Some reduction theorems on the functional equation of generalized distributivity

By V. D. BELOUSOV (Kišinev) and M. HOSSZÚ (Miskolc)

1. We consider the functional equation

$$(1) \qquad F[G(x, y), u] = H[K(x, u), L(y, u)]$$

of generalized distributivity, where the independent variables x, y as well as the values of the unknown function G are elements of a set Q and the variable u is chosen from another set P while the values of the other unknown functions F, H, K, L belong to Q'. It is convenient to use the notations

$$xy = G(x, y), \quad x \circ y = H(x, y),$$

$$F_u x = F(x, u), \quad K_u x = K(x, u), \quad L_u x = L(x, u)$$

by which (1) can be written as

$$(1') \qquad F_u(xy) = K_u x \circ L_u y; \qquad x, y \in Q; \; u \in P.$$

Using the terminology of groupoids ([2]), (1′) means that the triples of the mappings (F_u, K_u, L_u) are forming a system of *homotopies* of the groupoid $(Q, \cdot)$ into another groupoid $(Q', \circ)$.

If the mappings

$$x \to F_u x, \quad K_u x, \quad L_u x$$

are $1-1$ and onto, then we speak about *isotopy*. The special case $F_u = K_u = L_u$ leads to homomorphism and isomorphism, respectively. If $(Q', \circ)$ is the same groupoid as $(Q, \cdot)$ then we get the notion of *endotopy* and *autotopy*, respectively.

In the present paper we reduce (1) to simpler equations. We use only certain weakened solvability conditions for equations of the type $xy = z$, $y = F_u x$, etc.

2. First we reduce (1) to the special case $(Q,.) = (Q', \circ)$.

Theorem 1. *Suppose that there exists at least one* $u_0 \in P$ *such that* $x \to F_u x$, $K_u x$, $L_u x$ *are* $1-1$ *mappings of* Q *onto* Q', *then* (1) *is equivalent to the functional equation*

$$(2) \qquad \Phi_u(xy) = \Psi_u x \Lambda_u y; \qquad x, y \in Q; \; u \in P,$$

where

$$(3) \qquad F_u = F_0 \Phi_u, \quad K_u = K_0 \Psi_u, \quad L_u = L_0 \Lambda_u$$

F_0, K_0, L_0 *being particular solutions of* (1).

PROOF. The equivalence is clear since (2) is just (1) if we substitute both sides of (2) into F_0 and take into consideration that F_0, K_0, L_0 are particular solutions of (1).

Corollary. *The most general solutions of* (1′) *are given by* (3) *and*

$$x \circ y = F_0(K_0^{-1} x L_0^{-1} y), \tag{4}$$

where $xy, \Phi_u, \Psi_u, \Lambda_u$ *are arbitrary solutions of* (2) *and* F_0, K_0, L_0 *are arbitrary* $1-1$ *mappings of* Q *onto* Q' *(while the inverse mappings are denoted by the exponent* -1*).*

Thus it is enough to consider (2). Cf. [1,5].

3. Supposing that $(Q, \cdot)$ is a quasigroup, i. e. the mappings

$$x \to xy_0, \quad y \to x_0 y$$

are $1-1$ and onto for every fixed x_0, y_0 in Q, we give a further reduction. In this reduction the principal loop isotopes of $(Q, \cdot)$ play an important role. A quasigroup having a two sided universal identity element is called a *loop*. The notion of *principal* isotopy can be defined by (1′) as a special isotopy, where F is the identical mapping. Clearly, principal isotopy is an equivalence relation as well as isotopy. E. g. the quasigroup $(Q, +)$ defined on Q by

$$xb + ay = xy, \qquad x, y \in Q \tag{5}$$

for fixed $a, b \in Q$ is a principal isotope of $(Q, \cdot)$. This is a loop with identity $0 = ab$. This can be seen by (5) if we put $x = a$ resp. $y = b$. On the other hand, it is clear that every principal loop isotope of $(Q, \cdot)$ can be defined by (5) by means of certain fixed elements a, b. Indeed, if we have

$$xy = \psi x + \lambda y, \tag{6}$$

then we can choose elements a, b such that $\psi a, \lambda b$ are just the unit 0 of $(Q, +)$ and so by keeping $y = b$ resp. $x = a$ constant, we necessarily have

$$\psi x = xb, \quad \lambda y = ay.$$

If two quasigroups are related by (5), then we shall use the notation

$$(Q, +) = (Q, \cdot)^{[a, b]}. \tag{5′}$$

The following lemma might be interesting in itself:

Lemma. *Let* $(Q, +)$ *be a loop with identity* 0 *and* $(Q, \square) = (Q, +)^{[c, d]}$. *Then we have* $(Q, +) = (Q, \square)^{[d, c]}$. *The identity of* $(Q, \square)$ *is* $e = c + d$ *and, conversely,* $0 = d \square c$.

PROOF. The first part of the lemma states that

$$(x + d) \square (c + y) = x + y, \qquad x, y \in Q, \tag{7}$$

$$(x \square c) + (d \square y) = x \square y, \qquad x, y \in Q \tag{7′}$$

are equivalent relations.

For example let us suppose (7). By putting $x=0$ resp. $y=0$ we obtain

$$d\Box(c+y) = y, \quad (x+d)\Box c = x, \qquad x, y \in Q.$$

On the other hand, if we put here $d\Box y$ resp. $x\Box c$ instead of y resp. x, then, by cancellation, we get

$$c+(d\Box y) = y, \quad (x\Box c)+d = x, \qquad x, y \in Q. \tag{8}$$

Now, let us write $x\Box c$ and $d\Box y$ instead of x and y in (7). Then, taking also (8) into account, we have (7′).

The remaining statements of the Lemma follow from (7) and (7′) by substituting $x=y=0$ resp. $x=y=e$.

4. By means of a loop isotope $(Q, +)$ of $(Q, \cdot)$ we can reduce (2) to a simpler equation. Namely, let xy be of the form (6), where $+$ is a loop operation. Putting this into (2) we get

$$\Phi_u(\psi x+\lambda y) = \psi \Psi_u x + \lambda \Lambda_u y. \tag{9}$$

By choosing $\lambda \Lambda_u y = 0$ i. e. $y = \Lambda_u^{-1}\lambda^{-1}0$, this yields

$$\psi \Psi_u x = \Phi_u(\psi x + b_u),$$

where

$$b_u = \lambda \Lambda_u^{-1} \lambda^{-1} 0$$

s a function of u. In a similar way we obtain also

$$\lambda \Lambda_u y = \Phi_u(a_u + \lambda y).$$

Thus Ψ_u and Λ_u must be of the form

$$\Psi_u x = \psi^{-1}\Phi_u(\psi x + b_u), \quad \Lambda_u x = \lambda^{-1}\Phi_u(a_u + \lambda x). \tag{10}$$

Putting this back in (2) [or equivalently in (9)], we have

$$\Phi_u(s+t) = \Phi_u(s+b_u) + \Phi_u(a_u+t), \qquad s, t \in Q; \ u \in P \tag{11}$$

by the notations $s = \psi x$, $t = \lambda y$.

So we have proved the following

Theorem 2. *Let $(Q, \cdot)$ be a quasigroup with a system of autotopismus. Then these must be of the form* (6), (10), *where ψ, λ are certain $1-1$ mappings of Q onto itself, $(Q, +)$ is a loop, further, Φ_u, a_u, b_u and $(Q, +)$ are related (to each other) by* (11).

In other words, the most general form of the solutions of the functional equation (2), under the suppositions of Theorem 2, is given by (6), (10), where ψ, λ are arbitrary $1-1$ and onto mappings while Φ_u, a_u, b_u are arbitrary solutions of (11) in a loop $(Q, +)$.

Remark. (11) is equivalent to

$$\Phi_u(x\Box y) = \Phi_u x + \Phi_u y, \qquad x, y \in Q; \quad u \in P, \tag{12}$$

where we have $(Q, \square) = (Q, +)^{[a_u, b_u]}$, i. e.

$$s+t = (s+b_u)\square(a_u+t), \qquad s, t\in Q;\ u\in P. \tag{13}$$

This can be seen by introducing the variables $x = s+b_u$, $y = a_u+t$.

Observe that the operation $x\square y$ may depend on u. This is a consequence of its definition (13). Observe further that (12)—(13) imply

$$b_u\square a_u = 0,\quad a_u+b_u = e_u = \Phi_u 0$$

(see also the lemma proved above). Therefore, if a_u is a solution for a given $\square$, Φ_u, then b_u is already determined.

5. Without supposing the invertibility of the mappings we can prove only the following

Theorem 3. *Suppose that $(Q, \cdot)$ is a quasigroup the endotopisms of which are the triples $(\Phi_u, \Psi_u, \Lambda_u)$, i. e. (2) is fulfilled. Then Φ_u is a homotopism of $(Q, +) = (Q, \cdot)^{[a,b]}$ into $(Q, \square) = (Q, \cdot)^{[\Phi_u a,\, \Lambda_u b]}$.*

In other words, then the functional equation

$$\Phi_u(x+y) = \Phi_u x\square\Phi_u y, \qquad x, y\in Q;\ u\in P \tag{12'}$$

must be fulfilled (cf. [6]), where $(Q, +)$ is defined by (5) and $(Q, \square)$ is defined by a similar equation:

$$x\Psi_u b\square(\Lambda_u a)y = xy, \qquad x, y\in Q.$$

PROOF. By putting $x=a$ resp. $y=b$ into (2) we get

$$\Phi_u(ay)=\Psi_u a\Lambda_u,\quad \Phi_u(xb)=\Psi_u x\Lambda_u b$$

from which we can see the connection between Φ an Ψ resp. Φ and Λ. Thereafter we can verify (12′) so that we write xb and ay instead of x and y, respectively. In fact, then

$$\Phi_u(xb+ay) = \Phi_u(xb)\square\Phi_u(ay)$$

is a consequence of

$$\Phi_u(xy)=\Psi_u x\Lambda_u y=\Psi_u x\Lambda_u b\square\Psi_u a\Lambda_u b,$$

if we take the definitions of $(Q, +)$ and $(Q, \square)$ into account.

Observe that there $(Q, +)$ and $(Q, \square)$, being loop isotopes of the same $(Q, \cdot)$, are principal loop isotopes of each other. Thus we can find certain elements $\bar{a}_u$, $\bar{b}_u$ such that $(Q, \square) = (Q, +)^{[\bar{a}_u, \bar{b}_u]}$, i. e., these loops satisfy an equation of the form (13).

Observe further that we can express the operation $\square$ from (12′), supposed that $x\to\Phi_u x$ is invertible. Then (13) implies

$$\Phi_u^{-1}(s+t) = \Phi_u^{-1}(s+\bar{b}_u)+\Phi_u^{-1}(a_u+t), \qquad s, t\in Q;\ u\in P. \tag{11'}$$

6. By the above reduction we have reduced (1) i. e. (2) to (11) resp. to (12)—(13). Now let us express the orioginal functions figuring in (1) by means of the new functions:

$$(14)\qquad \begin{cases} F_u x = F_0 \Phi_u x, \\ xy = F_0(K_0^{-1} x L_0^{-1} y) = F_0(\psi K_0^{-1} x + \lambda L_0^{-1} y), \\ K_u x = K_0 \Psi_u x = K_0 \psi^{-1} \Phi_u(\psi x + b_u), \\ L_u x = L_0 \Lambda_u x = L_0 \lambda^{-1} \Phi_u(a_u + \lambda x). \end{cases}$$

It is easy to verify that (6), (14) really satisfy (1) with arbitrary $F_0, K_0, L_0, \psi, \lambda$ and with an arbitrary loop operation $+$, further, with arbitrary Φ_u, a_u, b_u satisfying (11). Hence the most general form of the solutions can be constructed as follows:

1° we choose arbitrary (1—1 and onto) mappings $F_0, K_0, L_0, \psi, \lambda$;

2° we choose an arbitrary loop $(Q, +)$;

3° we define the sets $A, B \subseteqq Q$ with the property

(π) $(Q, \square) = (Q, +)^{[a\, b]}$ is isomorphic to $(Q, +)$ for every $a \in A, b \in B$;

4° we choose arbitrary mappings $a_u: P \to A$, $b_u: P \to B$;

5° we choose arbitrary isomorphisms $\Phi_u: (Q, +)^{[a_u\, b_u]} \to (Q, +)$.

Note that A, B are not empty since they contain at least the identity element 0 of $(Q, +)$, $(Q, \square) = (Q, +)^{[0, 0]} = (Q, +)$ being certainly isomorphic to $(Q, +)$. It is an open question how to find the sets A, B having the property (π) for a given loop $(Q, +)$. If e. g. $(Q, +)$ is a group, then $A = B = Q$. Indeed, then all the isotopes $(Q, \square) = (Q, +)^{[a, b]}$ are isomorphic groups.

7. Finally, in the special case $K_u = L_u$ we show a reduction theorem under suppositions of another type.

Theorem 4. *Let $(Q, \cdot)$ be a grupoid for which there exists at least one $c \in Q$ such that*

(i) *$u \to \Phi_u(cc)$ is a $1-1$ mapping of P onto Q;*

(ii) *$u \to \Psi_u c$ is onto Q.*

Then the most general form of the solutions of the functional equation

$$(15)\qquad \Phi_u(xy) = \Psi_u x \Psi_u y, \qquad x, y \in Q; \quad u \in P$$

is

$$(16)\qquad \Psi_u x = \theta^{-1} \Phi_u \theta x,$$

$$(17)\qquad xy = \theta x + \theta y,$$

where $(Q, +)$ is an arbitrary groupoid, θ is an arbitrary $1-1$ and onto mapping of Q, further, Φ_u is an arbitrary endomorphism of $(Q, +)$ restricted only by (i) *and* (ii).

PROOF. With $x = y = c$ (15) gives that

$$u \to \Phi_u(cc) = \Psi_u c \Psi_u c = \theta \Psi_u c$$

is a $1-1$ mapping onto Q, consequently,

$$s \to \theta s = ss$$

must be onto and $\Psi_u c$ must be $1-1$. But, by (ii), $u \to \Psi_u c$ is onto therefore it has an inverse. Thus it follows that θ too has an inverse mapping.

On the other hand, (15) with $x=y$ gives

$$\Phi_u \theta x = \theta \Psi_u x$$

and this is equivalent to (16).

Let us substitute (16) into (15) then it becomes

$$\Phi_u(xy) = (\theta^{-1}\Phi_u \theta x)(\theta^{-1}\Phi_u \theta y)$$

i. e., writing $\theta^{-1}x, \theta^{-1}y$ instead of x and y, respectively,

$$\Phi_u(\theta^{-1}x\theta^{-1}y) = (\theta^{-1}\Phi_u x)(\theta^{-1}\Phi_u y).$$

But this means that Φ_u is an endomorphism of the groupoid $(Q, +)$ defined by

$$x+y = \theta^{-1}x\theta^{-1}y$$

i. e. by (17).

Conversely, it is easy to verify that (16)–(17) satisfy (15) if Φ_u is an endomorphism. This completes proof of the theorem.

Note that $(Q, +)$ is idempotent as we have

$$x+x = \theta^{-1}x\theta^{-1}x = \theta\theta^{-1}x = x$$

for every $x \in Q$. However, this does not play any role in the form (16)—(17) of the solution as they satisfy (15) also with non-idempotent $(Q, +)$.

Under the suppositions of theorems 1 and 4 we can get the solution of

$$F_u(xy) = K_u x \circ K_u y, \qquad x, y \in Q; \quad u \in P$$

in the form

$$xy = \theta x + \theta y, \quad x \circ y = F_0(K_0^{-1}x + K_0^{-1}y),$$

$$F_u x = F_0 \Phi_u x, \quad K_u x = K_0 \theta^{-1} \Phi_u \theta x,$$

where F_0, K_0, θ are arbitrary ($1-1$ and onto) mappings, further, $(Q, +)$ is an arbitrary groupoid with endomorphisms Φ_u restricted by the suppositions of the respective theorems (cf. [4]).

Bibliography

[1] J. Aczél, Vorlesungen über Funktionalgleichungen und ihre Anwendungen, *Basel–Stuttgart*, 1960.

[2] V. D. Belousow, Systems of quasigroups, *Uspekhi mat. nauk* XX, 1 (121), (1965), 75–146.

[3] M. Hosszú, On the functional equation of distributivity, *Acta Math. Acad. Sci. Hungar.* **4** (1953), 157–167.

[4] M. Hosszú, A generalization of the functional equation of distributivity, *Acta Sci. Math. Szeged* **20** (1959), 67–80.

[5] M. Hosszú, Functional equations and algebraic methods in the theory of geometric objects *Publ. Math. Debrecen* **5** (1958), 294–329.

[6] M. Hosszú, Über eine Verallgemeinerung der Distributivitätsgleichung, *Acta Sci. Math Szeged* **26** (1965), 103–196.

(Received August 22, 1964.)