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On integervalued generalized q-additive solutions of linear recursions

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Abstract. Let be given a natural number $q>1$. The system $\{R_0, R_1, \dots\}$ of sets

$$R_i = q^i\{0,1,\dots,q-1\} = \{q^i \cdot m \mid m = 0,1,\dots,q-1\} \quad (i=0,1,\dots)$$

is called the *numbers system of basis q*. Fuctions being additive with respect to the numbers systsem of base q are called *q-additive* functions. (A. O. GELFOND. [1]) Let

$$P(x) = a_k x^k + \dots + a_1 x + a_0, \quad P(x) \in \mathbb{Z}[x],$$

and

$$P(E)f(n) := a_k f(n+k) + \dots + a_1 f(n+1) + a_0 f(n),$$

where $f : \mathbb{N}_0 \to \mathbb{Z}$.

We give necessary and sufficient condition to existence of an integervalued generalized q-additive solution of the linear recursion

$$P(E)f(n) = c \cdot n \quad (\forall n \in \mathbb{N}_0)$$

Introduction. We write $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{C}$ for the sets of integers, positive integers, non-negative integers and complex numbers, respectively.

Definition 1.1. Let $R_i \subset \mathbb{N}_0$ $(i=1,\dots)$. The system $R=\{R_0, R_1, \dots\}$ is called an *R-system* if

(a) $0 \in R_i$ and $1 < \operatorname{card} R_i < \infty$ $(i=0,1,\dots)$;

(b) for every $0 \le i < j$, the least positive element of R_i is less than the least positive element of R_j;

(c) each $n \in \mathbb{N}_0$ admits a unique decomposition of the form

$$n = \sum_{i=0}^{s} r_i \qquad (r_i \in R_i\ ;\ s \ge 1). \tag{1.2}$$

Definition 1.3. Given an R-system system R, a function $f:\mathbb{N}_0\to\mathbb{C}$ is said to be *R-additive* if

$$f(0)=0 \quad \text{and} \quad f(n)=f(r_0)+\cdots+f(r_s)$$

with the decomposition (1.2) of the number n. Given a natural number $q>1$, the system $\{R_0,R_1,\dots\}$ of the sets

$$R_i=q^i\{0,1,\dots,q-1\}=\{q^i\cdot m \mid m=0,1,\dots,q-1\} \quad (i=0,1,\dots)$$

is called the *number system of basis* q. Functions being additive with respect to the number system of base q are called *q-additive* functions.

The concept of q-additivity goes back to A. O. Gelfond [1]. The concept of R-additivity was introduced by J. Fejér [2] as a generalization of q-additivity. For any complex constant c, the function $f(n)=c\cdot n$ is R-additive with respect to every R-system. Let

$$P(x)=a_kx^k+\cdots+a_1x+a_0\ , \quad P(x)\in\mathbb{Z}[x]\ .$$

For any function $f:\mathbb{N}_0\to\mathbb{Z}$ we write

$$\text{(1.4)} \quad P(E)f(n) := a_kf(n+k)+a_{k-1}f(n+k-1)+\cdots+a_1f(n+1)+a_0f(n).$$

Consider the condition

$$P(E)f(n)\equiv 0 \pmod{n} \qquad (\forall n\in\mathbb{N}).$$

Hence, for R-additive functions $f:\mathbb{N}_0\to\mathbb{Z}$, it follows

$$\text{(1.5)} \qquad P(E)f(n)=c\cdot n \qquad (\forall n\in\mathbb{N})$$

with a suitable rational integer c (see [3], [4]). In the case $c=0$, (1.5) can obviously be solved and the only interesting task is to investigate the structure of the solutions. Our aim in this paper will be the investigation of the solvability of (1.5) for $c\neq 0$.

Theorem. *Given an R-system R, let $d>2$, $P\in\mathbb{Z}[x]$ with $\deg P<d-2$ and let $R_0=\{0,1,\dots,d-1\}$ in the system R. Furthermore let*

$$P_1(x)=\left(P(x),\frac{x^d-1}{x-1}\right), \qquad P_2(x)=\frac{P(x)}{P_1(x)}$$

and let $c\neq 0$ be a fixed constant. Then the linear recursion

$$\text{(2.1)} \qquad P(E)f(n)=c\cdot n \qquad (\forall n\in\mathbb{N}_0)$$

admits an integervalued R-additive solution if and only if

$P(1) \neq 0$ *and* $P_2(1) \mid c$, *furthermore*

(a) $2P_2(1) \mid c$ *whenever* $\deg P_1(x) \geq 1$ *and* $P'(1) = 0$ *and* $(x+1) \mid P_1(x)$,

(b) $P_2^2(1) \mid P_2'(1) \cdot c$ *whenever* $\deg P_1(x) \geq 1$ *and* $P'(1) = 0$,

(c) $P'(1) = 0$ *whenever* $P_1(x) = 1$.

We shall make use of the following lemma.

Lemma. *Let $c \neq 0$ be a complex constant, $R_0 = \{0, 1, 2, \dots, d-1\}$ and let $P^*(x) \in \mathbb{C}[x]$ be a polynomial of degree at most $d-2$. Then there exists an R-additive function $f(n)$ satisfying the condition*

$$P^*(E)f(n) = c \qquad (\forall n \in \mathbb{N}_0)$$

if and only if the polynomial $P^*(x)$ is of the form $P^*(x) = (x-1)P(x)$ where $P(1) \neq 0$. The solutions are precisely the following functions $f(n)$:

$$f(n) = \frac{c}{P(1)} \cdot n + \sum_{j=0}^{d-1} b_j \rho^{jn} = \frac{c}{P(1)} \cdot n + g(n) \tag{2.2}$$

where $\rho = \exp(2\pi i/d)$ and the coefficients b_j $(j = 0, \dots, d-1)$ have the following properties

(i) $b_j = 0$ *if* $P^*(\rho^j) \neq 0$ $\qquad (j = 0, 1, \dots, d-1)$,

(ii) $\displaystyle\sum_{j=0}^{d-1} b_j = 0$.

Proof. See [2].

Proof of the Theorem. If $f(n)$ fulfills the equation (2.1) then

$$P^*(E)f(n) = \Delta P(E)f(n) = c \qquad (\forall n \in \mathbb{N}_0) \tag{2.3}$$

where $P^*(x) = (x-1)P(x)$. According to the Lemma, the linear recursion (2.3) has an R-additive solution $f(n)$ if and only if $P(1) \neq 0$. The solutions have the form (2.2) and satisfy conditions (i) and (ii).

Let $P(n)$ be of the form (1.4) and let $f(n)$ be a solution of (2.3). Then, on the one hand, we have

$$\begin{aligned}P(E)f(n) &= P(E)\left(\frac{c}{P(1)}n + g(n)\right) = P(E)\frac{c}{P(1)}n + P(E)g(n)\\ &= a_k\frac{c}{P(1)}(n+k) + a_{k-1}\frac{c}{P(1)}(n+k-1) + \cdots + a_1\frac{c}{P(1)}(n+1)\\ &\quad + a_0\frac{c}{P(1)}n + 0 + b_0(a_k + \cdots + a_0)\\ &= P(E)\frac{c}{P(1)}n + P'(1)\frac{c}{P(1)} + b_0P(1) = c\cdot n,\end{aligned}$$

that is

$$\frac{P'(1)}{P(1)}\cdot c + b_0P(1) = 0. \tag{2.4}$$

On the other hand, since $f(n)$ is integervalued,

$$P_1(E)f(n) = P_1(1)\frac{c}{P(1)}\cdot n + \frac{P_1'(1)}{(1)}c + b_0P_1(1) \in \mathbb{Z}.$$

Hence for $n = 0$ we get

$$\frac{P_1'(1)}{P(1)}c + b_0P_1(1) \in \mathbb{Z}, \tag{2.5}$$

and for $n = 1$ we get

$$P_1(1)\cdot\frac{c}{P(1)} + \frac{P_1'(1)}{P(1)}\cdot c + b_0P_1(1) \in \mathbb{Z}. \tag{2.6}$$

From (2.5) and (2.6) it follows

$$P_1(1)\frac{c}{P(1)} = \frac{c}{P_2(1)} \in \mathbb{Z}.$$

Thus necessarily

$$P_2(1) \mid c.$$

I. a) If $P'(1)=0$ then by (2.4) we have $b_0=0$. Then (2.6) implies

$$P_1(1)\cdot\frac{c}{P(1)}+\frac{P_1'(1)}{P(1)}c=\frac{c}{P_2(1)}+\frac{P_1'(1)}{P_1(1)}\cdot\frac{c}{P_2(1)}\in\mathbb{Z},$$

thus

$$\frac{P_1'(1)}{P_1(1)}\cdot\frac{c}{P_2(1)}\in\mathbb{Z}.$$

Since $P_1(x)$ is a symmetric reciprocal polynomial, it is easy to see that

$$\frac{h}{2}P_1(1)=P_1'(1)$$

where h denotes the degree of $P_1(x)$. If $(x+1)\mid P_1(x)$ then h is odd, otherwise h is even. Therefore, by the relation

$$\frac{P_1'(1)}{P_1(1)}\cdot\frac{c}{P_2(1)}=\frac{c}{P_2(1)}\cdot\frac{h}{2}\in\mathbb{Z},$$

necessarily $2P_2(1)\mid c$ whenever $(x+1)\mid P_1(x)$.

b) If $P'(1)\neq 0$ then, by (2.4),

$$b_0=-\frac{P'(1)\cdot c}{P^2(1)}$$

and, by (2.5),

$$\frac{P_1'(1)\cdot c}{P(1)}-\frac{P'(1)\cdot c}{P^2(1)}\cdot P_1(1)$$

$$=\frac{P_1'(1)\cdot c}{P(1)}-\frac{P_1'(1)P_2(1)+P_1(1)\cdot P_2'(1)}{P_1(1)\cdot P_2^2(1)}\cdot c=-\frac{P_2'(1)}{P_2(1)\cdot c}=\cdot\frac{c}{P_2(1)}\in\mathbb{Z},$$

or equivalently

$$P_2^2(1)\mid P_2'(1)\cdot c.$$

We show that some integervalued solution exists whenever the conditions are fulfilled.

We have to prove now that some integervalued solution satisfying (2.4) always exists. Namely, if (2.4) holds then the solutions of (2.3) satisfy also (2.3).

Let the solutions of (2.3) be the functions

$$f(n)=\frac{c}{P(1)}n+\sum_{j=0}^{d-1}b_j\rho^{jn}=A_n\qquad(\forall n\in\mathbb{N}_0)\tag{2.7}$$

where $A_n \in \mathbb{Z}$ ($\forall n \in \mathbb{N}_0$). It is not hard to see that $f(n) \in \mathbb{Z}$ ($\forall n \in \mathbb{N}_0$) if and only if $f(n) \in \mathbb{Z}$ for $n = 0, 1, \ldots, d-1$. From (2.7) we deduce

$$\sum_{s=0}^{d-1} b_s \rho^{sn} = A_n - \frac{c}{P(1)} n \qquad (n = 0, 1, \ldots, d-1). \tag{2.8}$$

Multiplying the respective equations in (2.8) by ρ^{-jn} for $n = 0, 1, \ldots, d-1$ and then summing up, we get

$$\sum_{n=0}^{d-1} \sum_{s=0}^{d-1} b_s \rho^{(s-j)n} = \sum_{\substack{s=0 \\ s \neq j}}^{d-1} b_s \sum_{n=0}^{d-1} \rho^{(s-j)n} + b_j \sum_{n=0}^{d-1} \rho^0 = \sum_{n=0}^{d-1} \left(A_n - \frac{c}{P(1)} n \right) \rho^{-jn}$$
$$= \sum_{n=0}^{d-1} c_n \rho^{-jn} \qquad (j = 0, 1, \ldots, d-1;\ c_n = A_n - \frac{c}{P(1)} n).$$

Hence

$$d b_j = C(\rho^{-j}) \qquad (j = 0, 1, \ldots, d-1)$$

where $C(\rho^{-j})$ is a polynomial with rational coefficients in ρ^{-j}. If $P(\rho^j) \neq 0$ then $b_j = 0$ ($j = 0, 1, \ldots, d-1$). Thus then we have $C(\rho^{-j}) = 0$ whence $C(\rho^j) = 0$. Therefore, with the notations

$$K(x) = \frac{x^d - 1}{x - 1}\ , \quad Q(x) = \frac{K(x)}{P_1(x)},$$

the polynomial

$$C(x) = A_0 + (A_1 - \frac{c}{P(1)}) x + \cdots + (A_{d-1} - \frac{(d-1)c}{P(1)}) x^{d-1}$$

where $A_0 = 0$ (since $f(0) = 0$) satisfies

$$Q(x) \mid C(x)$$

and $Q(x)$ is a product of circle division polynomials.

a) If $P'(1) = 0$ then, according to (2.4), the condition $b_0 = 0$ is necessary and sufficient for a solution of (2.3) in order to be also solution of (2.1). Then necessarily $(x-1) | C(x)$. Thus, with the notation $C(x) = xS(x)$, we have $S(x) = (x-1)Q(x)B^*(x)$ where $B^*(x)$ is a polynomial with rational coefficients. Let c_0 be an integer satisfying condition (a) and let

$$\lambda = \frac{c_0}{P_2(1)}$$

where $\lambda = 2\lambda_1$, $\lambda_1 \in \mathbb{Z}$ whenever $(x+1) \mid P_1(x)$. Then

$$C(x) = xS(x) = A_1 x + \cdots + A_{d-1}x^{d-1} - \frac{c_0}{P(1)}(x + \cdots + (d-1)x^{d-1}).$$

Hence, with the notation $A(x) = A_1 + A_2 x + \cdots + A_{d-1}x^{d-2}$,

$$\begin{aligned} S(x) &= A(x) - \frac{\lambda}{P_1(1)}(1 + 2x + \cdots + (d-1)x^{d-2}) \\ &= A(x) - \frac{\lambda}{P_1(x)}K'(x) = (x-1)Q(x)B^*(x). \end{aligned}$$

Therefore

$$A(x)(x-1) - \frac{\lambda d}{P_1(1)}K'(x)(x-1) = (x-1)^2 Q(x)B^*(x).$$

Since $K'(x)(x-1) + K(x) = dx^{d-1}$, we have

$$A(x)(x-1) - \frac{\lambda d}{P_1(1)}x^{d-1} = Q(x)\left[(x-1)^2 B^*(x) - \frac{\lambda}{P_1(1)}P_1(x)\right].$$

Here $Q(x)$ is a principal polynomial with rational coefficients and $P_1(1)|d$. Consequently, the polynomial

$$L(x) = (x-1)^2 B^*(x) - \frac{\lambda}{P_1(1)}P_1(x)$$

has integer coefficients. We show that the polynomial $B^*(x)$ can be chosen in a manner such that we have

$$L(x) = ax + b$$

with suitable constants a, b. We have then

$$L(1) = a + b = -\lambda\ ; \quad L'(1) = a = -\frac{\lambda}{P(1)}P_1'(1) \in \mathbb{Z}$$

and hence $a, b \in \mathbb{Z}$. Therefore $B^*(x)$ and then the polinomial $A(x)$ for the solution can be constructed.

b) Case $P'(1) \neq 0$. Now let

$$P(1)C(x) = B(x)Q(x) \quad \text{where} \quad B(x) = x \cdot \overline{B}(x) = x \cdot P(1)B^*(x).$$

Then $(x-1) \nmid Q(x)B(x)$ since $B(1)Q(1) = P(1) \cdot d \cdot b_0 \neq 0$. Taking into account that (by (2.4)) $b_0 = -P'(1)c/P^2(1)$, we have

$$B(1)Q(1) = P(1) \cdot d \cdot b_0 = -\frac{P(1) \cdot cP'(1)d}{P^2(1)} = -\frac{cdP'(1)}{P(1)}.$$

Since $d = P(1) \cdot Q(1)$,

$$B(1) \cdot Q(1) = -\frac{cP_1(1)Q(1)P'(1)}{P_1(1)P_2(1)},$$

$$B(1) = -\frac{c}{P_2(1)} \cdot P'(1) = -\frac{c}{P_2(1)}(P_1'(1)P_2(1) + P_1(1)P_2'(1)),$$

$$B(1) = -c \cdot P_1'(1) - c \cdot P_1(1)P_2(1)\frac{P_2'(1)}{P_2^2(1)}.$$

Thus we have obtained that

$$B(1) = -cP_1'(1) - P(1) \cdot c \cdot \frac{P_2'(1)}{P_2^2(1)}. \tag{2.9}$$

On the other hand, now we have

$$S(x) = Q(x) \cdot B^*(x); \quad C(x) = x \cdot S(x) = x \cdot Q(x) \cdot B^*(x).$$

Hence, with the transformations used in a),

$$A(x)(x-1) - \frac{\lambda d}{P_1(x)}x^{d-1} = Q(x)\left[(x-1)B^*(x) - \frac{\lambda}{P_1(1)}P_1(x)\right],$$

$$L(x) = (x-1)B^*(x) - \frac{\lambda}{P_1(1)}P_1(x) \qquad (L(x) \in \mathbb{Z}[x]).$$

We shall seek the functions $L(x)$ in the form of a constant. We have $L(1) = -\lambda$ and hece $L(x) = \lambda$ identically. Since $\lambda = c/P_2(1)$, with the notation $\overline{B}(x) = P(1)B^*(x)$ it follows

$$(x-1)\overline{B}(x) = P(1)\left(\lambda\frac{P_1(x)}{P_1(1)} - \lambda\right) = P(1)\frac{c}{P_2(1)}\left(\frac{P_1(x)}{P_1(1)} - 1\right).$$

Thus

$$(x-1)\overline{B}(x) = c(P_1(x) - P_1(1))$$

$$\overline{B}(x) = c\frac{P_1(x) - P_1(1)}{x-1}.$$

Therefore $\lim_{x\to 1}\overline{B}(x)=\overline{B}(1)=cP_1'(1)$. Since $B(x)=x\overline{B}(x)$,

$$B(1)=1\cdot\overline{B}(1)=cP_1'(1). \tag{2.10}$$

The coefficients of $B(x)$ are (mod $P(1)$)-uniquely determined, since

$$P(x)C(x)=\sum_{i=1}^{d-1}(P(1)A_i-ic)x^i=Q(x)\cdot B(x).$$

However, we shall show that the polynomial $B(x)$ satisfying (2.9) can be constructed by modifying the coefficients of a polynomial $B(x)$ satisfying (2.10) in a manner such that $B_i^*\equiv B_i \pmod{P(1)}$ should be preserved for $i=1,\dots,d-1$.

Observe that we have

$$-2cP_1'(1)\equiv 0 \pmod{P(1)}$$

because $P_2(1)\mid c$ and $2P_1'(1)/P_1(1)\in\mathbb{Z}$. Let $kP(1)=P_1'(1)(-2c)$. Then

$$k=\frac{P_1'(1)(-2c)}{P_1(1)P_2(1)}=\frac{P_1'(1)(-2)}{P_1(1)}\cdot\frac{c}{P_2(1)};$$
$$\frac{P_1'(1)(-2)}{P_1(1)}=-h\ ,\implies k=-h\frac{c}{P_2(1)}=-h\lambda.$$

Thus

$$P_1'(1)(-2c)=-h\lambda P(1)=h(-\lambda P(1)).$$

Since $\deg P_1(x)=\deg B(x)=h$, $B_0=0$ and $c\cdot P_2'(1)/P_2^2(1)\in\mathbb{Z}$, we can construct the polynomial $B(x)$ satisfying (2.9) by subtsracting $P(1)$ from every coefficient of a polynomial $B(x)$ satisfying (2.10) and then modifying the coefficients by appropriate multiples of $P(1)$.

II. It is easy to see that if $P_1(x)=1$ identically then $b_0=0$ because of (ii). Thus necessarily $P'(1)=P_2'(1)=0$. Hence necessarily

$$P_1(E)f(n)=1\frac{c}{P(1)}n+0=\frac{c}{P(1)}n=\frac{c}{P_2(1)}n\in\mathbb{Z}\qquad(\forall n\in\mathbb{N}).$$

Hence for $n=1$ we get

$$\frac{c}{P_2(1)}\in\mathbb{Z}.$$

Since $g(n)=0$ identically, the only existing solution is the trivial

$$f(n)=\frac{c}{P(1)}n=\frac{c}{P_2(1)}n.$$

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