Some functional equations in connection with a theorem of Dubourdieu

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1. Let us consider four systems of surfaces depending on the parameters x, y, z, u, respectively. In the u=constant surfaces there is determined a web by the lines x=constant resp. y=constant resp. z=constant. Such a web is called a hexagonal one if it is topologically equivalent with a web determined by the lines $x =$ $=$const. resp. y=const. resp. z=const. on the plane $x+y+z = 0$ in the orthogonal coordinate system x, y, z, i. e. the relation

$$U_u^1(x)+U_u^2(y)+U_u^3(z) = 0 \tag{1}$$

is satisfied with suitable functions U^i for every fixed u=const. Similarly, the relations

$$X_x^1(y)+X_x^2(z)+X_x^3(u) = 0, \tag{2}$$

$$Y_y^1(x)+Y_y^2(z)+Y_y^3(u) = 0, \tag{3}$$

$$Z_z^1(x)+Z_z^2(y)+Z_z^3(u) = 0 \tag{4}$$

mean that the webs in the x=const. resp. y=const. resp. z=const. surfaces are hexagonal webs. A theorem of Dubourdieu states that under certain differentiability conditions any three of the relations (1)—(4) imply the fourth.

In the present paper we raise some problems in connection with certain functional equations arising in the examinition of Dubourdieurs webs.

Problem 1. Which are the solutions U^i, X^i, Y^i, Z^i of the system of functional equations (1)—(4)?

Problem 2. What is the most general form of a ternary operation $u=u(x, y, z)$ satisfying (2)—(4)?

Problem 3. Which are the solutions X^i, Y^i, Z^i of the system of functional equations (2)—(4) for a given $u=u(x, y, z)$?

In the present paper we reduce the problems 2 and 3 to the solution of a system of functional equations containing only three unknown functions of two variables, further, we answer the problems 2 and 3 in the special case where

$$X^1 = X^2 = -X^3, \quad Y^1 = Y^2 = -Y^3, \quad Z^1 = Z^2 = -Z^3.$$

2. Let us consider the functional equations (2)—(4). Suppose that all the functions figuring there are invertible. Without loss of generality, we may suppose that

$$xyz = u(x, y, z)$$

is a loop operation, i. e. there exists a unit element e with the properties:

$$xee = exe = eex = x. \tag{5}$$

In the contrary case we would consider an isotope $I(x, y, z)$ defined by

$$xyz = I(xbc, ayc, abz)$$

with arbitrarily fixed a, b, c. Clearly, this $I(x, y, z)$ is a loop operation with unit element $e = abc$, further, since the hexagonal property is an isotopy invariant one, also $I(x, y, z)$ can be written in the form (2)—(4) with suitable functions X^i, Y^i, Z^i. Thus, in what follows, we assume (5).

With the notation

$$X(t) = -X^3(t)$$

(2) can be written as

$$X(xyz) = X^1(y) + X^2(z). \tag{6}$$

Let us substitute here $z = e$ resp. $y = e$. Then we get

$$X^1(y) = X(xye) - X^2(e), \quad X^2(z) = X(xez) - X^1(e)$$

by which (6) can be written as

$$X(xyz) = X(xye) + X(xez) - X^1(e) - X^2(e).$$

Consequently, with the notation

$$F_x^1(t) = X_x(t) - X_x^1(e) - X_x^2(e)$$

we arive at

$$F_x^1(xyz) = F_x^1(xye) + F_x^1(xez). \tag{7}$$

Similarly, (3)—(4) imply

$$F_y^2(xyz) = F_y^2(xye) + F_y^2(eyz), \tag{8}$$

$$F_z^3(xyz) = F_z^3(xez) + F_z^3(eyz). \tag{9}$$

By putting here $z = e$ resp. $x = e$, we have by (5) the initial conditions:

$$F_t^i(t) = 0, \qquad i = 1, 2, 3. \tag{10}$$

Moreover, by putting $x = e$, $y = e$, and $z = e$ in (7), (8) and (9) respectively, we see that eyz, xez, xye are quasi-additions:

$$\begin{cases} F_e^1(eyz) = F_e^1(y) + F_e^1(z), \\ F_e^2(xez) = F_e^2(x) + F_e^2(z), \\ F_e^3(xye) = F_e^3(x) + F_e^3(y). \end{cases} \tag{11}$$

Let

$$F_t^i(u\nabla_t^i v) = F_t^i(u) + F_t^i(v), \qquad i = 1, 2, 3 \tag{12}$$

define three systems of binary operations ∇_t^i which depend upon a parameter t. Then, e. g., (7) can be written as

$$F_x^1(xyz) = F_x^1(xye) + F_x^1(xez) =$$
$$= F_x^1[(xye)\nabla_x^1(xez)] = F_x^1[(x\nabla_e^3 y)\nabla_x^1(x\nabla_e^2 z)]$$

i. e.

$$xyz = (x\nabla_e^3 y)\nabla_x^1(x\nabla_e^2 z).$$

We obtain similar formulae from (8)—(9). Thus (7)—(11) can be summed up as

$$\begin{aligned} xyz &= (x\nabla_e^3 y)\nabla_x^1(x\nabla_e^2 z) = \\ &= (x\nabla_e^3 y)\nabla_y^2(y\nabla_e^1 z) = \\ &= (x\nabla_e^2 z)\nabla_z^3(y\nabla_e^1 z), \end{aligned} \tag{13}$$

where the operations ∇_t^i are quasi-additions of the form (12) with unit element t:

$$u\nabla_t^i t = t\nabla_t^i u = u, \qquad i = 1, 2, 3. \tag{10'}$$

Theorem 1. *Supposed that* $X^i(t), Y^i(t), Z^i(t)$ *are invertible and* (5) *is fulfilled, then*

(I) xyz *can be represented in the form* (13) *by means of the binary group operations* ∇_t^i *for which* (12), (10') *hold;*

(II) *the functions* X^i, Y^i, Z^i *can be given by* F^i *so that*

$$X_x^3(t) = -X_x(t) = -F_x^1(t) - X_x^1(e) - X_x^2(e),$$

$$X_x^2(t) = X(xet) - X_x^1(e) = F_x^1(xet) + X_x^2(e),$$

$$X_x^1(t) = X(xte) - X_x^2(e) = F_x^1(xte) + X_x^1(e),$$

$$Y_y^3(t) = -F_y^2(t) - Y_y^1(e) - Y_y^2(e),$$

$$Y_y^2(t) = F_y^2(eyt) + Y_y^2(e),$$

$$Y_y^1(t) = F_y^2(tye) + Y_y^1(e),$$

$$Z_z^3(t) = -F_z^3(t) - Z_z^1(e) - Z_z^2(e),$$

$$Z_z^2(t) = F_z^3(etz) + Z_z^2(e),$$

$$Z_z^1(t) = F_z^3(tez) + Z_z^1(e)$$

are true, where $X_s^i(e), Y_s^i(e), Z_s^i(e)$ *are arbitrary functions of the variables and* eyz, xez, xye *are arbitrary group operations experssible by means of* F_e^i *in the form* (11).

Thus problems 2 and (3) reduce to the determination of the functions $F_t^i(s)$ satisfying (7)–(11).[1])

Remark that the special case

$$u\nabla_t^i v = u\nabla_e^i t^{-1}\nabla_e^i v$$

of (13) was treated previously in [3].

3. Let us consider the special case

$$-X^3 = X^1 = X^2 = X, \quad -Y^3 = Y^1 = Y^2 = Y, \quad -Z^3 = Z^1 = Z^2 = Z.$$

Then (2)–(4) are specialized as

$$(14) \qquad xyz = X_x^{-1}[X_x(y)+X_x(z)] = Y_y^{-1}[Y_y(x)+Y_y(z)] = Z_z^{-1}[Z_z(x)+Z_z(y)].$$

From this it follows that

$$xyz = xzy = zyx = yxz.$$

Hence (14) can be simplified as

$$(15) \qquad xyz = F_x^{-1}[F_x(y)+F_x(z)] = F_y^{-1}[F_y(x)+F_y(z)] = F_z^{-1}[F_z(x)+F_z(y)].$$

Since $x * y = xty$ is a binary group operation for every fixed t, we have

$$(xty)tz = xt(ytz),$$

i. e.

$$F_z^{-1}\big(F_z\{F_x^{-1}[F_x(t)+F_x(y)]\}+F_z(t)\big) = F_x^{-1}\big(F_x(t)+F_x\{F_z^{-1}[F_z(y)+F_z(t)]\}\big)$$

which, by the new variables

$$u = F_x(t), \ v = F_x(y)$$

and by the notation

$$(16) \qquad f(s) = F_z[F_x^{-1}(s)],$$

can be written as

$$f^{-1}[f(u+v)+f(u)] = u+f^{-1}[f(u)+f(v)].$$

Now we define

$$(17) \qquad M(u,v) = f^{-1}[f(u)+f(v)]$$

by which we obtain the functional equation

$$(18) \qquad M(u+v,u) = u+M(u,v), \ M(u,v) = M(v,u).$$

Lemma. *The most general continuous solution of* (18) *is*

$$(19) \qquad M(u,v) = u+v+c,$$

where c is an arbitrary constant.

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[1]) A similar result can be established also for the more general case where $x+y$ is a group operation but not necessarily abelian ([2]).

PROOF. First we write (18) in the symmetric form

$$M(u,v)+u = M(u,v+u), \quad M(u,v)+v = M(u+v,v). \tag{20}$$

The repeated application of (20) gives

$$M(u,v)+nu = M(u,v+nu), \quad M(u,v)+mv = M(u+mv,v) \tag{21}$$

for every integer $n, m \geqq 0$. If we write new variables $v_1 = v+nu, u_1 = u+mv$, then (21) becomes

$$M(u,v_1-nu) = M(u,v_1)-nu, \quad M(u_1-mv,v) = M(u_1,v)-mv,$$

hence (21) remains valid for every integer n, m.

Both equations in (21) can be united in the form

$$M[u+mv, v+n(u+mv)] = M(u,v)+mv+n(u+mv), \quad n, m = 0, \pm 1, \pm 2, \dots \tag{22}$$

Let us introduce

$$N(u,v) = M(u,v)-u-v, \tag{23}$$

then (22) becomes

$$N[u+mv, v+n(u+mv)] = N(u,v). \tag{24}$$

Now, choosing incommensurable u and v, by means of suitable integers m_k, n_k we can define a sequence

$$u_1 = u+m_1v, v_1 = v+n_1u_1, \dots, u_{k+1} = u_k+m_{k+1}v_k, v_{k+1} = v_k+n_ku_k, \dots$$

such that

$$|u_{k+1}| < |u_k|, \quad |v_{k+1}| < |v_k|.$$

Then, taking (24) into account, we get

$$N(u,v) = N(u_1,v_1) = N(u_k,v_k) = \dots = N\big(\lim_{k\to\infty} u_k, \lim_{k\to\infty} v_k\big) = N(0,0) = c.$$

Thus, by (23), we have proved

$$M(u,v) = u+v+c$$

for any incommensurable u, v. However, the set of pairs of such u, v-s is everywhere dense in the plane (u,v), therefore our lemma is proved, since $M(u,v)$ is supposed to be continuous.

By our lemma and by (17) we obtain

$$f(u)+f(v) = f(u+v+c),$$

therefore,

$$f(u-c) = F_z[F_x^{-1}(u-c)] = a_{z,x}(u) \quad \big(c = c(z,x)\big)$$

is an additive function for every fixed z, x. Thus we get

$$F_z(t) = a_{z,x}[F_x(t)+c] = a_{z,x}[F_x(t)]+b_{z,x} = a_{z,0}[F_0(t)]+b_{z,0},$$

$$F_z^{-1}(t) = F_0^{-1}[a_{z,0}^{-1}(t-b_{z,0})], \quad b_{z,x} = a_{z,x}[c(z,x)],$$

by which (15_3) becomes

$$xyz = F_z^{-1}[F_z(x)+F_z(y)] = F_0^{-1}\big(a_{z,0}^{-1}\{a_{z,0}[F_0(x)]+b_{z,0}+a_{z,0}[F_0(y)]+$$

$$+b_{z,0}-b_{z,0}\}\big) = F_0^{-1}[F_0(x)+F_0(y)+a_{z,0}^{-1}(b_{z,0})] = F_0^{-1}[F_0(x)+F_0(y)+G(z)],$$

where

$$G(z) = a_{z,0}^{-1}(b_{z,0}) = c(z,0).$$

Here, because of $xyz=xzy$, i. e.

$$F_0^{-1}[F_0(x)+F_0(y)+G(z)] = F_0^{-1}[F_0(x)+F_0(z)+G(y)],$$

we have

$$F_0(y)+G(z) = F_0(z)+G(y),$$

thus

$$G(z)-F_0(z) = a, \quad G(z) = F_0(z)+a.$$

So xyz can be written as

$$xyz = F_0^{-1}[F_0(x)+F_0(y)+F_0(z)+a] = F^{-1}[F(x)+F(y)+F(z)],$$

where

$$F(t) = F_0(t)+a/2.$$

Theorem 2. *Every continuous ternary quasigroup operation xyz satisfying* (14) *is a quasi-addition:*

(25) $$xyz = F^{-1}[F(x)+F(y)+F(z)].$$

4. Finally, let us examine the solution of (14) e. g. for $X_x(t)$ in the special case where (25) holds. Then we must solve the functional equation

(26) $$F^{-1}[F(x)+F(y)+F(z)] = X_x^{-1}[X_x(y)+X_x(z)].$$

With other variables and with the notation

$$G_x(y)=X_x(y'), \quad s'=F^{-1}(s)$$

we get

$$G_x(x+y+z) = G_x(y)+G_x(z).$$

By putting $y = s-x$, $z = t-x$, this shows that

$$G_x(t-x) = a_x(t)$$

is an additive function of t for every fixed x. Taking the definition of G into account, we get

$$X_x(y) = G_{F(x)}[F(y)] = a_{F(x)}[F(x)+F(y)] = a_x^*[F(x)+F(y)].$$

This function $X_x(y)$ with an arbitrary additive $a_x^*(t)$ satisfies (26) as

$$X_x(xyz) = a_x^*[F(x)+F(x)+F(y)+F(z)] = a_x^*[F(x)+F(y)]+a_x^*[F(x)+F(z)] =$$

$$= X_x(y)+X_x(z)$$

holds.

Theorem 3. *The most general invertible solution of the functional equation* (26) *is*

$$X_x(t) = a_x[F(x) + F(y)],$$

where $a_x(t)$ *is an arbitrary invertible and additive function of the variable* t.

Bibliography

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