On residuals in partially ordered semigroups*

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Dedicated to Professor L. Kalmár on his 60[th] birthday

§ 1. Introduction

Let L be a residuated lattice-ordered semigroup and S a closed interval of L which has a maximum element s. In this paper we prove a lemma and a theorem which give some sufficient conditions in order that the mapping

$$x \to x:s \qquad (x \in S)$$

be an isomorphism with respect to the order relation and the operations of L.

The results of §§ 2 and 3 are needed in the proof of the lemma and the theorem. First we consider a naturally ordered semigroup in which unique prime factorization holds. In § 2, some consequences of these hypotheses are enumerated. In § 3, we prove two lemmas concerning the residuals $p:s$ $(q:s)$, where $p(q)$ is a prime (primary) element in S. These lemmas generalize some previous results of the author.

In § 5 we apply the results of § 4 to the residuated lattice-ordered semigroup of all ideals of an associative ring.

§ 2. Some consequences of unique prime factorization

Let S be a *partially ordered semigroup*, i. e. a set with an associative multiplication and with a partial order $\leqq$ such that, for all $a, b, c\ (\in S)$ $a \leqq b$ implies $ac \leqq bc$ and $ca \leqq cb$. S will be called *negatively ordered* if, for all $a, b\ (\in S)$, $ab \leqq a$ and $ab \leqq b$ hold.

We shall say that S is *naturally ordered*[1]) if it is negatively ordered and

(2. 1) $\qquad a < b$ implies $a = bx = yb$ for some $x, y \in S$.

Assume that S is a partially ordered semigroup (with at least three elements) satisfying:

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*) On the 6[th] Austrian Mathematical Congress the author lectured on some results of this paper.

[1]) In Fuchs [2] a positively ordered semigroup satisfying (2.1) is called naturally ordered.

(i) S is naturally ordered and has a minimum element[2]) 0 and a maximum element [2]) s;

(ii) every element $a \in S$ $(0 < a < s)$ may be represented as the product of a finite number of prime elements[3])

$$a = p_1 p_2 \dots p_k \tag{2. 2}$$

and two representations of an element a can differ only in the order of prime factors, that is, in S the unique prime factorization holds;

(iii) in S there exists an element $z (0 < z < s)$ which is not a zero-divisor.

From hypotheses (i), (ii), (iii) we conclude in turn:

(A) *The element s is the identity of S.*

The element s is idempotent because $s^2 < s$ would imply the existence of a prime element $p\,(\in S)$ with $s^2 \leqq p < s$, which is impossible.

It suffices to prove that $sp = ps = p$ for all primes $p \in S$. By (2. 1) and $p < s$ there exists $q\,(\in S)$ with $p = sq$. This implies, because of $s^2 = s$

$$sp = s^2 q = sq = p.$$

Naturally $ps = p$ holds too.

(B) *Every prime p of S is a maximal element*[4]) *in S.*[5])

Let m be an element of S with $p < m < s$. The property (2. 1) guarantees the existence of an element h such that $p = mh$. Because of the primeness of p this implies

$$h \leqq p$$

and because of the negative order in S

$$p \leqq h.$$

Thus $p = mp$ which contradicts hypothesis (ii).

(C) *S is commutative.*[6])

In view of (A) and (ii) if suffices to prove that the prime elements of S commute. Let p, q be different primes of S. By (2. 1) there exists an element $x\,(\in S)$ such that $qx = pq\,(< p)$. The prime property of p and (B) show that $x \leqq p$. Therefore $pq \leqq qp$. Changing the roles of p and q we get the converse inequality, whence $pq = qp$, in fact.

Because of (C) every element $a \in S\,(0 < a < s)$ can be written up to the order of factors *uniquely* in the form

$$a = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r} \tag{2. 3}$$

with prime powers $p_i^{k_i}$ belonging to different (pairwise commuting) bases p_i.

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[2]) We use the term „minimum element" to mean an element which is smaller than any other element. Obviously the minimum element 0 is unique and satisfies $0 \cdot a = a \cdot 0 = 0$ for all $a \in S$. The meaning of „maximum element" is clear. The maximum element s is in general not an identity in S.

[3]) An element $p \in S$ $(0 \leqq p < s)$ is said to be *prime* if $ab \leqq p$ $(a, b \in S)$ implies either $a \leqq p$ or $b \leqq p$.

[4]) An element m is called *maximal* in S if $m \leqq m' < s$ $(m' \in S)$ implies $m' = m$.

[5]) Here we follow the argument of ŠULGEĬFER [5].

[6]) Here we follow the argument of FUCHS—STEINFELD [3].

(D) *S is free of divisors of zero.*

The assumption $xy=0$ $(0<x, y<s)$ implies that 0 has at least one prime factorization. Among the prime factorizations of 0 let us consider a shortest one:

$$0=p_1 \dots p_r,$$

that is $p_1 \dots p_{i-1} p_{i+1} \dots p_r \neq 0$ holds for $1 \leqq i \leqq r$. Let p denote a prime element of S satisfying $z \leqq p$ where z is no divisor of zero. As $p_1 \dots p_r = 0 \leqq z \leqq p$ holds, by (B) the prime element p must occur among $p_1, \dots, p_r$. E. g. let $p=p_1$. Thus $zp_2 \dots \dots p_r \leqq p_1 p_2 \dots p_r = 0$, which contradicts hypothesis (iii).

(E) *Every prime power p^k $(k=1, 2, \dots)$ is a primary element*[7] *in S.*

Let a, b be two elements of S such that $ab \leqq p^k$. We can assume that $0<a, b<s$ and $ab \neq 0$. Because of (i) and (A) there exists an element d with $ab = p^k d$. If $a \not\leqq p^k$ then by (ii) the prime element p must occur among the prime factors of b. Thus $b \leqq p$ and so $b^k \leqq p^k$.

§ 3. Some results on residuals

Let L be a *residuated* partially ordered semigroup, i. e. a partially ordered semigroup having the following property: for every pair of elements $a, b \in L$ there exists a *right-residual* $a:b \in L$ and a *left-residual* $a::b \in L$ defined by

$$xb \leqq a \ (bx \leqq a) \text{ if and only if } x \leqq a:b \quad (x \leqq a::b). \tag{3. 1}$$

We need to know:

(a) If L is a residuated partially ordered semigroup and $\inf(a, b) = a \cap b$ $(a, b \in L)$ exists in L, then so do $(a:x) \cap (b:x)$, $(a::x) \cap (b::x)$ for all $x \in L$ and

$$(a:x) \cap (b:x) = (a \cap b):x, \quad (a::x) \cap (b::x) = (a \cap b)::x \tag{3. 2}$$

(See FUCHS [2].)

(b) A residuated partially ordered semigroup L which is at the same time a lattice is a lattice-ordered semigroup, that is,

$$(a \cup b)c = ac \cup bc \quad \text{and} \quad c(a \cup b) = ca \cup cb \tag{3. 3}$$

hold for all $a, b, c \in L$ (See FUCHS [2].)

(c) If the maximum element e of a lattice-ordered semigroup L is an identity in L, then

$$a \cup b = e \quad \text{implies} \quad a \cap b = ab \cup ba \quad (a, b \in L), \tag{3. 4}$$

$$a \cup b = e, a \cup c = e \quad \text{imply} \quad a \cup bc = e \quad (a, b, c \in L) \tag{3. 5}$$

and

$$a_1 a_2 \dots a_r \leqq a_1 \cap a_2 \cap \dots \cap a_r \qquad (a_i \in L). \tag{3. 6}$$

(See BIRKHOFF [1] or FUCHS [2].)

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[7]) An element $q \in S$ $(0 \leqq q < s)$ is called *primary* if $ab \leqq q$ and $a \not\leqq q$ $(a, b \in S)$ imply $b^r \leqq q$ and if $ab \leqq q$ and $b \not\leqq q$ imply $a^t \leqq q$ for suitable positive integers r, t.

We remark that (3. 6) implies that L is negatively ordered.

Let L be a lattice-ordered semigroup with the minimum element 0 and maximum element e which is an identity in L. For every element $s\,(\in L)$ the *interval* $[0, s] = S$ (consisting of all elements $x(\in L)$ with $0 \leqq x \leqq s$) is a lattice-ordered subsemigroup of L.

Lemma 3.1. (cf. STEINFELD [4]). *Let L be a residuated lattice-ordered semigroup with the minimum element* 0 *and maximum element e which is an identity in L. Let $[0, s] = S$ denote an interval of L. If $p\,(\in S, p \neq s)$ is a prime element in S then $p{:}s$ is a prime element in L and*

$$(3.7) \qquad (p{:}s) \cap s = p,$$

$$(3.8) \qquad p{:}s = p{::}s$$

hold.

Furthermore $p{:}s$ is the only prime element in L whose intersection with s is equal to p. For every $r \in L$, $r \cap s = p$ implies $r \leqq p{:}s$. The prime p of S is a prime element in L too if and only if $p = p{:}s$ holds.

PROOF. First we show that $p{:}s$ is a prime element in L. Let m, n be elements of L satisfying $mn \leqq p{:}s$. By the definition of $p{:}s$ we have

$$ms \cdot ns \leqq mns \leqq p.$$

As $ms \leqq s$, $ns \leqq s$, the prime property of p in S implies $ms \leqq p$ or $ns \leqq p$. Therefore $m \leqq p{:}s$ or $n \leqq p{:}s$.

$p \leqq (p{:}s) \cap s$ holds trivially, so instead of (3. 7) it suffices to prove $(p{:}s) \cap s \leqq p$. The relations

$$((p{:}s) \cap s)s \leqq (p{:}s)s \leqq p \text{ and } s \not\leqq p$$

imply $(p{:}s) \cap s \leqq p$, in fact.

Let r be an element of L such that $r \cap s = p$. This implies $rs \leqq r \cap s \leqq p$, thus, $r \leqq p{:}s$.

If r is a prime element in L with $r \cap s = p$, then because of

$$(p{:}s)s \leqq p \leqq r \text{ and } s \not\leqq r$$

the relation $p{:}s \leqq r$ must hold. Thus $r = p{:}s$.

For the left residual $p{::}s$ the same statements hold, thus $p{:}s = p{::}s$, that is (3. 8) is true.

If $p = p{:}s$, then p is a prime element in L. If however $p \neq p{:}s$, then because of $(p{:}s)s \leqq p$, $p{:}s \not\leqq p$ and $s \not\leqq p$ the element p is not prime in L.

For the primary elements an analogous result is true.

Lemma 3. 2. *Let L be a residuated lattice-ordered semigroup with the minimum element* 0 *and maximum element e which is an identity in L. Let s denote an element in the center of L. If q is a primary element of the interval $[0, s] = S$ such that $s^i \not\leqq q$*

for $i=1, 2, \ldots$, *then* $q:s$ *is a primary element in* L *and*

$$(q:s)\cap s = q, \tag{3. 9}$$

$$q:s=q::s \tag{3. 10}$$

hold. $q:s$ *is the only primary element in* L *whose intersection with* s *is* q.

From $r\cap s = q$ $(r\in L)$ *it follows* $r\leqq q:s$. *A primary element* q *of* S *is primary in* L *too if and only if* $q=q:s$.

PROOF. First we show that $q:s$ is primary in L. Let m, n be elements of L such that $mn\leqq q:s$. By the definition of $q:s$ we obtain

$$ms\cdot ns\leqq mns\leqq q.$$

If $n\not\leqq q:s$, that is $ns\not\leqq q$, the primary property of q implies $(ms)^k\leqq q$ for a suitable integer $k\geqq 1$. As s commutes with the elements of L we can write $(ms)^k=m^k s^k\leqq q$. In the case $k=1$ we get $m\leqq q:s$. If $k>1$, then $m^k ss^{k-1}\leqq q$ holds. Hence $s^{k-1}\not\leqq q$ implies $m^k s\leqq q$, that is $m^k\leqq q:s$.

Similarly $mn\leqq q:s$ and $m\not\leqq q:s$ imply $n^l\leqq q:s$ for a suitable integer $l\geqq 1$. Thus $q:s$ is primary indeed.

Because s belongs to the center, statement (3. 10) is trivial.

As $q\leqq(q:s)\cap s$ is always true, we have to prove only $(q:s)\cap s\leqq q$. The relations

$$((q:s)\cap s)s\leqq(q:s)s\leqq q \text{ and } s^i\not\leqq q \quad (i=1, 2, \ldots)$$

imply $(q:s)\cap s\leqq q$.

Let r be an element of L such that $r\cap s=q$. This implies $rs\leqq r\cap s\leqq q$ and so $r\leqq q:s$.

Furthermore if r is primary in L, then from $r\cap s = q$ and $s^i\not\leqq q$ it follows $s^i\not\leqq r$ $(i=1, 2, \ldots)$. Hence $(q:s)s\leqq q\leqq r$ implies $q:s\leqq r$, and so $r=q:s$.

If $q=q:s$ holds, then q is a primary element in L. If $q\neq q:s$, then because of $(q:s)s\leqq q$, $s^i\not\leqq q$ $(i=1, 2, \ldots)$ and $q:s\not\leqq q$, the element q is not primary in L.

§ 4. The mapping $x\to x:s$

First we shall prove

Lemma 4. 1. *Let* L *be a residuated lattice-ordered semigroup with a minimum element* 0 *and a maximum element* e *which is the identity of* L. *Let* s *be an element of* L *satisfying the following conditions*

(α) $s^2=s$ *and for every element* $x\in S=[0, s]$ *the relation* $(x:s)s=s(x:s)=$ $=(x:s)\cap s$ *holds,*

(β) $(0:s)\cup s=e$.

Then

$$x\to x:s \quad (\text{for } x\in S) \tag{4. 1}$$

is a one-to-one mapping from the lattice-ordered subsemigroup $S=[0, s]$ *onto the set* Q *of all right-residuals* $x:s$ $(x\in S)$ *which preserves intersections and products.* [8])

[8]) We remark that condition (β) implies $x\neq x:s$ for all $x\in[0, s]$.

Proof. As L is negatively ordered, $x \leqq (x:s) \cap s$ holds for every $x \in S$. On the other hand, condition (α) implies $(x:s) \cap s = (x:s)s \leqq x$. Therefore

(4. 2) $$(x:s) \cap s = x \qquad (\text{for every } x \in S).$$

The implication

(4. 3) $$a \leqq b \Rightarrow (a:s) \leqq (b:s) \qquad (0 \leqq a, b \leqq s)$$

follows immediately from the definition of right-residuals. Conversely, because of (4. 2), the implication

(4. 4) $$a:s \leqq b:s \Rightarrow a = (a:s) \cap s \leqq (b:s) \cap s = b \qquad (0 \leqq a, b \leqq s)$$

holds too. (4. 3) and (4. 4) imply that (4. 1) is a one-to-one mapping from S onto Q.

In view of (a) we get

$$a \cap b \to (a \cap b):s = (a:s) \cap (b:s) \qquad (0 \leqq a, b \leqq s)$$

and so (4. 1) is a homomorphic mapping with respect to the operation $\cap$.

Now we can verify

(4. 5) $$(ab):s = (a:s)(b:s) \qquad (0 \leqq a, b \leqq s).$$

(4. 2) and (α) imply

(4. 6) $$ab = \big((a:s) \cap s\big)\big((b:s) \cap s\big) = (a:s)s(b:s)s = (a:s)(b:s)s^2 = (a:s)(b:s)s.$$

Hence in view of (3. 1)

(4. 7) $$(a:s)(b:s) \leqq (ab):s.$$

On the other hand, $\big((ab):s\big)(b:s) \leqq (a:s)(b:s)$, therefore

(4. 8) $$\big((a:s)(b:s)\big)::\big((ab):s\big) \geqq b:s.$$

Furthermore $\big((ab):s\big)s \leqq ab \leqq (a:s)(b:s)$ implies

(4. 9) $$\big((a:s)(b:s)\big)::\big(ab):s\big) \geqq s.$$

From (4. 8), (4. 9) and (β) we obtain

(4. 10) $$\big((a:s)(b:s)\big)::\big((ab):s\big) \geqq (b:s) \cup s = e$$

and hence

(4. 11) $$ab:s \leqq (a:s)(b:s).$$

(4. 7) and (4. 11) complete the proof of (4. 5) and Lemma 4. 1.

We say that in a partially ordered semigroup S *the strict unique prime factorization* holds, if in S conditions (i), (ii), (iii) are fulfilled.

Theorem 4. 2. *Let L be a residuated lattice-ordered semigroup with a minimum element* 0 *and a maximum element e which is the identity of L. Let s be an element of L satisfying the following conditions:*

(I) *the strict unique prime factorization holds in the interval* $[0, s]=S$,

(II) $(0:s)\cup s=e$,

(III) *s is in the center of L.*

Then

$$x \to x:s \qquad (\text{for } x \in S) \tag{4. 12}$$

is an isomorphic mapping from the lattice-ordered semigroup S onto the set Q of all right-residuals $x:s$ $(x \in S)$; *thus the strict unique prime factorization holds in Q too.*

Proof. As in S the assumptions (i), (ii), (iii) are fulfilled, we can use the results (A)–(E). From the conditions and (A), (C), it follows that S is a commutative lattice-ordered subsemigroup of L with the identity s. Thus (B), (3. 4), (3. 5) and (I) imply that every element x $(0<x<s)$ of S can be written uniquely in the form

$$x = p_1^{k_1}\dots p_r^{k_r} = p_1^{k_1}\cap\dots\cap p_r^{k_r} \qquad (k_i \geqq 1) \tag{4. 13}$$

with different prime elements $p_1, \dots, p_r$ of S. In view of (E) the prime powers $p_i^{k_i}$ $(i=1, 2, \dots, r)$ are primary elements in S.

By making use of Lemma 3. 2, (E), (a) and (4. 13) we obtain $x = p_1^{k_1}\cap\dots$ $\dots\cap p_r^{k_r} = ((p_1^{k_1}:s)\cap s)\cap\dots\cap((p_r^{k_r}:s)\cap s) = (p_1^{k_1}:s)\cap\dots\cap(p_r^{k_r}:s)\cap s =$ $= ((p_1^{k_1}\cap\dots\cap p_r^{k_r}):s)\cap s = (x:s)\cap s$, that is,

$$(x:s)\cap s = x \qquad (\text{for all } 0<x<s). \tag{4. 14}$$

Because of (D), 0 is a prime element of S, therefore from Lemma 3. 1 it follows $0:s=0::s$. Hence $s(0:s)=s(0::s)=0=(0:s)s$. In view of (a), this and (II) imply

$$(0:s)\cap s = (0:s)s = 0. \tag{4. 15}$$

Because of $(s:s)\cap s = s$ one can write instead of (4. 14) more generally

$$(a:s)\cap s = a \qquad (\text{for all } 0\leqq a\leqq s). \tag{4. 14'}$$

Now we can prove that (I), (II) and (III) imply (α) and (β).

Let y denote an element $y=a:s\in Q$ $(a\in S)$. (4. 14'), (III) and (3. 6) imply $a=(a:s)\cap s=y\cap s\geqq ys=sy=b$. Hence $y\leqq b:s$. By making use of (4. 14') again we obtain $b=(b:s)\cap s\geqq y\cap s=(a:s)\cap s=a$. Thus

$$sy=ys=b=a=y\cap s \qquad (y\in Q)$$

holds. Because (A) implies $s^2=s\cap s=s$, the implication

$$\text{(I), (II) and (III)} \Rightarrow (\alpha) \text{ and } (\beta)$$

is proved.

In view of Lemma 4. 1 we have to show only

$$(a \cup b):s = (a:s) \cup (b:s) \qquad (a, b \in S). \tag{4. 16}$$

The cases $a=0$ or $b=0$ and $a=s$ or $b=s$, respectively, are trivial. Let us consider the elements a, b $(0<a, b<s)$ satisfying

$$\begin{aligned} a &= p_1^{m_1} \dots p_r^{m_r} = p_1^{m_1} \cap \dots \cap p_r^{m_r}, \\ b &= p_1^{n_1} \dots p_r^{n_r} = p_1^{n_1} \cap \dots \cap p_r^{n_r} \end{aligned} \qquad (m_i + n_i \geqq 1;\ p_i^0 = s). \tag{4. 17}$$

First we shall prove

$$a \cup b = \prod_{i=1}^{r} p_i^{\min(m_i, n_i)}. \tag{4. 18}$$

(4. 17) implies

$$a \cup b = \prod_{i=1}^{r} p_i^{m_i} \cup \prod_{i=1}^{r} p_i^{n_i} = \prod_{i=1}^{r} p_i^{\min(m_i, n_i)} (a' \cup b'),$$

where

$$a' = \prod_{i=1}^{r} p_i^{m_i - \min(m_i, n_i)} \quad \text{and} \quad b' = \prod_{i=1}^{r} p_i^{n_i - \min(m_i, n_i)}$$

are elements of S. As a' and b' cannot have common prime factors, because of (B) and (3. 5) $a' \cup b' = s$ holds. Thus (4. 18) is proved.

By using (4. 18) and (4. 5) we get

$$(a \cup b):s = \left(\prod_{i=1}^{r} p_i^{\min(m_i, n_i)} \right):s = \prod_{i=1}^{r} (p_i:s)^{\min(m_i, n_i)}. \tag{4. 19}$$

On the other hand, there results

$$(a:s) \cup (b:s) = \prod_{i=1}^{r} (p_i:s)^{m_i} \cup \prod_{i=1}^{r} (p_i:s)^{n_i} = \prod_{i=1}^{r} (p_i:s)^{\min(m_i, n_i)} (A' \cup B') \tag{4. 20}$$

where $A' = \prod_{i=1}^{r} (p_i:s)^{m_i - \min(m_i, n_i)}$ and $B' = \prod_{i=1}^{r} (p_i:s)^{n_i - \min(m_i, n_i)}$. By virtue of (II) this implies as above $A' \cup B' = e$. So statement (4. 16) has been verified.

Let us consider an element $x \in S$ $(0<x<s)$ written in the form (4. 13). We can write

$$x:s = (p_1^{k_1} \dots p_r^{k_r}):s = (p_1^{k_1}:s) \dots (p_r^{k_r}:s) = (p_1:s)^{k_1} \dots (p_r:s)^{k_r},$$

where because of Lemma 3. 1 the elements $p_i:s$ $(i=1, 2, \dots, r)$ are different prime elements in L.

Because (4. 12) is an isomorphic mapping of S onto Q the properties (i), (ii) and (iii) are fulfilled in Q too.

The proof of Theorem 4. 2 is thereby completed.

§ 5. Application to ideal theory

It is known that the set of all ideals of an associative ring is a residuated lattice-ordered semigroup. (See FUCHS [2], Third Part.) Thus we can apply the results of § 4 to the ideals of an associative ring.[9])

Lemma 5. 1. *Let* $\mathfrak{a}$ *be an ideal of an associative ring* R *satisfying the following conditions*

(α') $\mathfrak{a}^2=\mathfrak{a}$ *and for every ideal* $\mathfrak{x}$ *of* $\mathfrak{a}$ *the relation* $(\mathfrak{x}:\mathfrak{a})\mathfrak{a}=\mathfrak{a}(\mathfrak{x}:\mathfrak{a})=\mathfrak{a}\cap(\mathfrak{x}:\mathfrak{a})$ *holds,*

(β') $(0:\mathfrak{a})+\mathfrak{a} = R$.

Then

$$\mathfrak{x} \to \mathfrak{x}:\mathfrak{a} \qquad (\text{for ideals } \mathfrak{x} \text{ of } \mathfrak{a}) \tag{5. 1}$$

is a one-to-one mapping from the lattice-ordered semigroup S *of all ideals of* $\mathfrak{a}$ *onto the set* Q *of all right ideal quotients* $\mathfrak{x}:\mathfrak{a}$ $(\mathfrak{x}\in S)$ *which preserves intersections and products.*

For the proof it is enough to remark that because of (β') the ideals of $\mathfrak{a}$ are ideals of R too.

We shall say that *strict unique prime factorization holds* for the ideals of an associative ring A if the following conditions are fulfilled:

(i′) for the ideals $\mathfrak{a}$, $\mathfrak{b}$ of A

$\mathfrak{a}\subset\mathfrak{b}$ implies $\mathfrak{a}=\mathfrak{b}\mathfrak{c}=\mathfrak{d}\mathfrak{b}$ for some ideals $\mathfrak{c}, \mathfrak{d}$ of A;

(ii′) every ideal $\mathfrak{x}$ of A $(0\subset\mathfrak{x}\subset A)$ may be represented as the product of a finite number of prime ideals of A and two representations of $\mathfrak{x}$ can differ only in the order of the factors,

(iii′) 0 is a prime ideal of A.

Now we can apply Theorem 4. 2 to the ideals.

Theorem 5. 2. *Let* $\mathfrak{a}$ *be an ideal of an associative ring* R *satisfying the following conditions:*

(I′) *the strict unique prime factorization holds for the ideals* $\mathfrak{x}$ *of* $\mathfrak{a}$.

(II′) $(0:\mathfrak{a})+\mathfrak{a} = R$,

(III′) $\mathfrak{a}$ commutes with every ideal $\mathfrak{y}$ of R and $R\mathfrak{y}=\mathfrak{y}R=\mathfrak{y}$ holds.

Then

$$\mathfrak{x}\to\mathfrak{x}:\mathfrak{a} \qquad (\text{for ideals } \mathfrak{x} \text{ of } \mathfrak{a})$$

is an isomorphic mapping from the lattice-ordered semigroup S *of all ideals* $\mathfrak{x}$ *of* $\mathfrak{a}$ *onto the set* Q *of all (right) ideal quotients* $\mathfrak{x}:\mathfrak{a}$ $(\mathfrak{x}\in S)$.

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[9]) Naturally it would be possible to extend these results to the ideals of a semigroup or a semiring.

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