On the summability of the Fourier series of L^2 integrable functions, II

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§ 1. Introduction

Let π_n and π_n' be the classes of the not identically vanishing n'th order trigonometrical polynomials

$$f(x) = \frac{a_0}{2} + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{1. 1}$$

and

$$f(x) = \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{1. 2}$$

respectively, $s_k(x) = s_k(x; f)$ the k'th partial sum of $f(x)$ and $C_n^{(m)}$ the least positive quantity for which the inequality

$$\frac{1}{m} \left| \sum_{r=1}^{m} s_{k_r}\left(\frac{2\pi r}{m}\right) \right| \leqq C_n^{(m)} \left\{ \frac{|a_0|^2}{2} + \sum_{\nu=1}^{n} (|a_\nu|^2 + |b_\nu|^2) \right\}^{\frac{1}{2}} \tag{1. 3}$$

holds for each set of indices $k_1, k_2, \ldots, k_m$ satisfying the inequalities $0 \leqq k_r \leqq n$ $(r = 1, 2, \ldots, m)$ and for each function $f(x) \in \pi_n$.

The existence of two sequences of positive integers $\{m_s\}$ and $\{n_s\}$ $(s = 1, 2, 3\ldots)$ both tending to infinity, with the sequence $\{C_{n_s}^{(m_s)}\}$ remaining bounded, would imply that the Fourier series of each L^2 integrable function would converge almost everywhere. (Cfr. [4].) In [4] I have shown that if

$$E_l(x) = \cos x + \cos 2x + \ldots + \cos lx, \quad E_0(x) = 0 \tag{1. 4}$$

then

$$C_n^{(m)} = \frac{1}{m} \max_{\substack{k_r = 0, 1, \ldots, n \\ r = 1, 2, \ldots, m}} \left\{ \frac{m^2}{2} + \sum_{p=1}^{m} \sum_{q=1}^{m} E_{\min(k_p, k_q)} \left(\frac{2\pi}{m} (p-q) \right) \right\}^{\frac{1}{2}} \tag{1. 5}$$

and I have formulated the following

Conjecture (C): *If* $m | n$ (including the trivial case $n = 0$) *then*

$$C_n^{(m)} = \sqrt{\frac{1}{2} + \frac{n}{m}}. \tag{1. 6}$$

Moreover equality occurs in (1. 3) *if and only if*

$$k_1 = k_2 = \dots = k_m = n$$

and $f(x)$ *is a multiple of*

$$\tag{1. 7} \tfrac{1}{2} + \cos mx + \cos 2mx + \dots + \cos nx.$$

Note that the conjecture implies that $C_n^{(n)}$ is bounded. One of the purposes of this paper is to verify this conjecture for $m \leqq 53$; for $m > 53$ its truth or falsity remains an open question.

The statement of the conjecture in the cases $m \leqq 53$ will follow from the somewhat stronger

Statement (C′) *If* $m|n$, $m \leqq 53$ *and* $\lambda_n^{(m)}$ *is the least positive quantity for which the inequality*

$$\tag{1. 8} \left| \sum_{r=1}^{m} s_{k_r}\left(\frac{2\pi}{m} r\right) \right| \leqq \lambda_n^{(m)} \left\{ \sum_{\nu=1}^{n} (|a_\nu|^2 + |b_\nu|^2) \right\}^{\frac{1}{2}}$$

holds with each set of indices $k_1, \dots, k_m$ *satisfying the inequalities* $0 \leqq k_r \leqq n$ $(r = 1, 2, \dots, m)$ *and for each function* $f(x) \in \pi_n'$ *then*

$$\tag{1. 9} \lambda_n^{(m)} = \sqrt{nm}.$$

Moreover equality occurs in (1. 8) *if and only if*

$$k_1 = k_2 = \dots = k_m = n$$

and $f(x)$ *is a multiple of*

$$\tag{1. 10} E_{n/m}(mx) = \cos mx + \cos 2mx + \dots + \cos nx.$$

In the meanwhile we shall have occasion to solve the following

Problem. *If* π_n'' *is the class of real trigonometrical polynomials of the form* (1. 2) *normed by the condition*

$$\sum_{\nu=1}^{n} (a_\nu^2 + b_\nu^2) = 1$$

and $1 \leqq r_1 < r_2 < \dots < r_\mu \leqq n \leqq 38$ ($r_\varkappa$ integer, $\varkappa = 1, 2, \dots, \mu$) *to find the least quantity* $\lambda_n'(r_1, r_2, \dots, r_\mu)$ *for which*

$$\tag{1. 11} \sum_{\varkappa=1}^{\mu} \max_{k=0,1,\dots,n} s_k\left(\frac{2\pi}{n} r_\varkappa\right) \leqq \lambda_n'(r_1, r_2, \dots, r_\mu) \qquad (f \in \pi_n'')$$

and to find the extremal functions for which (1. 11) *holds with sign of equality.*

The solution is:

$$\tag{1. 12} \lambda_n'(r_1, r_2, \dots, r_\mu) = \sqrt{n\mu}$$

irrespectively of the distribution of the integers $r_\varkappa$; they may be in a cluster or may be distributed more or less uniformly in the interval $(1, n)$.

This problem has one and only one extremal polynomial $\hat{f}(x)$ characterized by the following peculiar properties:

(i) $$\hat{f}\left(\frac{2\pi}{n}r\right)=\sqrt{\frac{n}{\mu}} \quad \text{if} \quad r\in(r_1, \dots, r_\mu);$$

(ii) $$\hat{f}\left(\frac{2\pi}{n}r\right)=0 \quad \text{if} \quad r\in(1, 2, \dots, n) \text{ and } r\notin(r_1, \dots, r_\mu);$$

(iii) $$\hat{f}\left(\frac{2\pi}{n}\left(r+\frac{1}{2}\right)\right)=-\sqrt{\frac{\mu}{n}} \quad \text{if} \quad r\in(1, 2, \dots, n);$$

(iv) its Fourier coefficient b_n vanishes.

So the extremal polynomial assumes only three different values in the points $\pi j/n$ $(j=1, 2, \dots, 2n;\ n\leqq 38)$.

With the method of this paper it may be possible to extend this result to some other n's, too, though this method falls short in treating the case of unrestricted n's [1]. If the Problem's solution would be of the form indicated above for each n, it would imply Statement (C′) for any m and Conjecture (C). Indeed Statement (C′), in the case of unrestricted n's would follow by Lemma 2 from the special case $\mu|n$, $r_\varkappa=\varkappa n/\mu$. However, there exist examples showing that the solution of the Problem in the case $n>38$ is not always given by formula (1. 12) nor do the extremal functions always possess properties (i)—(iv).

On the other hand this Problem is solved in a host of cases. For, in the case of a fixed n, there exists 2^n-1 possibilities of choosing μ numbers $r_1, r_2, \dots, r_\mu$ $(\mu=1, 2, \dots, n)$ from the set $1, 2, \dots, n$ and so the number of special cases in which the Problem is solved is $2^{39}-39$.*).

The most important special case is, however $\mu=n$. Then we have the exceedingly simple results

$$\lambda_n'(1, 2, \dots, n)=n$$

and $\hat{f}(x)=\cos nx$ $(n\leqq 38)$. If this solution of the particular case $\mu=n$ would hold without the restriction $n\leqq 38$ or at least for an infinity of n's, this would again imply that the Fourier series of any L^2 integrable function would converge almost everywhere.

Now if $r_1, r_2, \dots, r_n$ is any permutation of the numbers $1, 2, \dots, n$ $(n>38)$, little can be said about the sequence

$$\lambda_n'(r_1), \lambda_n'(r_1, r_2), \dots, \lambda_n'(r_1, r_2, \dots, r_{n-1}), \lambda_n'(r_1, r_2, \dots, r_n)$$

except that it consists of nondecreasing numbers. Indeed if $\hat{f}(x)\in\pi_n''$ is one of the extremals defined above, i. e.:

$$\lambda_n'(r_1, r_2, \dots, r_\mu)=\sum_{\varkappa=1}^{\mu}\max_{k=0,\dots,n} s_k\left(\frac{2\pi}{n}r_\varkappa; \hat{f}\right)$$

*) If one identifies those special cases which are trivially equivalent, this number must be reduced by a factor of about 1/75.

($\mu < n$, n unrestricted) then one has

$$\lambda_n'(r_1, \dots, r_\mu, r_{\mu+1}) \geqq \sum_{\varkappa=1}^{\mu+1} \max_{k=0,1,\dots,n} s_k\left(\frac{2\pi}{n} r_\varkappa; \hat{f}\right) \geqq$$

$$\geqq \sum_{\varkappa=1}^{\mu} \max_{k=0,\dots,n} s_k\left(\frac{2\pi}{n} r_\varkappa; \hat{f}\right) = \lambda_n'(r_1, \dots, r_\mu)$$

since $s_0(2\pi r_{\mu+1}/n, \hat{f})=0$.

It may be possible that this trivial inequality could be sharpened to

$$\lambda_n'(r_1, \dots, r_\mu) < \lambda_n'(r_1, \dots, r_{\mu+1}).$$

If this would be true for $\mu = n-1$ only, this would be decisive. For we shall prove the following

Theorem 1. *If for some n the inequality*

$$\lambda_n'(1, 2, \dots, n-1) < \lambda_n'(1, 2, \dots, n)$$

holds, then for this n one has $\lambda_n'(1, 2, \dots, n)=n$.

Again, one sees from the solution of the Problem that in the case $\mu=n\leqq 38$, the extremal function $\hat{f}(x)=\cos nx$ displays the following three features: (a) it is positive on the places $x=2\pi j/n$ $(j=1, 2, \dots, n)$, (b) it is an even function and (c) it is unique. In connection with this we have

Theorem 2. *If for some n each extremal function of* (1. 11) *in the case $\mu=n$ has property* (a) *or* (b), *or there exists only one extremal function, then for this n the equality $\lambda_n'(1, 2, \dots, n)=n$ holds.*

§ 2. Definitions, notations and lemmas

The notation $\|f\|$ will be used for any polynomial of the form (1. 1) to denote the quantity

$$\left\{\frac{|a_0|^2}{2} + \sum_{\nu=1}^{n} (|a_\nu|^2 + |b_\nu|^2)\right\}^{\frac{1}{2}}.$$

The classes π_n, π_n' and π_n'' of trigonometrical polynomials have been already defined in § 1.

For sake of simplicity the m'tuple of integers $k_1, k_2, \dots, k_m$ will be called the vector $\mathbf{k}$ and the set of all vectors admissible in (1. 5) (i. e. the set of all vectors the coordinates of which are nonnegative integers not exceeding n) the set K. Then if

(2. 1) $$E(k_1, \dots, k_m) = E(\mathbf{k}) = \sum_{p=1}^{m} \sum_{q=1}^{m} E_{\min(k_p, k_q)}\left(\frac{2\pi}{m}(p-q)\right),$$

the conjectured equality (1. 6) is by virtue of (1. 5) equivalent to

(2. 2) $$\max_{\mathbf{k}\in K} E(\mathbf{k}) = mn \qquad (m|n).$$

A vector $\mathbf{k}'$ will be called a *maximal vector* if

$$E(\mathbf{k}') = \max_{\mathbf{k} \in K} E(\mathbf{k})$$

and the set of maximal vectors for a given m and n will be denoted by K^*.

Finally it will be convenient to introduce the $m \times m$ matrix $\mathbf{E}(\mathbf{k}) = [e_{pq}]_{p,q=1}^{m}$ with the elements

$$e_{pq} = E_{\min(k_p, k_q)}\left(\frac{2\pi}{m}(p-q)\right). \tag{2. 3}$$

The scalar quantity $E(\mathbf{k})$ is equal to the sum of the elements of the matrix $\mathbf{E}(\mathbf{k})$.

We shall need the following lemmas.

Lemma 1. *If for some m and n $(m|n)$ the statement* (C') *holds, then for the same particular values of m and n the conjecture* (C) *is also true.*

Lemma 2. *Let* $\mathbf{x}$ *be a vector, the elements of which are the real numbers* $x_1, x_2, \dots, x_m$, *n a natural number* (not necessarily a multiple of m), $\mathbf{k} = \{k_1, k_2, \dots, k_m\}$ *a vector the elements of which are nonnegative integers not exceeding n, $\lambda'(\mathbf{x}, \mathbf{k})$ and $\lambda''(\mathbf{x}, \mathbf{k})$ the least quantities for which the inequalities*

$$\left| \sum_{r=1}^{m} s_{k_r}(x_r; f) \right| \leqq \lambda'(\mathbf{x}, \mathbf{k}) \qquad (f \in \pi_n', \|f\| = 1) \tag{2. 4}$$

and

$$\sum_{r=1}^{m} s_{k_r}(x_r; f) \leqq \lambda''(\mathbf{x}, \mathbf{k}) \qquad (f \in \pi_n'') \tag{2. 5}$$

respectively, hold. Then

$$\lambda'(\mathbf{x}, \mathbf{k}) = \lambda''(\mathbf{x}, \mathbf{k}) = \left\{ \sum_{p=1}^{m} \sum_{q=1}^{m} E_{\min(k_p, k_q)}(x_p - x_q) \right\}^{\frac{1}{2}}.$$

If $\lambda'(\mathbf{x}, \mathbf{k}) \neq 0$, then there exists but one extremal function in π_n'' for wich equality holds in (2. 5). *This is the polynomial*

$$f_{\mathbf{x}, \mathbf{k}}(x) = [\lambda'(\mathbf{x}, \mathbf{k})]^{-1} \sum_{r=1}^{m} E_{k_r}(x - x_r) \tag{2. 6}$$

and any extremal function in (2. 4) *is of the form $e^{i\alpha} f_{\mathbf{x}, \mathbf{k}}(x)$ where α is real.*

Lemma 3. *If $m|n$,* $\mathbf{k}$ *is an admissible vector,*

$$k_1 \equiv k_2 \equiv \dots \equiv k_m \equiv 0 \qquad (\mathrm{mod}\ m)$$

and $\mathbf{k}$ *is not equal to the vector*

$$\mathbf{k}^* = \{n, n, \dots, n\} \tag{2. 7}$$

then $\mathbf{k}$ *cannot be a maximal vector.*

Lemma 4. *If, for a particular pair of quantities m and n $(m|n)$ the set K^* contains the only element* $\mathbf{k}^*$, *then* (i) (1. 6) *and* (1. 9) *hold,* (ii) *in* (1. 3) *and* (1. 8) *equalities stand only if $f(x)$ is a multiple of* (1. 7) *or* (1. 10), *respectively.*

Lemma 5. *If* $m|n$, $\mathbf{k}\in K^*$ *then* $\min_{r=1,2,\dots,m} k_r \equiv 0 \pmod{m}$.

Lemma 6. *If for a particular* $m>2$ *the quantity*

$$H(m)=\max_{l=1,2,\dots,m-1}\left\{-m+l+4\sum_{r=1}^{\left[\frac{m-1}{2}\right]}\max\left\{E_l\left(\frac{2\pi}{m}r\right),0\right\}\right\} \tag{2. 8}$$

satisfies the inequality

$$H(m)<\frac{11}{3} \tag{2. 9}$$

and n *is any multiple of* m, *then the set* K^* *consists of the only vector* $\mathbf{k}^*$.

Lemma 7. *If* $H(n)<0$ *for a particular* n, *then for this* n *the solution of the Problem of the Introduction is that given in* § 1.

§ 3. Proof of Lemmas 1—5 and of the Theorems

If, by virtue of the supposition of Lemma 1 one has for some m and n $(m|n)$ and for any $f\in\pi_n'$

$$\frac{1}{m}\left|\sum_{r=1}^{m}\sum_{\nu=1}^{k_r}\left(a_\nu\cos\nu\frac{2\pi}{m}r+b_\nu\sin\nu\frac{2\pi}{m}r\right)\right|\leqq\sqrt{\frac{n}{m}}\left\{\sum_{\nu=1}^{n}(|a_\nu|^2+|b_\nu|^2)\right\}^{\frac{1}{2}}$$

with equality only if $f=c(\cos mx+\cos 2mx+\dots+\cos nx)$ and $k_1=k_2=\dots$ $\dots=k_m=n$, then for any a_0 we have

$$\frac{1}{m}\left|\sum_{r=1}^{m}\left\{\frac{a_0}{2}+\sum_{\nu=1}^{k_r}\left(a_\nu\cos\nu\frac{2\pi}{m}r+b_\nu\sin\nu\frac{2\pi}{m}r\right)\right\}\right|\leqq$$

$$\leqq\left|\frac{a_0}{2}\right|+\frac{1}{m}\left|\sum_{r=1}^{m}\sum_{\nu=1}^{k_r}\left(a_\nu\cos\nu\frac{2\pi}{m}r+b_\nu\sin\nu\frac{2\pi}{m}r\right)\right|\leqq$$

$$\leqq\frac{1}{\sqrt{2}}\left|\frac{a_0}{\sqrt{2}}\right|+\sqrt{\frac{n}{m}}\left\{\sum_{\nu=1}^{n}(|a_\nu|^2+|b_\nu|^2)\right\}^{\frac{1}{2}}\leqq\sqrt{\frac{1}{2}+\frac{n}{m}}\left\{\frac{|a_0|^2}{2}+\sum_{\nu=1}^{n}(|a_\nu|^2+|b_\nu|^2)\right\}^{\frac{1}{2}}$$

by Cauchy's inequality, with equality everywhere only if $k_1=k_2=\dots=k_m=n$ and $f=c(\frac{1}{2}+\cos mx+\cos 2mx+\dots+\cos nx)$. This proves Lemma 1.

Lemma 2 can be proved with the method of Kolmogoroff and Seliverstoff ([3]) just as it was done in a similar case in § 4 of Part I of this paper; we do not repeat it here.

To prove Lemma 3 we note that one has in this particular case with the notation of (2. 3)

$$e_{pp}=k_p,\ e_{pq}=\sum_{\varkappa=1}^{a_{pq}m}\cos\varkappa\frac{2\pi}{m}(p-q)=0\qquad(p\neq q) \tag{3. 1}$$

where the integer a_{pq} is defined by $\min(k_p, k_q) = a_{pq}m$ and so

$$E(\mathbf{k}) = \sum_{p=1}^{m} k_p \leqq \sum_{p=1}^{m} n = E(\mathbf{k}^*) \tag{3. 2}$$

with equality only if $\mathbf{k} = \mathbf{k}^*$.

In view of the foregoing and of formula (1. 5) the first half of the statement of Lemma 4 is evident. As to its second half we treat the case $f \in \pi_n$ only. Then we have

$$\frac{1}{m}\left|\sum_{r=1}^{m} s_{k_r}\left(\frac{2\pi}{m} r\right)\right| < \sqrt{\frac{1}{2} + \frac{n}{m}}\, \|f\|$$

if $\mathbf{k}$ is an admissible vector different from $\mathbf{k}^*$ and $f \in \pi_n$.

If, however $\mathbf{k} = \mathbf{k}^*$ one has by Cauchy's inequality

$$\frac{1}{m}\left|\sum_{r=1}^{m} s_{k_r}\left(\frac{2\pi}{m} r\right)\right| = \frac{1}{m}\left|\sum_{r=1}^{m} f\left(\frac{2\pi}{m} r\right)\right| = \left|\frac{a_0}{2} + a_m + a_{2m} + \dots + a_n\right| \leqq$$

$$\leqq \sqrt{\frac{1}{2} + \frac{n}{m}} \left\{\frac{|a_0|^2}{2} + |a_m|^2 + |a_{2m}|^2 + \dots + |a_n|^2\right\}^{\frac{1}{2}} \leqq \sqrt{\frac{1}{2} + \frac{n}{m}}\, \|f\|$$

and equality stands on both places only if $a_0 = a_m = a_{2m} = \dots = a_n$, $a_\nu = 0$ for $\nu \not\equiv 0 \pmod{m}$, $b_\nu = 0$ for each ν.

The case $f \in \pi_n'$ can be dealt with similarly.

Lemma 5 is an obvious version of Theorem 5 of Part I of this paper. Its proof is incorporated alongside with similar statements in § 6. (Formula (6. 3)).

Theorem 1 may be deduced from Lemmas 2 and 5. Lemma 5 states in the particular case $m = n$ that if $\mathbf{k} \in K^*$ and the vector $\mathbf{k}^*$ defined by (2. 7) is not contained in K^* then at least one element of $\mathbf{k}$ vanishes. We shall show that the assumption $\mathbf{k}^* \notin K^*$ is in contradiction with the supposition of Theorem 1, hence $\mathbf{k}^* \in K^*$ and $\lambda_n'(1, 2, \dots, n) = [E(\mathbf{k}^*)]^{\frac{1}{2}} = n$.

Indeed if $\mathbf{k} \notin K^*$ and $\mathbf{k}^\circ \in K^*$ then we may suppose without loss of generality that the last element of the vector $\mathbf{k}^\circ = \{k_1^\circ, \dots, k_{n-1}^\circ, k_n^\circ\}$ vanishes: $k_n^\circ = 0$. Then by Lemma 2 and by the definition (1. 11) of $\lambda_n'(1, 2, \dots, \mu)$

$$[\lambda_n'(1, 2, \dots, n)]^2 = \sum_{p=1}^{n} \sum_{q=1}^{n} E_{\min(k_p^0, k_q^0)}\left(\frac{2\pi}{n}(p-q)\right) =$$

$$= \sum_{p=1}^{n-1} \sum_{q=1}^{n-1} E_{\min(k_p^0, k_q^0)}\left(\frac{2\pi}{n}(p-q)\right) \leqq [\lambda_n'(1, 2, \dots, n-1)]^2$$

which contradicts the supposition of Theorem 1.

Finally turning to the proof of Theorem 2 let $f_\mu(x)$ be an extremal function of (1. 11) in the case $r_\varkappa = \varkappa$ $(\varkappa = 1, 2, \dots, \mu)$.

(a) If any partial sum (up to the n'th) of $f_{n-1}(x)$ is positive on the place $x=2\pi$, then $\lambda_n'(1, 2, \ldots, n)$ is by definition greater than $\lambda_n'(1, 2, \ldots, n-1)$:

$$\lambda_n'(1, 2, \ldots, n-1) = \sum_{\varkappa=1}^{n-1} \max_k s_k\left(\frac{2\pi}{n}\varkappa; f_{n-1}\right) <$$

$$< \sum_{\varkappa=1}^{n} \max_k s_k\left(\frac{2\pi}{n}\varkappa; f_{n-1}\right) \leqq \lambda_n'(1, 2, \ldots, n).$$

If however each of these partial sums in non-positive, then $\max\limits_k s_k(2\pi, f_{n-1}) = s_0(2\pi, f_{n-1}) = 0$ and by supposition (a) of Theorem 2 $f_n(x) \neq f_{n-1}(x)$, hence

$$\lambda_n'(1, 2, \ldots, n-1) = \sum_{\varkappa=1}^{n} \max_k s_k\left(\frac{2\pi}{n}\varkappa; f_{n-1}\right) < \lambda_n'(1, 2, \ldots, n).$$

In both cases we have $\lambda_n'(1, 2, \ldots, n-1) < \lambda_n'(1, 2, \ldots, n)$ and Theorem 1 can be applied.

Now from the definition of $\lambda_n'(1, 2, \ldots, n)$ it is clear that if

$$f_n(x) = \sum_{\nu=1}^{n} (a_\nu^* \cos \nu x + b_\nu^* \sin \nu x) \tag{3. 3}$$

is an extremal function, then

$$f_n\left(x - \frac{2\pi}{n}\right) = \tag{3. 4}$$

$$= \sum_{\nu=1}^{n} \left\{\left(a_\nu^* \cos\frac{2\pi}{n}\nu - b_\nu^* \sin\frac{2\pi}{n}\nu\right)\cos \nu x + \left(b_\nu^* \cos\frac{2\pi}{n}\nu + a_\nu^* \sin\frac{2\pi}{n}\nu\right)\sin \nu x\right\}$$

must also be an extremal.

(b) If, by supposition $f_n(x)$ and $f_n\left(x - \frac{2\pi}{n}\right)$ are both even functions, we have by (3. 3) and (3. 4)

$$b_\nu^* = 0, \quad a_\nu^* \sin\frac{2\pi}{n}\nu = 0 \qquad (\nu = 1, 2, \ldots, n)$$

hence for odd n's $a_\nu^* = 0$ for $\nu = 1, 2, \ldots, n-1$. Finally from $\|f_n(x)\| = 1$ we have $f_n(x) = \pm\cos nx$. Since only the positive sign yields a maximum, we can apply the first part of Theorem 2.

If, however, n is even we are led to the conclusion that any extremal must be of the form $f_n(x) = a_n^* \cos nx + a_{n/2}^* \cos nx/2$. Now we have to distinguish several cases.

Case 1: $a_n^* \leqq 0$. Then

$$\sum_{\varkappa=1}^{n} \max_{k=0,\ldots,n} s_k\left(\frac{2\pi}{n}\varkappa;\, a_{n/2}^* \cos\frac{n}{2}x + a_n^* \cos nx\right) =$$

$$= \sum_{\varkappa=1}^{n} \max_{k=0,\ldots,n/2} s_k\left(\frac{2\pi}{n}\varkappa;\, a_{n/2}^* \cos\frac{n}{2}x\right) = \frac{n}{2}|a_{n/2}^*| < n$$

since $|a_{n/2}^*| \leqq 1$.

Case 2. 1: $a_n^* > |a_{n/2}^*| \geqq 0$. Then

$$\sum_{\varkappa=1}^{n} \max_{k=0,\dots,n} s_k\left(\frac{2\pi}{n}\varkappa;\, a_{n/2}^* \cos\frac{n}{2}x + a_n^* \cos nx\right) = \sum_{\varkappa=1}^{n} [a_n^* + (-1)^{\varkappa} a_{n/2}^*] = na_n^* \leqq n$$

with equality only if $a_n^* = 1,\ a_{n/2}^* = 0$ since $a_n^{*2} + a_{n/2}^{*2} = 1$.

Case 2. 2: $|a_{n/2}^*| \geqq a_n^* > 0$. Now

$$\sum_{\varkappa=1}^{n} \max_{k=0,\dots,n} s_k\left(\frac{2\pi}{n}\varkappa;\, a_{n/2}^* \cos\frac{n}{2}x + a_n^* \cos nx\right) = \sum_{f_n\left(\frac{2\pi}{n}j\right)\geqq 0} f_n\left(\frac{2\pi}{n}j\right) =$$

$$= \frac{n}{2}(a_n^* + |a_{n/2}^*|) \leqq n\sqrt{\frac{a_n^{*2} + a_{n/2}^{*2}}{2}} < n \qquad (j = 1, 2, \dots, n).$$

Summing up the different cases the maximum is attained only for $f_n(x) = \cos nx$ and we are led again to the first part of Theorem 2.

(c) Turning to the last part of this theorem if $f_n(x)$ is unique for some n, then $f_n(x) = f_n\left(x - \frac{2\pi}{n}\right)$ and from (3. 3) and (3. 4) one has

$$a_\nu^* \cos\frac{2\pi}{n}\nu - b_\nu^* \sin\frac{2\pi}{n}\nu = a_\nu^*$$

$$a_\nu^* \sin\frac{2\pi}{n}\nu + b_\nu^* \cos\frac{2\pi}{n}\nu = b_\nu^*.$$

This system has only the trivial solution, save if $\nu = n$, i. e. the extremal function is necessarily of the form $f_n(x) = a_n^* \cos nx + b_n^* \sin nx$ $(a_\nu^{*2} + b_\nu^{*2} = 1)$. Now

$$\sum_{\varkappa=1}^{n} \max_{k=0,1,\dots,n} s_k\left(\frac{2\pi}{n}\varkappa;\, a_n^* \cos nx + b_n^* \sin nx\right) = \begin{cases} 0, & \text{if } a_n^* \leqq 0 \\ na_n^*, & \text{if } a_n^* \geqq 0 \end{cases}$$

and the left hand side is equal to n only if $a_n^* = 1,\ b_n^* = 0$, in all other cases it is less than n, hence $f_n(x) = \cos nx$, and $\lambda_n'(1, 2, \dots, n) = n$.

§ 4. A generalization of the problem of the introduction

Our next aim is to deal with the case $m \leqq 38$ in Statement (C′) of the Introduction and to solve simultaneously the problem given in § 1.

The solution of both problems is contained in the following

Lemma 8. *If* $m|n,\ k_r = 0, 1, \dots, n$ $(r = 1, 2, \dots, m)$, $m \leqq 38$, $1 \leqq r_1 < r_2 < \dots$ $\dots < r_\mu \leqq m$ ($r_\varkappa$ integer for $\varkappa = 1, 2, \dots, \mu$) *and* $\lambda_n^{(m)}(r_1, r_2, \dots, r_\mu)$ *is the least positive quantity for which*

$$\left|\sum_{\varkappa=1}^{\mu} s_{k_{r_\varkappa}}\left(\frac{2\pi}{m} r_\varkappa; f\right)\right| \leqq \lambda_n^{(m)}(r_1, r_2, \dots, r_\mu) \qquad (f \in \pi_n',\ \|f\| = 1) \tag{4. 1}$$

for any set $k_{r_1}, k_{r_2}, \dots, k_{r_\mu}$ *then*

$$\lambda_n^{(m)}(r_1, r_2, \dots, r_\mu) = \sqrt{n\mu}. \tag{4.2}$$

Equality occurs only if $k_{r_1} = k_{r_2} = \dots = k_{r_\mu} = n$ *and* $f(x)$ *is equal to*

$$\frac{1}{\sqrt{n\mu}} \sum_{\varkappa=1}^{\mu} E_n\left(x - \frac{2\pi}{m} r_\varkappa\right) \tag{4.3}$$

multiplied by a constant of modulus 1.

Hence statement (C′) follows for $m \leqq 38$ by taking $\mu = m$, $r_\varkappa = \varkappa$ $(\varkappa = 1, 2, \dots, m)$ since

$$\frac{1}{m} \sum_{r=1}^{m} E_n\left(x - \frac{2\pi}{m} r\right) = \frac{1}{m} \sum_{\nu=1}^{n} \sum_{r=1}^{m} \cos \nu \left(x - \frac{2\pi}{m} r\right) = \cos mx + \cos 2mx + \dots + \cos nx.$$

On the other hand, one has by Lemma 2, that if Lemma 8 holds, then with the notations employed there,

$$\sum_{\varkappa=1}^{\mu} s_{k_{r_\varkappa}}\left(\frac{2\pi}{m} r_\varkappa; f\right) \leqq \sqrt{n\mu} \qquad (f \in \pi_n'') \tag{4.4}$$

and equality occurs only if $k_{r_1} = k_{r_2} = \dots = k_{r_\mu} = n$ and $f(x)$ is equal to the function (4. 3) or with the notations of Lemma 2

$$\lambda''(\mathbf{x}, \mathbf{k}) = \sqrt{n\mu} \quad \text{if} \quad x = \left\{\frac{2\pi}{m} r_1, \dots, \frac{2\pi}{m} r_\mu\right\}, \quad \mathbf{k} = \{\overset{1}{n}, \overset{2}{n}, \dots, \overset{\mu}{n}\}.$$

In the special case $n = m$ the inequality (4. 4) is equivalent to (1. 11)—(1. 12). Since for $x \not\equiv 0 \pmod{2\pi}$

$$E_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)x}{2 \sin \frac{x}{2}} - \frac{1}{2},$$

one has

$$E_n(x) = \begin{cases} n & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x = \frac{2\pi}{n} r \qquad (r = 1, 2, \dots, n-1) \\ -1 & \text{if} \quad x = \frac{\pi}{n}(2r+1) \qquad (r = 0, 1, \dots, n-1) \end{cases}$$

further

$$E_n\left(x - \frac{2\pi}{n} r\right) = \sum_{\nu=1}^{n-1} \cos \nu \left(x - \frac{2\pi}{n} r\right) + \cos nx,$$

therefore in our special case the function (4. 3), the only function in π_n'' for which (4. 4) is satisfied with the sign of equality, is by virtue of (2. 6) the one characterized by conditions (i)—(iv) in § 1.

§ 5. Proof of Lemma 6 in the case $H(m)<0$ and of Lemmas 7 and 8

In order to prove Lemma 6 we shall give an upper estimation for the quantities $E(\mathbf{k})=\Sigma e_{pq}$ (see 2. 3). Suppose that γ_l of the elements of the vector $\mathbf{k}$ satisfy the congruences $k_p \equiv l \pmod m$ $(l=0, 1, \ldots, m-1)$ so that with $\gamma=\gamma_0$

(5. 1)
$$\gamma+\gamma_1+\gamma_2+\ldots+\gamma_{m-1} = m$$

and the elements $k_{p_1}, k_{p_2}, \ldots, k_{p_\gamma}$ should be divisible by m.

Putting $h=n/m$ we have

$$\sum_{p=1}^{m} e_{pp} = \sum_{p=1}^{m} k_p \leqq \sum_{i=1}^{\gamma} k_{p_i} + \sum_{l=1}^{m-1} \gamma_l \{(h-1)m+l\}.$$

On the other hand if $p \neq q$ and $x_r=2\pi r/m$, $e_{pq}=E_{\min(k_p, k_q)}(x_p-x_q) \leqq \leqq \max\{E_{k_p}(x_p-x_q), 0\}+\max\{E_{k_q}(x_p-x_q), 0\}$ and we can estimate the sum of the off-diagonal elements of $\mathbf{E}(\mathbf{k})$:

$$\sum_{p=1}^{m} \sum_{q \neq p} e_{pq} \leqq \sum_{p=1}^{m} \sum_{q \neq p} \max\{E_{k_p}(x_p-x_q), 0\} + \sum_{q=1}^{m} \sum_{p \neq q} \max\{E_{k_q}(x_q-x_p), 0\} =$$

$$= 2 \sum_{p=1}^{m} \sum_{r=1}^{m-1} \max\{E_{k_p}(x_r), 0\} = 4 \sum_{p=1}^{m} F(k_p, m)$$

where for $m>2$

$$F(k_p, m) = \frac{1}{2} \sum_{r=1}^{m-1} \max\{E_{k_p}(x_r), 0\} = \sum_{r=1}^{\left[\frac{m-1}{2}\right]} \max\{E_{k_p}(x_r), 0\}$$

since

$$E_{k_p}(x_r)=E_{k_p}(x_{m-r})$$

and in the case of an even m

$$E_{k_p}(x_{m/2}) = \cos \pi + \cos 2\pi + \ldots + \cos k_p \pi \leqq 0.$$

We remark that

(5. 2)
$$F(k_p, m) = F(k_p+m, m) \text{ and } F(0, m)=0.$$

Summing up we have

(5. 3)
$$E(\mathbf{k}) \leqq \sum_{i=1}^{\gamma} k_{p_i} + \sum_{l=1}^{m-1} \gamma_l \{(h-1)m+l\} + 4 \sum_{l=1}^{m-1} \gamma_l F(l, m) \leqq$$

$$\leqq \sum_{i=1}^{\gamma} k_{p_i} + \sum_{l=1}^{m-1} \gamma_l n + \sum_{l=1}^{m-1} \gamma_l H(m) = M(\mathbf{k})$$

by the definition (2. 8) of $H(m)$.

Table of the values of the function $H(m)$ for $33 \leqq m \leqq 60$
(If $3m \leqq 32$, then $H(m) = -1$.) [2]

m	$H(m)$	m	$H(m)$	m	$H(m)$	m	$H(m)$
33	−0,8370	40	0,3252	47	1,3644	54	4,1527
34	−0,8346	41	0,4410	48	2,5628	55	3,9579
35	−0,5882	42	1,0522	49	2,2586	56	3,9382
36	−0,1857	43	0,9025	50	2,5996	57	4,9962
37	−0,2992	44	0,8997	51	3,2941	58	4,5419
38	−0,4837	45	1,8167	52	2,9881	59	4,5401
39	0,4421	46	1,5694	53	2,8764	60	5,8271

The accompanying table of the values of $H(m)$ [2] shows that $H(m)<0$, if $2<m\leqq 38$ and so it follows immediately that

$$E(\mathbf{k}) \leqq M(\mathbf{k}) < \sum_{i=1}^{\gamma} k_{p_i} + \sum_{l=1}^{m-1} \gamma_l n \leqq \gamma n + \sum_{l=1}^{m-1} \gamma_l n = mn \tag{5.4}$$

if $\gamma_1+\gamma_2+\ldots+\gamma_{m-1}>0$ and $2<m\leqq 38$.

On the other hand if $m\leqq 38$ and $\gamma_1+\gamma_2+\ldots+\gamma_{m-1}=0$, then $\gamma=m$ and Lemma 3 can be applied:

$$E(\mathbf{k})\leqq E(\mathbf{k}^*)=M(\mathbf{k}^*)=mn \tag{5.5}$$

with equality only if $\mathbf{k}=\mathbf{k}^*$.

So we found that if $2<m\leqq 38$ the set K^* contains the only element $\mathbf{k}^*$ and Lemma 6 is proved for these m's. Using Lemmas 4 and 1 we see that Conjecture (C) and Statement (C′) are also proved for these m's.

Turning to Lemma 7 we regard it as a consequence of Lemma 8 in the particular case $m=n\leqq 38$ (cfr. the end of § 4) since we know that $H(n)<0$ for $n\leqq 38$ and shall prove the more general Lemma 8.

Lemma 8 was already proved in the particular case $\mu=m$ ($r_1=1, r_2=2, \ldots, r_m=m$) for we have shown that if $m\leqq 38$, $m|n$, then

$$E(\mathbf{k})\leqq M(\mathbf{k}),\ M(\mathbf{k})<mn \quad \text{if} \quad \mathbf{k}\in K \quad \text{and} \quad \mathbf{k}\neq\mathbf{k}^*$$

and

$$[\lambda_n^{(m)}(1, 2, \ldots, m)]^2 = \max_{\mathbf{k}\in K} E(\mathbf{k}) = E(\mathbf{k}^*) = mn.$$

Since K^* contains the only element $\mathbf{k}^*$ there is a unique extremal function which can be found by applying Lemma 2.

Now we turn to the case $\mu<m$. To any sequence of μ numbers $k_{r_1}, k_{r_2}, \ldots, k_{r_\mu}$ we adjoin two m-dimensional vectors $\mathbf{k}_\mathbf{r}$ and $\mathbf{k}^\mathbf{r}$ of the set K. The $r_\varkappa$'th components of both of these vectors will be equal to $k_{r_\varkappa}$ ($\varkappa=1, 2, \ldots, \mu$) whereas if ϱ is not equal to any of the numbers $r_1, r_2, \ldots, r_\mu$ then the ϱ'th component of $\mathbf{k}_\mathbf{r}$ will be 0 and the same component of $\mathbf{k}^\mathbf{r}$ will be n. The set of all vectors $\mathbf{k}_\mathbf{r}\in K$ will be denoted by $K_\mathbf{r}$, i. e. $K_\mathbf{r}$ is the set of all admissible vectors whose ϱ'th components are 0 if ϱ is not equal to any of the numbers $r_1, r_2, \ldots, r_\mu$.

We have by Lemma 2

$$(5.6) \qquad [\lambda_n^{(m)}(r_1, \ldots, r_\mu)]^2 = \max_{\substack{k_{r_\varkappa}=0,1,\ldots,n \\ \varkappa=1,2,\ldots,\mu}} \sum_{\sigma=1}^{\mu} \sum_{\tau=1}^{\mu} E_{\min(k_{r_\sigma}, k_{r_\tau})}\left(\frac{2\pi}{m}(r_\sigma - r_\tau)\right) =$$

$$= \max_{\mathbf{k}\in K_{\mathbf{r}}} \sum_{p=1}^{m} \sum_{q=1}^{m} E_{\min(k_p, k_q)}\left(\frac{2\pi}{m}(p-q)\right) = \max_{\mathbf{k}\in K_{\mathbf{r}}} E(\mathbf{k})$$

since in the case $\mathbf{k}\in K_{\mathbf{r}}$

$$E_{\min(k_p,k_q)}\left(\frac{2\pi}{m}(p-q)\right) = E_0\left(\frac{2\pi}{m}(p-q)\right) = 0$$

if at least one of the quantities p and q is not contained in the set $r_1, r_2, \ldots, r_\mu$.
Further, it follows from (5. 5) and (5. 3) that for $m \leqq 38$, $m|n$

$$(5.7) \qquad mn \geqq M(\mathbf{k}^{\mathbf{r}}) = M(\mathbf{k}_{\mathbf{r}}) + (m-\mu)n$$

with equality only if $\mathbf{k}^{\mathbf{r}} = \mathbf{k}^*$ i. e. the components of $\mathbf{k}^{\mathbf{r}}$ are either n's or 0's. This last vector will be denoted by $\mathbf{k}_{\mathbf{r}}^*$ and one has by (3. 2)

$$(5.8) \qquad E(\mathbf{k}_{\mathbf{r}}^*) = \mu n.$$

On the other hand from (5. 7) and (5. 3) we have that

$$\mu n > M(\mathbf{k}_{\mathbf{r}}) \geqq E(\mathbf{k}_{\mathbf{r}}),$$

if $\mathbf{k}_{\mathbf{r}} \in K_{\mathbf{r}}$, $\mathbf{k}_{\mathbf{r}} \neq \mathbf{k}_{\mathbf{r}}^*$. Hence by (5. 6)

$$(5.9) \qquad [\lambda_n^{(m)}(r_1, r_2, \ldots, r_\mu)]^2 \geqq E(\mathbf{k}) \qquad (\mathbf{k}\in K_{\mathbf{r}})$$

and equality stands only if $\mathbf{k} = \mathbf{k}_{\mathbf{r}}^*$. In view of (5. 8) this is the first part of the statement of Lemma 8. Its second part, formula (4. 3), is derived from the unicity property of the vector $\mathbf{k}_{\mathbf{r}}^*$ in connection with inequality (5. 9), and from the statement of Lemma 2 regarding the extremal function. With this, we have got the solution of the Problem of § 1, too.

§ 6. Proof of Lemma 6 in the case $0 \leqq H(m) \leqq 11/3$

In dealing with the case $0 \leqq H(m) \leqq 11/3$ (this contains by virtue of the Table the cases $39 \leqq m \leqq 53$), we shall need the estimation

$$(6.1) \qquad E_l(x) \geqq -C(l-1) - 1 \quad \text{with} \quad C = \frac{1-(3-2\sqrt{2})^2}{\sqrt{2}\pi} = 0.21845.$$

This estimation is obviously valid if $l=1$. Hence we may restrict ourselves to the cases $l>1$. Obviously it is sufficient to regard the interval $(0, \pi)$ and in this only the subintervals

$$\left(\frac{\pi}{l'}, \frac{2\pi}{l'}\right), \left(\frac{3\pi}{l'}, \frac{4\pi}{l'}\right), \left(\frac{5\pi}{l'}, \frac{6\pi}{l'}\right), \dots$$

where $l' = l+\frac{1}{2}$. Now if x lies in one of these subintervals but not in the first one, then

$$E_l(x) = \frac{\sin l' x}{2\sin\frac{1}{2}x} - \frac{1}{2} > \frac{\sin l'\left(x-\frac{2\pi}{l'}\right)}{2\sin\frac{1}{2}\left(x-\frac{2\pi}{l'}\right)} - \frac{1}{2} = E_l\left(x-\frac{2\pi}{l'}\right).$$

Hence the place x_0 of the absolute minimum of $E_l(x)$ in $(0, \pi)$ lies in $(\pi/l', 2\pi/l')$. Moreover it is easily seen that

$$\frac{d}{dx}E_l(x) > 0 \quad \text{if} \quad \frac{3\pi}{2l'} \leqq x \leqq \frac{2\pi}{l'}$$

and so $\pi/l' < x_0 < 3\pi/(2l')$.

We estimate $E_l(x)$ in this last interval as follows:

$$E_l(x) = -\left\{\cos\left(l'x - \frac{3}{2}\pi\right)\right\}\left\{2\sin\frac{x}{2}\right\}^{-1} - \frac{1}{2} \geqq$$

$$\geqq -\left\{1-\left(\frac{2}{\pi}\right)^2\left(l'x-\frac{3}{2}\pi\right)^2\right\}\left\{x\,\frac{\sin 3\pi/(4l')}{3\pi/(4l')}\right\}^{-1} - \frac{1}{2} = E_l^*(x).$$

Here we used the inequalities

$$\cos\alpha < 1-\left(\frac{2}{\pi}\right)^2\alpha^2 \quad \text{if} \quad 0<\alpha<\frac{\pi}{2} \quad \text{and} \quad \frac{\sin\beta}{\beta} > \frac{\sin\gamma}{\gamma} \quad \text{if} \quad 0<\beta<\gamma<\frac{\pi}{2}.\,^{*)}$$

The place and value of the minimum of the function $E_l^*(x)$ can be calculated directly and so we have

$$E_l(x) \geqq \min E_l^*(x) = -C\,\frac{3\pi/4}{\sin 3\pi/(4l')} - \frac{1}{2} > -C(l+1) - \frac{1}{2} > -C(l-1) - 1$$

since for $l \geqq 2$

$$\sin\frac{3\pi}{4l'} > \frac{3\pi}{4l'}\left[1-\frac{1}{6}\left(\frac{3\pi}{4l'}\right)\right]^2 > \frac{3\pi}{4(l+1)}.$$

*) The second of these inequalities is obvious. If one puts in it $\beta=\alpha/2$, $\gamma=\pi/4$ one has

$$\frac{\sin\alpha/2}{\alpha/2} > \frac{\sin\pi/4}{\pi/4} \quad \text{if} \quad 0<\alpha<\frac{\pi}{2}.$$

This is equivalent to the first inequality by virtue of the relation $1-\cos\alpha = 2\sin^2\alpha/2$.

Let now $\alpha_1, \alpha_2, \ldots, \alpha_m$ be such a permutation of the numbers $1, 2, \ldots, m$ that the coordinates of the m dimensional vector $\mathbf{k}$ satisfy the inequalities

$$k_{\alpha_1} \leqq k_{\alpha_2} \leqq \ldots \leqq k_{\alpha_m}.$$

Let further be

$$k_{\alpha_i} = k'_{\alpha_i} + k''_{\alpha_i} \qquad (i = 1, 2, \ldots, m)$$

where k'_{α_i} is the largest multiple of m not exceeding k_{α_i}.

We define the vector $\mathbf{k}^{(j)}$ as follows. All its coordinates are equal to the corresponding coordinates of $\mathbf{k}$, save the α_1'th, α_2'th, ..., α_j'th: these are $k'_{\alpha_1}, k'_{\alpha_2}, \ldots, k'_{\alpha_j}$, respectively.

Then by (2. 1)

$$E(\mathbf{k}) - E(\mathbf{k}^{(j)}) = -k''_{\alpha_1} + 2\sum_{r=1}^{m}\left\{E_{k_{\alpha_1}}\left(\frac{2\pi}{m}r\right) - E_{k'_{\alpha_1}}\left(\frac{2\pi}{m}r\right)\right\} -$$
$$- k''_{\alpha_2} + 2\sum_{r \neq \alpha_2 - \alpha_1}\left\{E_{k_{\alpha_2}}\left(\frac{2\pi}{m}r\right) - E_{k'_{\alpha_2}}\left(\frac{2\pi}{m}r\right)\right\} -$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$- k''_{\alpha_j} + 2\sum_{\substack{r \neq \alpha_j - \alpha_1 \\ r \neq \alpha_j - \alpha_2 \\ \ldots \\ r \neq \alpha_j - \alpha_{j-1}}}\left\{E_{k_{\alpha_j}}\left(\frac{2\pi}{m}r\right) - E_{k'_{\alpha_j}}\left(\frac{2\pi}{m}r\right)\right\}. \tag{6. 2}$$

Let us denote by S_i the i'th row of the right hand side of the last equality. Then

$$S_i = -k''_{\alpha_i} + 2\sum_{r=1}^{m}\left\{\cos\frac{2\pi}{m}r + \cos 2\cdot\frac{2\pi}{m}r + \ldots + \cos k''_{\alpha_i}\frac{2\pi}{m}r\right\} -$$
$$-2\sum_{\mu=1}^{i-1}\left\{\cos\frac{2\pi}{m}(\alpha_i - \alpha_\mu) + \cos 2\cdot\frac{2\pi}{m}(\alpha_i - \alpha_\mu) + \ldots + \cos k''_{\alpha_i}\frac{2\pi}{m}(\alpha_i - \alpha_\mu)\right\} =$$
$$= -k''_{\alpha_i} - 2\sum_{\mu=1}^{i-1} E_{k''_{\alpha_i}}(\alpha_i - \alpha_\mu) < -k''_{\alpha_i} + 2\sum_{\mu=1}^{i-1}\{C(k''_{\alpha_i} - 1) + 1\}$$

by (6. 1). From this follow

$$S_1 = -k''_{\alpha_1} \leqq 0$$
$$S_2 = (2C-1)k''_{\alpha_2} + 2(1-C) \leqq 2(1-C)$$
$$S_3 \leqq (4C-1)k''_{\alpha_3} + 4(1-C) \leqq 4(1-C)$$
$$S_4 \leqq (6C-1)(m-2) + 5.$$

Hence we have

$$E(\mathbf{k}) \leqq E(\mathbf{k}^{(1)}) \tag{6. 3}$$

$$E(\mathbf{k}) \leqq E(\mathbf{k}^{(2)}) + 2(1-C) < E(\mathbf{k}^{(2)}) + 2 \tag{6. 4}$$

$$E(\mathbf{k}) \leqq E(\mathbf{k}^{(3)}) + 6(1-C) < E(\mathbf{k}^{(3)}) + 5 \tag{6. 5}$$

$$E(\mathbf{k}) \leqq E(\mathbf{k}^{(4)}) + (6C-1)(m-2) + 5 + 6(1-C) < E(\mathbf{k}^{(4)}) + \frac{m}{3} + 10. \tag{6. 6}$$

We divide now the set K into the following five subsets.

(i) $K^{(1)}$: it contains the only element $\mathbf{k}^*$ defined by (2. 7).

(ii) $K^{(2)}$: its vectors are characterized by $k_{\alpha_1} < n, k_{\alpha_2} = \ldots = k_{\alpha_m} = n$.

(iii) $K^{(3)}$: its vectors are characterized by $k_{\alpha_1} \leqq k_{\alpha_2} < n, k_{\alpha_3} = \ldots = k_{\alpha_m} = n$.

(iv) $K^{(4)}$: its vectors are characterized by $k_{\alpha_1} \leqq k_{\alpha_2} \leqq k_{\alpha_3} < n,\ k_{\alpha_4} = \ldots = k_{\alpha_m} = n$.

(v) $K^{(5)}$: its vectors are characterized by $k_{\alpha_1} \leqq k_{\alpha_2} \leqq k_{\alpha_3} \leqq k_{\alpha_4} < n$.

Clearly $K = K^{(1)} + K^{(2)} + \ldots + K^{(5)}$.

Our purpose is now to show that if $m|n$ and $m \geqq 5$, then the sets $K^{(2)}, K^{(3)}$ and $K^{(4)}$ contain no maximal vectors and if $m|n$, $H(m) < 11/3$ the subset $K^{(5)}$, too, has no maximal element.

Suppose $\mathbf{k} \in K^{(2)}$. Then by (6. 3) and Lemma 3 $E(\mathbf{k}) < E(\mathbf{k}^*)$.

Again, if $\mathbf{k} \in K^{(3)}$, then

$$E(\mathbf{k}) < E(\mathbf{k}^{(2)}) + 2 = k_1' + k_2' + (m-2)n + 2 \leqq$$

$$\leqq 2(n-m) + (m-2)n + 2 = (n-2)m + 2 < E(\mathbf{k}^*)$$

and similarly if $\mathbf{k} \in K^{(4)}$, then

$$E(\mathbf{k}) \leqq E(\mathbf{k}^{(3)}) + 5 < (n-3)m + 5 < E(\mathbf{k}^*).$$

Finally if $\mathbf{k} \in K^{(5)}, m \geqq 5$, then by (6. 6) and (5. 3)

$$\begin{aligned} E(\mathbf{k}) &< E(\mathbf{k}^{(4)}) + \frac{m}{3} + 10 \\ &\leqq \gamma n - 4m + (m-\gamma)n + \sum_{l=1}^{m-1} \gamma_l H(m) + \frac{m}{3} + 10 \\ &\leqq mn - 4m + (m-4)H(m) + \frac{m}{3} + 10 \\ &= mn + \left\{H(m) - \frac{11}{3}\right\}(m-3) - H(m) - 1 \end{aligned}$$

where $\gamma, \gamma_1, \ldots, \gamma_{m-1}$ are the numbers defined in Section 5, referred to the vector $\mathbf{k}^{(4)}$.

The last row is less than $E(\mathbf{k}^*) = mn$, if $-1 \leqq H(m) \leqq 11/3$ i. e. by the Table if $2 < m \leqq 53$, and so via Lemma 4 Statement (C') of § 1 is verified, too, for these m's.

§ 7. Proof of Statement (C') in the cases $m = 1$ and $m = 2$

The remaining cases $m = 1$ and $m = 2$ can be treated quite simply. In the case $m = 1, k_1 \leqq n, f \in \pi_n'$

$$|s_{k_1}(2\pi)| = |a_1 + a_2 + \ldots + a_{k_1}| \leqq \sqrt{k_1}\,\{|a_1|^2 + |a_2|^2 + \ldots + |a_{k_1}|^2\}^{\frac{1}{2}} \leqq$$

$$\leqq \sqrt{n}\left\{\sum_{\nu=1}^{n} (|a_\nu|^2 + |b_\nu|^2)\right\}^{\frac{1}{2}}$$

and equality stands on both places only if $k_1=n$, $a_1=a_2=\ldots=a_n$, $b_1=b_2=\ldots$ $\ldots=b_n=0$.

In the case $m=2$ and $k_1 \leqq k_2 \leqq n$, say,

$$\frac{1}{2}\,|s_{k_1}(\pi)+s_{k_2}(2\pi)| =$$

$$= \left| a_2+a_4+\ldots+\frac{1+(-1)^{k_1}}{2}\,a_{k_1}+\frac{1}{2}\,a_{k_1+1}+\frac{1}{2}\,a_{k_1+2}+\ldots+\frac{1}{2}\,a_{k_2} \right| \leqq$$

$$\leqq \left\{ \left[\frac{k_1}{2}\right]+\frac{1}{2^2}(k_2-k_1) \right\}^{\frac{1}{2}} \{|a_2|^2+|a_4|^2+\ldots+|a_{2[k_1/2]}|^2+$$

$$+|a_{k_1+1}|^2+|a_{k_1+2}|^2+\ldots+|a_{k_2}|\}^{\frac{1}{2}} \leqq \left(\frac{k_1+k_2}{4}\right)^{\frac{1}{2}} \|f\| \leqq \sqrt{\frac{n}{2}}\,\|f\|$$

with equality everywhere only if $k_1=k_2=n$, n even, and $a_\nu=a_2$ if ν is even, $a_\nu=0$ if ν is odd and $b_\nu=0$ for each ν. *)

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The table of the values of $H(m)$ was prepared on the electronic computer FINAC by the Istituto Nazionale per le Applicazioni del Calcolo, Rome ([2]). I express my sincere thanks to this institution and its director, A. GHIZZETTI for the work done.

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*) (*Note added on proof*, September, 1965.) In the meantime I could find the following partial answer to the question posed on page 92: *If* $n>2$, *the sequence* $\lambda_n'(1)$, $\lambda_n'(1,2)$, $\ldots$, $\lambda_n'(1,2,\ldots,[n/2])$ *consists of strictly increasing numbers.*

Indeed denoting $\sum_{p,q=1}^{h} E_{\min(k_p,k_q)}\left(\frac{2\pi}{n}(p-q)\right)$ by $E_n(k_1,\ldots,k_h)$ we have

$$\lambda_n'^2(1,2,\ldots,l-1) = \max_{\substack{k_r=0,1,\ldots,n \\ r=1,2,\ldots,l-1}} E_n(k_1,\ldots,k_{l-1}) = E_n(\hat{k}_1,\ldots,\hat{k}_{l-1}),$$

say, and we may write

$$E_n(1,\hat{k}_1,\ldots,\hat{k}_{l-1})+E_n(\hat{k}_1,\hat{k}_2,\ldots,\hat{k}_{l-1},1) =$$

$$= 2E_n(\hat{k}_1,\ldots,\hat{k}_{l-1})+2+\sum_{\hat{k}_g>0}\left\{\cos\frac{2\pi}{n}(l-g)+\cos\frac{2\pi}{n}g\right\}.$$

If $l \leqq n/2$ then each term of the sum of the right-hand side is non-negative, since $\cos\alpha+\cos\beta \geqq 0$ if $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha+\beta \leqq \pi$ and so

$$\lambda_n'^2(1,2,\ldots,l-1)<\{E_n(1,\hat{k}_1,\ldots,\hat{k}_{l-1})+E_n(\hat{k}_1,\ldots,\hat{k}_{l-1},1)\}/2 \leqq$$

$$\leqq \max\{E_n(1,\hat{k}_1,\ldots,\hat{k}_{l-1}),\,E_n(\hat{k}_1,\ldots,\hat{k}_{l-1},1)\}$$

and, a fortiori

$$\lambda_n'^2(1,2,\ldots,l-1)<\lambda_n'^2(1,2,\ldots,l) \qquad (l=1,2,\ldots,[n/2])$$

by the definition of $\lambda_n'(1,2,\ldots,l)$.

For the solution of another special case of the problem treated in this paper, see the author's forthcoming article: A property of Dirichlet's kernel (Magyar Tud. Akadémia Mat. Kut. Int. Közl., in the press).

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