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A note on $\mathcal{M}$-harmonic Besov p-spaces

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Abstract. Two new characterizations of the $\mathcal{M}$-harmonic Besov space $\mathcal{M}B^p, 1 < p < \infty$, are given.

1. Introduction

The class of $\mathcal{M}B^p$-harmonic (and thus holomorphic) Besov p-functions was studied by several authors (see [2], [3], [4], [5], [7], [9], [10], [12]). Motivated by the characterizations of $\mathcal{M}$-harmonic Bloch space $\mathcal{M}B^\infty$ obtained in [7] we give two new characterizations of the space $\mathcal{M}B^p, 1 < p < \infty$, (Theorem 1).

The class of MB^p-functions, $1 < p < \infty$, which contains the space $\mathcal{M}B^p$, was defined and characterized by M. Pavlović and the author in [6]. We close this note by stating several additional characterizations of the class MB^p (Theorem 2 and Theorem 3). Since they are straightforward extensions of the well known results on $\mathcal{M}$-harmonic and holomorphic Besov p-functions, the proofs will be omitted.

Let B be the open unit ball in $\mathbb{C}^n$ and S the unit sphere in $\mathbb{C}^n$. We denote by ν the normalized Lebesgue measure on B and by σ the rotation invariant probability measure on S.

Let $\widetilde{\Delta}$ be the invariant Laplacian on B. That is, $\widetilde{\Delta}f(z) = \Delta(f \circ \varphi_z)(0)$, $f \in C^2(B)$, where Δ is the ordinary Laplacian and φ_z the standard automorphism of B taking 0 to z ([11]).

As in [11], we say that a function $f \in C^2(B)$ is $\mathcal{M}$-harmonic in B, $f \in \mathcal{M}$, if $\widetilde{\Delta}f(z) = 0$ for every $z \in B$.

For $f \in C^1(B)$, $Df = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$, denotes the complex gradient of f, $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{2n}}\right)$, $z_k = x_{2k-1} + ix_{2k}$, $k = 1, 2, \dots, n$, denotes the real gradient.

For $f \in C^1(B)$ let $\widetilde{D}f(z) = D(f \circ \varphi_z)(0)$, $z \in B$, and $\widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$, $z \in B$, be the invariant complex gradient of f and the invariant real gradient of f respectively.

Let $1 < p < \infty$ and let B^p be the linear space of all functions $f \in C^1(B)$ such that $\| |\widetilde{\nabla} f| \|_{L^p(\tau)} < \infty$, where $d\tau(z) = (1 - |z|^2)^{-n-1} d\nu(z)$. We shall call $\mathcal{M}B^p = \mathcal{M} \cap B^p$, $1 < p < \infty$, $\mathcal{M}$-harmonic Besov p-space.

Theorem 1. *Let $f \in \mathcal{M}$, $2n < p < \infty$. Then the following statements are equivalent:*

(i) $f \in \mathcal{M}B^p$,

(ii) $\int_B (1 - |z|^2)^{p-n-1} |\nabla f(z)|^p d\nu(z) < \infty$,

(iii) $\int_B (1 - |z|^2)^{p-n-1} (|Rf(z)|^p + |R\bar{f}(z)|^p) d\nu(z) < \infty$.

Here, as usual, $R = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$, $\bar{R} = \sum_{j=1}^{n} \bar{z}_j \frac{\partial}{\partial \bar{z}_j}$ are radial derivatives.

2. Proof of Theorem 1

For the proof of Theorem 1 the folowing lemmas will be needed.

Lemma 2.1 ([7]). *Let $0 < r < 1$. There is a constant C such that if $f \in \mathcal{M}$ then*

(a) $|T_{ij} Rf(w)| \le C(1 - |w|^2)^{-1/2} \int_{E_r(w)} |Rf(z)| d\tau(z),\ w \in B,$

(b) $|T_{ij} \bar{R}f(w)| \le C(1 - |w|^2)^{-1/2} \int_{E_r(w)} |\bar{R}f(z)| d\tau(z),\ w \in B,$

(c) $|T_{ij} f(w)| \le C(1 - |w|^2)^{-1/2} \int_{E_r(w)} |f(z)| d\tau(z),\ w \in B.$

Here, $T_{ij} = \bar{z}_i \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial z_i}$ denotes the tangential derivative.

For $z \in B$ and r between 0 and 1, $E_r(z) = \{w \in B : |\varphi_z(w)| < r\}$. We set $|E_r(z)| = \nu(E_r(z))$.

In this note we follow the custom of using the letter C to stand for a positive constant which changes its value from one appearance to another while remaining independent of the important variables.

Lemma 2.2. *If $s > 1$, then*

$$\int_0^1 \frac{dt}{|1 - t\langle z, w\rangle|^s} \leq \frac{C}{|1 - \langle z, w\rangle|^{s-1}}, \quad z, w \in B.$$

Lemma 2.3 ([11], p. 17)**.** *If $\alpha > 0$, then*

$$\int_S \frac{d\sigma(\xi)}{|1 - \langle \xi, z\rangle|^{n+\alpha}} = O\left(\frac{1}{(1 - |z|)^\alpha}\right), \quad z \in B.$$

Lemma 2.4. *For $0 < s < t$ we have*

$$\int_0^1 \frac{(1-\rho)^{s-1}}{(1 - r\rho)^t} d\rho \leq C(1 - r)^{s-t}, \quad 0 \leq r < 1.$$

Proof of Theorem 1. In [8] it is shown that

$$\begin{aligned} |\widetilde{\nabla} f(z)|^2 &= 2(|\widetilde{D} f(z)|^2 + |\widetilde{D} \bar{f}(z)|^2) \\ &= 2(1 - |z|^2)(|Df(z)|^2 - |Rf(z)|^2 + |D\bar{f}(z)|^2 - |R\bar{f}(z)|^2). \end{aligned} \tag{2.1}$$

An application of Cauchy–Schwarz inequality shows that $|\widetilde{\nabla} f(z)| \geq (1 - |z|^2)|\nabla f(z)|$. Thus, (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (iii). It is easy to see that if (ii) holds then

$$\int_B (1 - |z|^2)^{p-n-1} \left|\frac{\partial f}{\partial z_j}(z)\right|^p d\nu(z) < \infty, \quad 1 \leq j \leq n,$$

and

$$\int_B (1 - |z|^2)^{p-n-1} \left|\frac{\partial f}{\partial \bar{z}_j}(z)\right|^p d\nu(z) < \infty, \quad 1 \leq j \leq n,$$

which in turn implies that

$$\int_B (1 - |z|^2)^{p-n-1} |Rf(z)|^p d\nu(z) < \infty$$

and

$$\int_B (1 - |z|^2)^{p-n-1} |R\bar{f}(z)|^p d\nu(z) < \infty.$$

(iii) $\Longrightarrow$ (i). Assume, now, that

$$\int_B (1-|z|^2)^{p-n-1}(|Rf(z)|^p + |R\bar{f}(z)|^p)d\nu(z) < \infty.$$

It is easy to check hat

$$|z|^2|Df(z)|^2 = |Rf(z)|^2 + \sum_{i<j}|T_{ij}f(z)|^2.$$

Using this and (2.1) we find that

$$|z|^2|\widetilde{\nabla} f(z)|^2 = 2(1-|z|^2)\Big[(1-|z|^2)(|Rf(z)|^2 + |R\bar{f}(z)|^2) + \sum_{i<j}|T_{ij}f(z)|^2 + \sum_{i<j}|T_{ij}\bar{f}(z)|^2\Big].$$

Hence, to show that $f \in \mathcal{M}B^p$ it is sufficient to show that

$$\int_B (1-|z|^2)^{\frac{p}{2}-n-1}(|T_{ij}f(z)|^p + |T_{ij}\bar{f}(z)|^p)d\nu(z) < \infty, \quad 1 \le i < j \le n.$$

An integration by parts show that

$$f(z) = \int_0^1 [Rf(tz) + \bar{R}f(tz) + f(tz)]dt.$$

From this we conclude that it is sufficient to prove that

$$\int_B (1-|z|^2)^{\frac{p}{2}-n-1}\left(\int_0^1 |T_{ij}u(tz)|dt\right)^p d\nu(z) < \infty, \quad 1 \le i < j \le n,$$

where $u(z) = Rf(z)$ or $\bar{R}f(z)$ or $R\bar{f}(z)$ or $\bar{R}\bar{f}(z)$ or $f(z)$.

We will show that, for fixed $1 \le i < j \le n$,

$$I = \int_B (1-|z|^2)^{\frac{p}{2}-n-1}\left(\int_0^1 |T_{ij}Rf(tz)|dt\right)^p d\nu(z) < \infty.$$

The remaining cases may be treated analogously.

Using Lemma 2.1, Fubini's theorem and Lemma 2.2 we find that for any $s > 0$

$$\begin{aligned}\int_0^1 |T_{ij}Rf(tz)|dt &\leq C\int_0^1 \left(\int_{E_r(tz)} \frac{|Rf(w)|(1-|w|^2)^s}{|1-\langle tz, w\rangle|^{n+s+\frac{3}{2}}} d\nu(w)\right) dt \\ &\leq C\int_0^1 \left(\int_B \frac{|Rf(w)|(1-|w|^2)^s d\nu(w)}{|1-t\langle z, w\rangle|^{n+s+\frac{3}{2}}}\right) dt \\ &= C\int_B |Rf(w)|(1-|w|^2)^s \left(\int_0^1 \frac{dt}{|1-t\langle z, w\rangle|^{n+s+\frac{3}{2}}}\right) d\nu(w) \\ &\leq C\int_B \frac{|Rf(w)|(1-|w|^2)^s}{|1-\langle z, w\rangle|^{n+s+\frac{1}{2}}} d\nu(w).\end{aligned}$$

Applying the continuous form of Minkowski's inequality we obtain

$$\begin{aligned}I &\leq C\int_0^1 (1-r)^{\frac{p}{2}-n-1} \\ &\times \left(\int_0^1 \left(\int_S \left(\int_S \frac{|Rf(\rho\xi)|(1-\rho)^s d\sigma(\xi)}{|1-\langle r\zeta, \rho\xi\rangle|^{n+s+\frac{1}{2}}}\right)^p d\sigma(\zeta)\right)^{1/p} d\rho\right)^p dr. \end{aligned} \tag{2.2}$$

By Hölder's inequality

$$\begin{aligned}&\int_S \frac{|Rf(\rho\xi)|d\sigma(\xi)}{|1-\langle r\zeta, \rho\xi\rangle|^{n+s+\frac{1}{2}}} \\ &\leq \left(\int_S \frac{|Rf(\rho\xi)|d\sigma(\xi)}{|1-\langle r\zeta, \rho\xi\rangle|^{n+s+\frac{1}{2}}}\right)^{1/p} \left(\int_S \frac{d\sigma(\xi)}{|1-\langle r\zeta, \rho\xi\rangle|^{n+s+\frac{1}{2}}}\right)^{1/p'} \\ &\leq \frac{C}{(1-r\rho)^{(s+\frac{1}{2})/p'}} \left(\int_S \frac{|Rf(\rho\xi)|^p d\sigma(\xi)}{|1-\langle r\zeta, \rho\xi\rangle|^{n+s+\frac{1}{2}}}\right)^{1/p},\end{aligned} \tag{2.3}$$

by Lemma 2.3. Now we substitute (2.3) into (2.2) and use Fubini's theorem and Lemma 2.3 to get

$$\begin{aligned}I &\leq C\int_0^1 (1-r)^{\frac{p}{2}-n-1}\left(\int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{(s+\frac{1}{2})/p'}}\right. \\ &\times \left.\left(\int_S \left(\int_S \frac{|Rf(\rho\xi)|^p d\sigma(\xi)}{|1-\langle r\zeta, \rho\xi\rangle|^{n+s+\frac{1}{2}}}\right) d\sigma(\zeta)\right)^{1/p} d\rho\right)^p dr\end{aligned} \tag{2.4}$$

$$= C\int_0^1 (1-r)^{\frac{p}{2}-n-1}\left(\int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{(s+\frac{1}{2})/p'}}\right.$$

$$\left.\times\left(\int_S |Rf(\rho\xi)|^p d\sigma(\xi)\int_S \frac{d\sigma(\zeta)}{|1-\langle r\zeta,\rho\xi\rangle|^{n+s+\frac{1}{2}}}\right)^{1/p} d\rho\right)^p dr$$

$$\leq C\int_0^1 (1-r)^{\frac{p}{2}-n-1}\left(\int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{s+\frac{1}{2}}}\right.$$

$$\left.\times\left(\int_S |Rf(\rho\xi)|^p d\sigma(\xi)\right)^{1/p} d\rho\right)^p dr.$$

A simple observation shows that it is possible to select positive parameters $s,\ t_1,\ t_2,\ t_3,\ t_4$ such that

(i) $s = t_1 + t_2 = t_3 + t_4$,

(ii) $\frac{1}{p'} < t_3 - t_1 < \frac{3}{2} - \frac{n+1}{p}$,

(iii) $1 - \frac{n+1}{p} < t_2$.

Applying Hölder's inequality on (2.4) and Lemma 2.4 we obtain

$$I \leq C\int_0^1 (1-r)^{\frac{p}{2}-n-1}\left[\left(\int_0^1 \frac{(1-\rho)^{t_1 p'} d\rho}{(1-r\rho)^{t_3 p'}}\right)^{p/p'}\right.$$

$$\left.\times\left(\int_0^1 \frac{(1-\rho)^{t_2 p}}{(1-r\rho)^{(t_4+\frac{1}{2})p}}\left(\int_S |Rf(\rho\xi)|^p d\sigma(\xi)\right) d\rho\right)\right] dr$$

$$\leq C\int_0^1 (1-r)^{\frac{3p}{2}-n+(t_1-t_3)p-2}\left(\int_0^1 \frac{(1-\rho)^{t_2 p}}{(1-r\rho)^{(t_4+\frac{1}{2})p}}\right.$$

$$\left.\times\left(\int_S |Rf(\rho\xi)|^p d\sigma(\xi)\right) d\rho\right) dr$$

$$= C\int_0^1 \left[(1-\rho)^{t_2 p}\left(\int_S |Rf(\rho\xi)|^p d\sigma(\xi)\right)\right.$$

$$\left.\times\left(\int_0^1 \frac{(1-r)^{\frac{3p}{2}-n+(t_1-t_3)p-2} dr}{(1-r\rho)^{(t_4+\frac{1}{2})p}}\right)\right] d\rho$$

$$= C\int_B (1-|w|^2)^{p-n-1} |Rf(w)|^p d\nu(w) < \infty.$$

This finishes the proof of Theorem 1.

3. Class MB^p

For fixed $r,\ 0<r<1,\ 0<p\le\infty$ and $f\in C(B)$, we define

$$\widehat{f}(z,r)=\frac{1}{|E_r(z)|}\int_{E_r(z)} f(w)d\nu(w),$$

$$MO_pf(z,r)=\left(\frac{1}{|E_r(z)|}\int_{E_r(z)}|f(w)-\widehat{f}(z,r)|^p d\nu(w)\right)^{1/p},$$
$$0<p<\infty,$$

$$MO_\infty f(z,r)=\sup\left\{|f(w)-\widehat{f}(z,r)|:w\in E_r(z)\right\},$$

$$MO_p^*f(z,r)=\left(\frac{1}{|E_r(z)|}\int_{E_r(z)}|f(w)-f(z)|^p d\nu(w)\right)^{1/p},$$
$$0<p<\infty,$$

and

$$MO_\infty^*f(z,r)=\sup\big\{|f(w)-f(z)|:w\in E_r(z)\big\}.$$

A function $f\in C^2(B)$ is said to be of class $M_L,\ 0<L<\infty$, if $|\widetilde{\Delta}f(z)|\le Lr^{-2}MO_\infty^*f(z,r)$, for all $z\in B$, $0<r<1$, and of class M if $f\in M_L$, for some $L>0$.

The following characterizations of the M-Besov p-classes $MB^p = M\cap B^p$ are extensions of the well known results on $\mathcal{M}$-harmonic and holomorphic Besov p-spaces.

Theorem 2. *Let $f\in M$, $1<q<\infty$, $0<p\le q$. Then the following statements are equivalent:*

(i) $\|Q_pf\|_{L^q(\tau)}<\infty$, *where* $Q_pf(z)=\|f\circ\varphi_z-f(z)\|_{L^p(\nu)}$, $z\in B$,

(ii) $f\in MB^q$,

(iii) $\|Qf\|_{L^q(\tau)} < \infty$, *where*

$$Qf(z) = \sup_{|w|=1} \left\{ \frac{\left(|\langle Df(z), \bar{w}\rangle|^2 + |\langle D\bar{f}(z), \bar{w}\rangle|^2\right)^{1/2}}{\sqrt{\langle G_z w, w\rangle}} \right\}$$

and

$$G_z = \frac{1}{2}\left(\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log(1-|z|^2)^{-n-1}\right).$$

Theorem 3. *Let $f \in M$, $\alpha > -1$, $1 < q < \infty$ and $0 < p \le \infty$. Then the following statements are equivalent:*

(i) $f \in MB^q$,

(ii) $MO^*_p f(\,\cdot\,, r) \in L^q(\tau)$, *for all* r, $0 < r < 1$,

(iii) $MO^*_p f(\,\cdot\,, r) \in L^q(\tau)$, *for some* r, $0 < r < 1$,

(iv) $MO_p f(\,\cdot\,, r) \in L^q(\tau)$, *for all* r, $0 < r < 1$,

(v) $MO_p f(\,\cdot\,, r) \in L^q(\tau)$, *for some* r, $0 < r < 1$,

(vi) $\displaystyle\int_B\int_B \frac{|f(z)-f(w)|^q(1-|z|^2)^\alpha(1-|w|^2)^\alpha}{|1-\langle z, w\rangle|^{2n+2+2\alpha}} d\nu(z)d\nu(w) < \infty.$

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