On h-ideals and k-ideals in hemirings*)

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1. Introduction

If a non-empty set S is a semigroup under each of two binary relations $+$ and $\cdot$, and if $\cdot$ is distributive over $+$, then the system $(S, +, \cdot)$ is called a *semiring*. A zero element of a semiring S is an element 0 such that $0\cdot x = x\cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. By a *hemiring* we mean an additively commutative semiring with zero. The concept of ideal in semirings that is found most often in current literature is the following. A *left ideal* of a semiring S is a non-empty subset I closed under premultiplication by elements of S and under addition. *Right ideals* are defined dually and a *two-sided ideal*, or simply *ideal*, is both a left and a right ideal. We shall hereafter call these ideals left, right, and two-sided *semi-ideals*.

Although semi-ideals are useful for many purposes, they do not in general coincide with the usual ring ideals if S is a ring and, for this reason, their use is somewhat limited in trying to obtain analogues for semirings of ring theorems. Indeed, many results in rings apparently have no analogues in semirings using only semi-ideals.

HENRIKSEN ([5]) defined a more restricted class of ideals in semirings, which he called k-*ideals*, with the property that if the semiring S is a ring then a complex in S is a k-ideal if and only if it is a ring ideal. A still more restricted class of ideals in hemirings has been given by IIZUKA ([6]). However, a definition of ideal in any additively commutative semiring S can be given which coincides with Iizuka's definition provided S is a hemiring, and we call these ideals h-*ideals*.

The purpose of this paper is to investigate h-ideals and k-ideals in hemirings in an effort to obtain analogues of familiar ring theorems.

Definitions and basic concepts occupy section 2 and, in particular, we compare the BOURNE and IIZUKA congruence relations in hemirings.

Section 3 contains our main results, and begins with a discussion of hemirings of type (H). A hemiring is of type (H) provided that, under any natural homomorphism of the hemiring onto a Bourne factor hemiring modulo an h-ideal, the image of an h-ideal is an h-ideal. Specifically, we give analogues, for hemirings of type (H), of several theorems wellknown in ring theory.

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2. Basic concepts

Definition 2. 1. A *left k-ideal* I of a semiring S is a left semi-ideal such that if $a \in I$ and $x \in S$, and if either $a+x \in I$ or $x+a \in I$, then $x \in I$. A left semi-ideal I of an additively commutative semiring S is called a *left h-ideal* provided that if $x, z \in S$ and $i_1, i_2 \in I$, and $x+i_1+z = i_2+z$, then $x \in I$.

Right k- and h-ideals are defined dually, and a two-sided ideal of either type is both a left and a right ideal of that type. It is clear that every h-ideal is a k-ideal, but examples disprove the converse. It is routine to verify that any class of k-ideals (h-ideals) in a hemiring is closed under intersection.

As for rings, we define the semi-ideal (k-ideal, h-ideal) *generated by* a complex M of S as the intersection of all semi-ideals (k-ideals, h-ideals) that contain M. If M consists of a single element then the ideal it generates is called *principal*. Although a characterization of principal semi-ideals is well-known, we know of none for principal k- and h-ideals.

Bourne ([1]) has defined a relation, in an additively commutative semiring S, relative to a two-sided semi-ideal I; namely $a \equiv b(I)$ provided there are elements $i_1, i_2 \in I$ such that $a+i_1 = b+i_2$. This relation has been further investigated by Bugenhagen [4] and the following results are known. The Bourne relation is a congruence relation in S and hence partitions S into congruence classes $C_a, C_b, \ldots$, the class C_x containing x. Defining addition and multiplication by $C_a \oplus C_b = C_{a+b}$ and $C_a \odot C_b = C_{ab}$, these classes form an additively commutative semiring S/I. The class C_a is not necessarily of the form $a+I$. The semi-ideal I is contained in a congruence class C_I, which is a k-ideal of S, and $S/I = S/C_I$. If S has a zero then C_I is the zero of S/I. Finally, I is a congruence class mod I if and only if it is a k-ideal; in this case $C_I = I$.

Iizuka ([6]) has also defined a relation, in a hemiring, relative to any two-sided semi-ideal I, which we now give for any additively commutative semiring S; namely, $a[\equiv]b(I)$ provided there are elements $i_1, i_2 \in I$ and $z \in S$ such that $a+i_1+z = b+i_2+z$. The Iizuka relation is an additively cancellative congruence relation and we shall denote the congruence classes by $D_a, D_b, \ldots$. As before, the D_x form an additively commutative semiring $S[/]I$ that is also additively cancellative. The semi-ideal I is included in a class D_I, which is an h-ideal of S, and $S[/]I = S[/]D_I$. If S has a zero then D_I is the zero of $S[/]I$. Moreover, I is a congruence class mod I if and only if it is an h-ideal. In this case, $D_I = I$.

Theorem 2. 2. *If I is a two-sided semi-ideal in an additively commutative semiring S then the Bourne [Iizuka] congruence class $C_I[D_I]$ is the two-sided k-ideal [h-ideal] of S generated by I.*

Proof. We prove the bracked assertion. Let A be any two-sided h-ideal of S containing I, $b \in D_I$, and $i \in I$. Since $b[\equiv]i\ (I)$ there exist $i_1, i_2 \in I$ and $z \in S$ such that $b+i_1+z = i+i_2+z$. Since A is an h-ideal, the equation $b+i_1+z = (i+i_2)+z$ implies that $b \in A$, whence $D_I \subseteq A$. Since D_I is an h-ideal containing I, our result follows.

Clearly, for any semi-ideal I in any additively commutative semiring S, $x \equiv y(I)$ implies $x[\equiv]y(I)$. Examples show the converse false. But we shall later see that if I is any h-ideal in a finite hemiring S then these two relations coincide.

By the *natural homomorphism* of S onto S/I we mean the mapping that carries each $a \in S$ onto the Bourne class C_a containing it. Similarly, we speak of the natural homomorphism of S onto $S[/]I$.

HENRIKSEN ([5]) observed that a k-ideal I of a hemiring S is the kernel of the natural homomorphism of S onto S/I. If I is any semi-ideal of S then the Iizuka class D_I is the zero of $S[/]I$, whence the h-ideal D_I of S is the kernel of the natural homomorphism ϱ of S onto $S[/]I$. Thus, if I is an h-ideal then $D_I = I$, and I is the kernel of ϱ.

The following theorem gives a sufficient condition for the Bourne and Iizuka relations to coincide.

Theorem 2. 3. *If I is a semi-ideal of an additively commutative semiring S, and S/I is additively cancellative, then $S/I = S[/]I$, and if I is a k-ideal then it is an h-ideal.*

PROOF. Suppose $a, b \in S$ and $a[\equiv]b(I)$. Then there exist $i_1, i_2 \in I$ and $z \in S$ such that $a + i_1 + z = b + i_2 + z$. If v is the natural homomorphism of S onto S/I then, using cancellation, $(a+i_1)v = (b+i_2)v$. Thus there exist $i_3, i_4 \in I$ such that $a + i_1 + i_3 = b + i_2 + i_4$, whence $a \equiv b(I)$. Therefore $S/I = S[/]I$. Now suppose I is a k-ideal of S. If $x + i_1 + z = i_2 + z$, where $i_1, i_2 \in I$ and $z \in S$, then by cancellation $(x+i_1)v = i_2 v$, whence $x + i_1 + i_3 = i_2 + i_4$ for some $i_3, i_4 \in I$. Since I is a k-ideal, it follows that $x \in I$ and I is an h-ideal.

BOURNE [2] calls a semiring S with zero element 0 *semi-isomorphic* to a semiring T with zero provided there is a homomorphism from S onto T with kernel 0. It is not difficult to give an example of semiisomorphic semirings that fail to be isomorphic. Also, it is clear that semi-isomorphic rings are actually isomorphic. In this connection we have the following result.

Theorem 2. 4. *A semiring with zero that is semi-isomorphic to a ring is itself a ring.*

PROOF. Let φ be a semi-isomorphism from the semiring S with zero element $0'$ onto the ring R. For any $a \in S$, there exists $x \in S$ such that $-(a\varphi) = x\varphi$. Thus $0 = a\varphi - a\varphi = a\varphi + x\varphi = (a+x)\varphi$, whence $a + x = 0'$. It follows that $(S, +)$ is a group. If $a\varphi = b\varphi$ for $a, b \in S$ then $0 = a\varphi - b\varphi = (a-b)\varphi$, whence $a = b$. Therefore φ is $1-1$, and S is a ring isomorphic to R.

Semi-isomorphisms between semirings replace many of the isomorphisms between rings. The first example of this occurs in Theorem 2. 5, an analogue of the fundamental homomorphism theorem for rings. The unbracketed assertions are due to BOURNE ([2]).

Theorem 2. 5. *Let φ be a homomorphism of a hemiring S onto a [an additively cancellative] hemiring T. The kernel I of φ is a k-ideal [h-ideal] of S and there is a semi-isomorphism Ψ of S/I [$S[/]I$] onto T such that if v is the natural homomorphism of S onto S/I [$S[/]I$] then $\varphi = v\Psi$.*

For the proof of the bracketed assertions, note that if T is additively cancellative then its zero is an h-ideal, whence the kernel of φ is an h-ideal of S. Again using additive cancellation to show Ψ single-valued, the remainder of the proof is routine and is thus omitted. An example can be given to show that *semi-isomorphism* cannot be replaced by *isomorphism* in the unbracketed statement.

A semiring is called *additively regular* if its additive semigroup is regular in the sense that for each $a \in S$ there exists $x \in S$ such that $a+x+a = a$. The following theorem and its two corollaries are of some interest.

Theorem 2. 6. *An additively regular hemiring S that is semi-isomorphic to an additively cancellative semiring T with zero is a ring.*

PROOF. If φ is a semi-isomorphism of S onto T, and e is an additive idempotent of S, then, since $e\varphi$ is an additive idempotent in T and the only such element in T is zero, e is in the kernel of φ. Since φ has kernel zero, $e=0$. Thus $(S, +)$ is a regular semigroup with a unique idempotent and is hence a group.

Corollary 2. 7. *Let φ be a homomorphism from an additively regular hemiring S onto an additively cancellative hemiring T. If I is the kernel of φ then $S/I = S[/]I$ and S/I is a ring isomorphic to T.*

PROOF. By Theorem 2. 5, S/I is semi-isomorphic to T, whence, since S/I is also additively regular, Theorem 2. 6 shows that S/I is a ring. By Theorem 2. 3, $S/I = S[/]I$. Since the ring S/I is semi-isomorphic to T, our result is immediate.

Corollary 2. 8. *If I is an h-ideal of a hemiring S then S/I is semi-isomorphic to $S[/]I$. If S is additively regular then S/I is a ring and $S/I = S[/]I$.*

PROOF. If ϱ is the natural homomorphism of S onto $S[/]I$ then I is the kernel of ϱ. Thus Theorem 2. 5 (with $T=S[/]I$) shows that S/I is semi-isomoprhic to $S[/]I$. If S is additively regular then, since $S[/]$ is additively cancellative, Corollary 2. 7 shows that S/I is a ring and $S/I = S[/]I$.

An example can be given to show each assertion of Corollary 2. 8 false if I is only a k-ideal.

BOURNE and ZASSENHAUS ([3]) define the *zeroid* Z of a semiring S as $\{z \in S: z+x = x \text{ for some } x \in S\}$. If S has an additive idempotent then Z is non-empty, and the zeroid of any additively cancellative semiring with zero is zero. IIZUKA ([6]) points out that if S is a hemiring, $Z = \{z \in S: z[\equiv]0\,(0)\}$ and is the intersection of all h-ideals of S. The intersection of all k-ideals in any semiring with 0 is just 0. A frequent problem is that in a given hemiring the zeroid may not be zero. However, BOURNE and ZASSENHAUS proved that if Z is the zeroid of any hemiring S then S/Z has zeroid equal to zero, and if S is finite then S/Z is a ring. The first of these two results can be generalized as follows.

Theorem 2. 9. *If I is any h-ideal of a hemiring S then both S/I and $S[/]\,I$ have zeroid equal to zero.*

PROOF. Let ν be the natural homomorphism of S onto $T=S/I$ and ϱ the natural homomorphism of S onto $V=S[/]I$. Let $Z(T)$ and $Z(V)$ denote the zeroids of T and V, respectively. Since I is an h-ideal it is the zero of V, *i. e.*, $I=0_V$; but I is also a k-ideal and hence is the zero of T, *i. e.*, $I=0_T$. Suppose $z\varrho \in Z(V)$. Then $z\varrho + x\varrho = x\varrho$ for some $x \in S$. Thus there exist $i_1, i_2 \in I$ and $y \in S$ such that $(z+x)+ +i_1+y = x+i_2+y$, *i. e.*, $z+i_1+(x+y) = 0+i_2+(x+y)$, so that $z[\equiv]0\,(I)$. Hence $z\varrho = 0\varrho = I = 0_V$, whence $Z(V)=0_V$. Now if $z\nu \in Z(T)$ then $z\nu + x\nu = x\nu$ for some $x \in S$. Thus there exist $i_1, i_2 \in I$ such that $(z+x)+i_1 = x+i_2$, *i. e.*, $z+i_1+x = 0+i_2+x$, so that $z[\equiv]0\,(I)$. Therefore $z\varrho = 0\varrho = I$, whence $z \in I$. But $z \in I$ means $z\nu = I = 0_T$, whence $Z(T) = 0_T$.

A semiring S is called *additively periodic* provided the additive semigroup of S is periodic, *i. e.*, every element of $(S, +)$ has finite order. This is the case if and only if for each $x \in S$ there exist integers $0 < m < n$ such that $mx = nx$. In proving that for any finite hemiring S, S/Z is a ring, BOURNE and ZASSENHAUS actually proved that *every additively periodic hemiring with zeroid equal to zero is a ring.* This observation together with Theorems 2. 9 and 2. 3 provide a simple proof of the next theorem.

Theorem 2. 10. *If I is an h-ideal of an additively periodic hemiring S then S/I is a ring and $S/I = S[/]I$.*

Once again we remark that an example shows Theorem 2. 10 false if I is only a k-ideal.

3. Hemirings of type (H)

Although k-ideals and h-ideals in a ring coincide with the familiar ring ideals, these two kinds of ideals generally lack an important property enjoyed by both semi-ideals and ring ideals. Namely, k-ideals and h-ideals need not be preserved under homomorphisms; indeed, they need not be preserved under natural homomorphisms, as appropriate examples show. For some purposes, however, it is nice to have these ideals preserved under natural homomorphisms, and thus we are led to the following definition.

Definition 3. 1. An additively commutative semiring S is said to be of *type* (H) provided that if I is an h-ideal of S, and ν is the natural homomorphism of S onto S/I, then the image, under ν, of any h-ideal of S is an h-ideal of S/I.

If we replace *h-ideal* by *k-ideal* everywhere in this definition, S is said to be of *type* (K). To require that a hemiring S be of type (K) is actually weaker than requiring that its k-ideals be preserved under arbitrary homomorphisms of S. However, the corresponding question for type (H) is still unanswered. Clearly every ring is of type (K) and type (H), and suitable examples exist to show that not every hemiring is of either of these types.

The hemiring I^+ of all non-negative integers and the hemiring E^+ of all non-negative even integers are familiar examples of hemirings that are of type (K) and type (H). Indeed, since both are additively cancellative their h-and k-ideals coincide. The proofs of these assertions are straightforward once the k-ideals are shown to be all sets of the form (m), where (m) denotes all non-negative integral multiples of the element $m \in I^+[E^+]$. We cannot make any general statement regarding the occurence of hemirings of type (K), but for those of type (H) we have the following theorem.

Theorem 3. 2. *Every additively regular hemiring and every additively periodic hemiring (in particular every finite hemiring) is of type* (H).

The proof will be simplified by the following two lemmas.

Lemma 3. 3. (1) *Let φ be a homomorphism from an additively regular semiring S onto a ring R. If i is an element of any left k-ideal I of S then $-(i\varphi) = x\varphi$ for some $x \in I$.*

(2) *Let φ be a homomorphism from a semiring S onto an additively periodic ring R. If i is an element of any left semi-ideal I of S then $-(i\varphi) = x\varphi$ for some $x \in I$.*

PROOF. In case (1), I is an additively regular subsemiring of S. Thus if $i \in I$, there exists $x \in I$ such that $i+x+i = i$. Applying φ to this equation shows that $x\varphi = -(i\varphi)$.

In case (2), let $i \in I$ and $r \in i\varphi$. Now there exist integers $n > m > 0$ such that $mr = nr$, whence $(n-m)r = 0$, so that $-r = (n-m-1)r$. Thus $-(i\varphi) = -r = (n-m-1)r = (n-m-1)(i\varphi) = [(n-m-1)i]\varphi$, and $x = (n-m-1)i \in I$.

Lemma 3. 4. *If φ is a homomorphism from an additively regular (periodic) semiring S onto a ring R then φ preserves both k-ideals and h-ideals.*

PROOF. If I is a k-ideal of the semiring S, $I\varphi$ is a semi-ideal of R. Now suppose $a, b \in I\varphi$, and $r \in R$, and $r + a = b$. If $i_1, i_2 \in I$ with $i_1\varphi = a$ and $i_2\varphi = b$, then $r = i_2\varphi - i_1\varphi$. If S is additively regular, part (1) of Lemma 3. 3 shows that $-(i_1\varphi) = x\varphi$ for some $x \in I$, and if S is additively periodic then part (2) of that lemma applies. Thus, $r = i_2\varphi - i_1\varphi = (i_2 + x)\varphi \in I\varphi$. It follows that $I\varphi$ is a k-ideal of R. Since k-ideals, h-ideals and ring ideals coincide in R, h-ideals are preserved by φ also.

Now, *turning to the proof of Theorem* 3. 2., if I is any h-ideal of the additively regular or additively periodic hemiring S, then S/I is a ring by Corollary 2. 8 or Theorem 2. 10. Lemma 3. 4 then shows that the natural homomorphism of S onto S/I preserves h-ideals, whence S if of type (H).

We now give some results for hemirings of type (H) and type (K), including analogues of three well-known ring theorems. The following lemma is basic to what follows.

Lemma 3. 5. *Let I be a k-ideal of a hemiring S, and let v be the natural homomorphism of S onto S/I. If A is a k-ideal of S/I and $B = Av^{-1}$, then S/B is isomorphic to $(S/I)/A$.*

OUTLINE OF A PROOF. Now B is a k-ideal of S. Since Iv is the zero of S/I, and A is a k-ideal of S/I, we have $Iv \in A$. Thus $I = (Iv)v^{-1} \subseteq Av^{-1} = B$. Let ϱ be the natural homomorphism of S onto S/B and v' the natural homomorphism of S/I onto $(S/I)/A$. Consider the mapping $\varphi: a\varrho \to (av)v'$ of S/B into $(S/I)/A$. Now φ is single-valued, one-to-one, and maps S/B homomorphically onto $(S/I)/A$. The details can be found in [7].

Theorem 3. 6. *If S is a hemiring of type (K) (type (H)), and I is a k-ideal (h-ideal) of S, then S/I is of type* (K) (*type* (H)).

OUTLINE OF A PROOF. Let S be a hemiring of type (K) and I any k-ideal of S. We must show that if A is a k-ideal of S/I, then, under the natural homomorphism v' of S/I onto $(S/I)/A$, the image of any k-ideal A' of S/I is a k-ideal of $(S/I)/A$. Let v be the natural homomorphism of S onto S/I and let $B = Av^{-1}$; then B is a k-ideal of S. Also, $B' = A'v^{-1}$ is a k-ideal of S. Let ϱ be the natural homomorphism of S onto S/B. By Lemma 3. 5, the mapping $\varphi: a\varrho \to (av)v'$ is an isomorphism of S/B onto $(S/I)/A$. Consider the inverse image $(A'v')\varphi^{-1}$ of $A'v'$. If we can show that this is a k-ideal of S/B then clearly $A'v'$ is a k-ideal of $(S/I)/A$. Consider $(A'v^{-1})\varrho = B'\varrho$. Since B' is a k-ideal of S, and since S is of type (K), $B'\varrho$ is a k-ideal of S/B. Now $(A'v')\varphi^{-1}=B'\varrho$, and the reader is referred to [7] for the details. The proof of the bracketed assertion is parallel.

Lemma 3. 7. *If S is a hemiring, I a semi-ideal of S, and N a k-ideal of S that contains I, then, denoting by v the natural homomorphism of S onto S/I, we have $Nv=N/I$.*

PROOF. Now $Nv=\{av:a\in N\}$. Let $av\in Nv$, $a\in N$; then $av=\{x\in S:x\equiv a(I)\}$. If ϱ is the natural homomorphism of N onto N/I then $a\varrho=\{x\in N:x\equiv a(I)\}$. Clearly $a\varrho\subseteq av$. Conversely, if $x\in av$ then there exist $i_1, i_2\in I\subseteq N$ such that $x+i_1= a+i_2$. Since N is a k-ideal we have $x\in N$. Thus $x\in a\varrho$, so that $av=a\varrho\in N/I$. Therefore $Nv\subseteq N/I$. Now if $b\varrho\in N/I$ with $b\in N$, as above we have $b\varrho=bv\in Nv$, whence $N/I\subseteq Nv$.

We remark that examples exist that show Lemma 3. 7 false if N is only a semi-ideal. Our next theorem is an analogue of a basic result in ring theory.

Theorem 3. 8. *Let S be a hemiring and I any semi-ideal of S. Then any k-ideal (h-ideal) M of S/I is of the form N/I for some k-ideal (h-ideal) N of S that contains I. If I is an h-ideal, and S is of type* (H), *then the h-ideals of S/I are exactly the hemirings N/I where N is an h-ideal of S containing I.*

Before giving the proof we remark that the last sentence of this theorem holds if we replace type (H) by type (K) and h-ideals by k-ideals.

PROOF. Let I be a semi-ideal of S, v the natural homomorphism of S onto S/I; then $N=Mv^{-1}$ is a k-ideal [h-ideal] of S and $Nv=M$. Now $N=Mv^{-1}$ contains the kernel of v which, in turn, contains I, so by Lemma 3. 7 we have $M=Nv=N/I$. Now suppose I is an h-ideal of S and S is of type (H). If N is any h-ideal of S containing I and v is defined as before, then by Lemma 3. 7, $Nv=N/I$. Since S is of type (H) $Nv=N/I$ is an h-ideal of S/I.

As a final lemma we have

Lemma 3. 9. *If I is a k-ideal and M a semi-ideal of an additively commutative semiring S, and $M\subseteq I$, then $I/M=S/M$ if and only if $I=S$.*

PROOF. Suppose $I/M=S/M$ and $s\in S$. If v and ϱ are the natural homomorphisms of S onto S/M and of I onto I/M, respectively, then $sv=a\varrho$ for some $a\in I$. Since $s\in sv$ there exist $i_1, i_2\in M\subseteq I$ such that $s+i_1=a+i_2$, whence, since I is a k-ideal, $s\in I$. Thus $S=I$.

Definition 3. 10. A semiring S is called h-simple if it contains no h-ideal other than S itself and its zeroid; a semiring S with zero is called 0-*h-simple* if it contains no h-ideal except S itself and possibly zero. By Theorem 2. 9, if M is an h-ideal of a hemiring S then S/M has zeroid equal to zero, whence the two concepts of simplicity coincide in S/M.

The next theorem is familiar from ring theory. Its proof, facilitated by Theorem 3. 8 and Lemma 3. 9, is omitted since it parallels the usual proof given in rings.

Theorem 3. 11. *If S is a hemiring of type (H), and M is a proper h-ideal of S, then S/M is 0-h-simple if and only if M is maximal among the h-ideals of S.*

If we define a semiring S with 0 to be 0-k-*simple* provided it has no k-ideal except 0 and S itself, then the last theorem has an obvious analogue for hemirings of type (K) and k-ideals.

Our final result is an analogue of an isomorphism theorem for rings.

Theorem 3. 12. *If S is a hemiring of type (H) (type (K)), and I and M are h-ideals (k-ideals) of S such that $I \subseteqq M$, then S/M is isomorphic to $(S/I)/(M/I)$.*

PROOF. By Theorem 3. 8, M/I is an h-ideal of S/I. If ν is the natural homomorphism of S onto S/I then $(M/I)\nu^{-1} = M$. For, by Lemma 3. 7, $M \subseteqq (M/I)\nu^{-1}$. Conversely, if ϱ is the natural homomorphism of M onto M/I, and $x \in (M/I)\nu^{-1}$, we have $x\nu \in M/I$, so that $x\nu = m\varrho$ for some $m \in M$. Now $m\varrho \subseteqq M$, and, since $x \in x\nu = m\varrho$, it follows that $x \in M$. Thus $(M/I)\nu^{-1} = M$. An application of Lemma 3. 5 completes the proof.

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