha$. Therefore

$$\text{(II)} \qquad (x+a, a) \in U_\alpha, \text{ and } (a, y+a) \in U_\alpha \text{ for each } a \in G.$$

Let $a^* = y+a$; as a runs through the elements of G, a^* also runs through the elements of G. Hence substituting in (II) for a, we have $(x-y+a^*, a^*-y) \in U_\alpha$ and $(a^*-y, a^*) \in U_\alpha$ for each element $a^* \in G$. Hence $(x-y+a^*, a^*) \in U_\alpha \circ U_\alpha = U_\alpha$ (as the base is idempotent) for each $a^* \in G$. Therefore $(x-y, 0) \in V_\alpha$. Thus $(x-y) \in V_\alpha(0)$. Therefore each $V_\alpha(0)$ is a subgroup of G.

Given $V_\alpha(0)$ choose U_β as in (I). If $x \in V_\alpha(0)$, $y \in U_\beta(x)$ then $(x, y) \in U_\beta \subseteqq V_\alpha$ (from (I)) and $(x, 0) \in V_\alpha$. Therefore $(x+a, y+a) \in U_\alpha$ for each $a \in G$ and $(x+a, a) \in U_\alpha$ for each $a \in G$. Since $U_\alpha = U_\alpha^{-1}$ we have $(y+a, a) \in U_\alpha$ for each $a \in G$. Therefore $(y, 0) \in V_\alpha$. Hence $y \in V_\alpha(0)$. Therefore $U_\beta(x) \subseteqq V_\alpha(0)$ i. e. $V_\alpha(0)$ is open. Given $\alpha \in \Lambda$, since $V_\alpha \subseteqq U_\alpha$, $V_\alpha(0) \subseteqq U_\alpha(0)$. Therefore (G, T) contains arbitrarily small open subgroups i. e. it is a s-group.

(3. 17) Proposition. *Let (G, T) be a Hausdorff topological group such that*

(1) *(G, T) is uniformisable with a uniformity $\mathfrak{U}$ having a symmetric and idempotent base and*

(2) *The group translations of G form an equicontinuous family with respect to the uniformity $\mathfrak{U}$.*

Then (G, T) is $(B, \mathcal{V}, M)$-metrizable.

PROOF. This follows from (3. 16) and (3. 10).

Since algebraic structures like rings and vector spaces admit a group operation, it is seen that the notion of Boolean valuation in groups can also be extended to them so as to take into account their extra operations. Thus we have:

Definition. A group Boolean-valuation $\mathcal{V}$ of a ring (commutative) R into a Boolean algebra B is called a *Boolean-valuation on R* (or *ring B-valuation*) if it satisfies further the following conditions viz. $\mathcal{V}(xz) \leqq \mathcal{V}(x) \wedge \mathcal{V}(z)$ (where $\wedge$ is the lattice product in B).

Since a ring B-valuation is also a group B-valuation we recall that $d(x, y) = \mathcal{V}(x-y)$ is a B-metric if and only if $\mathcal{V}(x) = \mathcal{V}(-x)$. In this case the corresponding B-valuation is called a *ring B-norm*. (If no distinction need be made we call it simply a B-norm on R).

Remark (i). *Let R be a ring with the unit e. Then any B-valuation $\mathcal{V}$ on R is a B-norm on R.*

PROOF. Now $\mathcal{V}(-x) = \mathcal{V}(-xe) = \mathcal{V}((x)(-e)) \leqq \mathcal{V}(x) \wedge \mathcal{V}(-e) \leqq \mathcal{V}(x)$. Again $\mathcal{V}(x) = \mathcal{V}((-x)(-e)) \leqq \mathcal{V}(-x) \wedge \mathcal{V}(-e) \leqq \mathcal{V}(-x)$. Therefore $\mathcal{V}(x) = \mathcal{V}(-x)$. Thus $\mathcal{V}$ is a B-norm on R.

Remark (ii). *Let R be a ring with the unit e and let $\mathcal{V}$ be a B-valuation on R. Then $\mathcal{V}(x) \leqq \mathcal{V}(e)$ for each $x \in R$.*

PROOF. $\mathcal{V}(x) = \mathcal{V}(x \cdot e) \leqq \mathcal{V}(x) \wedge \mathcal{V}(e) \leqq \mathcal{V}(e)$.

The following propositions follow exactly the same pattern of proof as in the case of groups.

(3. 18) Proposition. *Let R be a ring admitting a B-metric d and let M be any dual ideal of B. Then the ring operations of R are uniformly continuous with respect to the (B, d, M)-uniformity A_M on R if and only if the following conditions $(*)'$ is satisfied: $(*)'$. Given $m \in M$ there exists an element $n \in M$ such that $d(x, a) \leqq n$, $d(y, b) \leqq n \Rightarrow d(x-y, a-b) \leqq m$ and $d(xy, ab) \leqq m$ whatever be the elements $x, y, a, b \in R$.*

Corollary. *If d is the B-metric determined by the B-norm $\mathcal{V}$ on R then the ring operations of R are uniformly continuous with respect to any (B, d, M)-uniformity A_M on R.*

PROOF. To prove this it suffices to show that the condition $(*)'$ of (3. 18) is satisfied. It has already been observed that for the continuity of the group operations given $m \in M$, $d(x, a) \leqq m$, $d(y, b) \leqq m \Rightarrow d(x-y, a-b) \leqq m$. Now $d(xy, ab) = \mathcal{V}(xy-ab) = \mathcal{V}(xy-ay+ay-ab) = \mathcal{V}[y(x-a)+a(y-b)] \leqq [\mathcal{V}(y) \wedge \mathcal{V}(x-a)] \vee [\mathcal{V}(a) \wedge \mathcal{V}(y-b)] \leqq \mathcal{V}(x-a) \vee \mathcal{V}(y-b) \leqq m \vee m = m$. Thus the condition $(*)'$ of (3. 18) is satisfied and hence the result.

Exactly as for groups we can define the concept of $(B, \mathcal{V}, M)$-metrizability for topological rings. We also make the following:

Definition. A topological ring (R, T) is said to be an *I-ring* if it has a nuclear base consisting of ideals.

Then:

(3. 19) Proposition. *Any Hausdorff I-ring (R, T) can be uniformly imbedded as a dense subring of a projective limit I of discrete rings.*

Again we have:

(3. 20) Proposition. *The* $(B, \mathcal{V}, M)$*-metrizable Hausdorff topological rings are precisely the I-rings.*

In the non-Hausdorff case we have:

(3. 21) Proposition. *Let* (R, T) *be a I-ring with a nuclear base* $[I_\lambda]$, $(\lambda \in \Lambda)$ *of ideals such that* $I = \bigcap_{\lambda \in \Lambda} I_\lambda$ *is a (algebraic) direct summand. Then* (R, T) *is* $(B, \mathcal{V}, M)$*-metrizable.*

In the case of a vector space V over a field F a group B-valuation will be called a *vector space B-valuation* if it satisfies further the condition $\mathcal{V}(ax) \leqq \mathcal{V}(x)$ for all $a \in K$ and $x \in V$ (as F is a field the inequality will become an equality. But the above definition can also be extended for modules over arbitrary commutative rings). In this case it is seen that any vector space B-valuation is also a B-norm. A theory of $(B, \mathcal{V}, M)$-metrizability similar to the case of groups and rings can be developed for vector spaces also. Further as any vector subspace of a vector space V is a direct summand of V we have the following result:

The $(B, \mathcal{V}, M)$*-metrizable topological vector spaces are precisely the s-"vector spaces" (i. e. vector spaces having a base of neighbourhoods of zero consisting of vector subspaces).*

In this connection I wish to express my gratitude to Dr. V. S. Krishnan and Dr. V. K. Balachandran for their valuable guidance in the preparation of this paper.

Bibliography

[1] L. M. Blumenthal, Theory and applications of distance geometry, *Oxford*, 1953.
[2] D. Ellis, Geometry in abstract distance spaces, *Publ. Math. Debrecen* **2** (1951), 1–25.
[3] J. L. Kelley, General topology, *New York–Toronto–London*, 1955.
[4] D. Montgomery and L. Zippin, Topological Transformation groups, *New York–London*, 1955.
[5] L. Pontrjagin, Topologiche Gruppen *Leipzig*, 1957.
[6] P. S. Rema, Boolean metrization and topological spaces (to appear).
[7] P. S. Rema, On sets lattices and groups with Boolean metrization, *doctorate dissertation, University of Madras* (1963).
[8] E. R. van Kampen, Locally bicompact abelian groups and their character groups, *Ann. of Math.* **36** (1935), 448–463.

(Received April 24, 1964.)