Soluble embeddings of group amalgams

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1. Introduction

Let $\mathfrak{A}=am(A, B; H)$ be an amalgam of two groups A, B with amalgamation $H=A\cap B$ normal in A and in B, and call such an amalgam normal. The main aim of this note is to prove (Theorem 2. 3) that $\mathfrak{A}$ is embeddable in a soluble group if — and of course only if — A and B are soluble and the automorphism groups induced by A and B on H together generate a soluble group. In a recent paper [1] GRAHAM HIGMAN has given a criterion for the embeddability of an amalgam of finite p-groups in a finite p-group, which has as a corollary for normal amalgams of finite p-groups the result analagous to Theorem 2. 3. Higman's proof uses wreath products; ours uses a technical lemma on generalised free products which is possibly of some independent interest. Professor HIGMAN has made some helpful suggestions for a wreath product treatment of the soluble case and at present Mr. R. B. J. T. ALLENBY is working on these suggestions. I thank Mr ALLENBY for some very useful remarks.

Suppose that $\mathfrak{A}=am(A, B; H)$ is a quite general amalgam. The CIM-algebra of $\mathfrak{A}$ (for full details see [2]) is the set of all subgroups of $\mathfrak{A}$ obtained from A, B, H and the unit subgroup E by means of the operations of commutation; intersection; and multiplication of subgroups in those cases where the product is again a subgroup. A CIM-criterion is a set of equations between the elements of the CIM-algebra. The central theme of [2] was the proof that no CIM-criterion can be necessary and sufficient for the embeddability of an amalgam in a soluble, or in a nilpotent group. In other words, in general one has to go "outside" the amalgam in order to find embeddability criteria; for instance in [3] the tensor product was used to formulate a criterion for embeddability of an amalgam in a nilpotent group of class 2. A glance at Higman's paper will show that his criterion in certainly "internal"; and we show in the final section that the criterion given by Theorem 2. 3 is likewise internal.

The notation used is as follows. For any group elements x, y the conjugate $y^{-1}xy$ is denoted by x^y and the commutator $x^{-1}x^y$ by $[x, y]$; longer simple commutators are left-normed; for subgroups X, Y the symbol $[X, Y]$ means the subgroup generated by all $[x, y]$ with obvious notation; and the derived series of X is $X=X^{(0)}\supseteqq X'=X^{(1)}\supseteqq X^{(2)}\supseteqq\ldots$.

2. Generalised free products and soluble embeddings

Our first theorem describes a connection between the cartesian $[A, B]$ and the amalgamation in the generalised free product of a normal amalgam am $(A, B; H)$.

Theorem 2. 1. *Let G be the generalised free product of two groups A and B amalgamating a normal subgroup H. Then in G,*

(i) $[A, B] \cap H = [A, H][B, H]$;

(ii) for $n \geqq 1$, $[A, B]^{(n)} \cap H = [[A, B]^{(n-1)} \cap H, [A, B]^{(n-1)}]$.

PROOF. (i) Firstly $[A, H]$ is a normal subgroup of A, $[B, H]$ is normal in B so they are both normal in H and $[A, H][B, H]$ is a group; evidently it is also a normal subgroup of G contained in $[A, B] \cap H$. To prove that $[A, B] \cap H$ is contained in $[A, H][B, H]$, proceed as follows. Let S be a left transversal (system of left coset representatives) of H in A and T a left transversal of H in B. Then $[A, B]$ is generated by elements of the form [sh, th′] with obvious notation. Using well-known commutator identities it is a matter of routine to show that [sh, th′] is congruent modulo $[A, H][B, H]$ to $[s, t]$. It follows that every element of $[A, B]$ is of the form $[s_1, t_1]^{\varepsilon_1}[s_2, t_2]^{\varepsilon_2}...[s_n, t_n]^{\varepsilon_n}u$, where the s_i are in S, the t_i in T, u in $[A, H][B, H]$ and the ε_i integers. Suppose this element also lies in H. Going over to the factor-group G/H and letting star denote image of element and subgroup of G in G/H, we get

$$[s_1^*, t_1^*]^{\varepsilon_1}[s_2^*, t_2^*]^{\varepsilon_2}...[s_n^*, t_n^*]^{\varepsilon_n} = 1 \tag{2. 2}$$

But G^* is the absolutely free product of A^* and B^* so that, as is well-known, $[A^*, B^*]$ is free on the generators $[s^*, t^*]$ with s in $S-H$ and t in $T-H$. Evidently we can so arrange things (before going over to the factor-group!) that in (2. 2) no pair $[s_i^*, t_i^*]$ and $[s_{i+1}^*, t_{i+1}^*]$ are identical; and it follows at once that all the ε_i are zero. This means that every element of $[A, B] \cap H$ is in $[A, H][B, H]$, completing the proof of (i).

(ii) This is by induction over n. For $n=1$ we have immediately that $[A, B]' \cap \cap H \supseteq [[A, B] \cap H, [A, B]]$. By the proof of the first part, a typical generator of $[A, B]'$ is of the form $[f_1x_1, f_2x_2]$, where f_1 and f_2 lie in the group K generated by all the $[s, t]$ and x_1, x_2 lie in $[A, H][B, H]$. Straightforward calculation gives the following sequence of equalities and congruences modulo $[[A, H][B, H], [A, B]]$: $[f_1x_1, f_2x_2] = = [f_1, f_2x_2][f_1, f_2x_2, x_1][x_1, f_2x_2] \equiv [f_1, f_2x_2] = [f_1, x_2][f_1, f_2][f_1, f_2, x_2] \equiv [f_1, f_2]$, where we remind the reader that commutators are left-normed. It follows that every element g of $[A, B]'$ is of the form ku for k in K' and u in $[[A, H][B, H], [A, B]]$. We now use the fact that K^* is free on all the $[s^*, t^*]$ different from the identity, so that $K^{*\prime} = [A^*, B^*]'$ is free, possibly of zero rank. Then, assuming that g is in H, its image in G^* is k^* and so expressible as a reduced word in a system of free generators for $K^{*\prime}$; we can so arrange things with k and with this system that the reduced word is obtained from k simply by changing every s to s^* and every t to t^*. But $g^*=1$ and $u^*=1$ so that as k^* is reduced, k must also be 1. Thus $g \in [[A, H][B, H], [A, B]]$ and we have completed the case $n=1$. The proof of the inductive step requires no essentially new ideas and we omit it.

In the following theorem we have not been at pains to obtain the best possible bound for the length of the embedding group.

Theorem 2. 3. *Let $\mathfrak{A}=am(A, B; H)$ be a normal amalgam where A is soluble of length s and B soluble of length t. If the groups of automorphisms induced by A and B on H together generate a soluble group of length n, then $\mathfrak{A}$ is embeddable in a soluble group of solubility length not exceeding $n+1+\max(s, t)$.*

PROOF. Let G be the generalised free product of $\mathfrak{A}$. Then our condition certainly implies that $[H, [A, B]^{(n)}]=E$ in G, and it follows at once from Theorem 2. 1 that $[A, B]^{(n+1)}\cap H=E$. Put $Y=[A, B]^{(n+1)}$ and $L=G/Y$. We shall show that L is a soluble embedding for $\mathfrak{A}$. Firstly the image of A in L is isomorphic with $A/Y\cap A$, so we look at an element a of $Y\cap A$. Again using star to denote image of element and subgroup in G/H, we have $a^*\in[A^*, B^*]^{(n+1)}\cap A^*$. But G^* is the absolutely free product of A^* and B^* so that A^* meets $[A^*, B^*]$ trivially. Thus $a^*=1$, $a\in Y\cap H = E$ so that A is isomorphically represented in L. So is B. Next, the intersection $AY/Y\cap BY/Y$ is at least HY/Y. Conversely suppose that $aY=bY$ so that $a=by$ with obvious notation. Then $a^*=b^*y^*$ so that a^* is in the normal closure of B^* in G^*, and therefore $a^*=1$. Thus $a\in H$ and AY/Y and BY/Y intersect precisely in HY/Y, and L embeds $\mathfrak{A}$. Lastly L is soluble of length at most $n+1+\max(s, t)$, since it is an extension of $[A, B]/[A, B]^{(n+1)}$ by $G/[A, B]$; this factor group is generated by elementwise permuting homomorphic images of A and B. This completes the proof of the theorem.

3. Internal criteria

Once again let $\mathfrak{A}=\mathrm{am}(A, B; H)$ be a normal amalgam and denote by X_n the subgroup $[A, B]^{(n)}\cap H$ of the generalised free product of $\mathfrak{A}$. Then the X_n are subgroups of H normal in G which, by Theorem 2. 1, are connected by the equations

$$X_n=[X_{n-1}, [A, B]^{(n-1)}], \qquad n=1, 2, \ldots.$$

We shall describe briefly how the X_n can be constructed using only the multiplications inside $\mathfrak{A}$. Firstly $X_0=[A, H][B, H]$ and here the construction is obvious since $[A, H]$ and $[B, H]$ are in H.

Suppose next that $a_1b_1\ldots a_kb_k$ is any formal "word" in elements of A and B, where a_1 and/or b_k may be absent. Then we define the formal commutator of an element h of H with this word to be the element

$$h^{-1}((\ldots((h^{a_1})^{b_1})\ldots)^{a_k})^{b_k}$$

of H. It is well-defined because of normality and is, of course, the *actual* commutator $[h, a_1b_1\ldots a_kb_k]$ of h with the *element* $a_1b_1\ldots a_nb_n$ of any group containing $\mathfrak{A}$. Sor for instance X_1 is the subgroup of H generated by all formal commutators of elements of X_0 with formal words of the form $(a_1^{-1}b_1^{-1}a_1b_1)^{\varepsilon_1}\ldots(a_k^{-1}b_k^{-1}a_kb_k)^{\varepsilon_k}$ where the ε_i are ±1 and $(a^{-1}b^{-1}ab)^{-1}$ means $b^{-1}a^{-1}ba$. The reader will readily supply definition of X_{n+1} from X_n and proofs for all the claims made about the X_i.

Our final result is:

Theorem 3. 1. *A normal amalgam $\mathfrak{A}=am(A, B; H)$ is embeddable in a soluble group if and only if A and B are soluble and X_n is the unit subgroup for some n.*

PROOF. If X_n is nontrivial for some n, then in any group containing $\mathfrak{A}$ the n-th term of the derived series is non-trivial; so if $X_n \neq E$ for all n, every group embedding $\mathfrak{A}$ is insoluble.

Conversely suppose $X_n = E$ for some n and that A and B are soluble. An argument similar to that employed in Theorem 2. 2 shows that $G/[A, B]^{(n)}$ is a soluble embedding for $\mathfrak{A}$.

Theorem 3. 1 gives the internal criterion promised in the introduction. It is pleasant in that it uses only intersection and commutation; the point is, of course, that the X_n are not necessarily in the CIM-algebra of $\mathfrak{A}$.

References

[1] GRAHAM HIGMAN, Amalgams of p-groups, *J. of Algebra* **1**, (1964), 301–305.

[2] B. H. NEUMANN–JAMES WIEGOLD, On certain embeddability criteria for group amalgams, *Publ. Math. Debrecen* **9** (1962), 57–64.

[3] JAMES WIEGOLD, Nilpotent products of groups with amalgamations, *Publ. Math. Debrecen* **6** (1959), 131–168.

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