On Hacque's E-mappings and on semi-topogenous orders

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In the two notes [3] as well as in [4] MICHEL HACQUE has introduced the concept of E-mapping by the following

Definition 1. *Let E be an arbitrary set. An E-mapping is a mapping ϱ of $\mathfrak{P}(E)$ into $\mathfrak{P}[\mathfrak{P}(E)]$, satisfying*[1]) *the following conditions for $A, B, X, Y \subseteq E$:*

(E0) $$\varrho(A) \neq \emptyset, \text{ and } Y \supseteq X \in \varrho(A) \Rightarrow Y \in \varrho(A);$$

(E1) $$\varrho(\emptyset) = \mathfrak{P}(E);$$

(E2) $$X \in \varrho(A) \Rightarrow A \subseteq X;$$

(E3) $$A \subseteq B \Rightarrow \varrho(A) \supseteq \varrho(B).$$

The set of all E-mappings possesses a natural partial order defined by

$$\varrho_1 \subseteq \varrho_2 \text{ if } \varrho_1(X) \subseteq \varrho_2(X) \text{ for any } X \subseteq E.$$

With respect to this partial order the set of all E-mappings constitutes a complete lattice having

$$\varrho_M\colon \varrho_M(X) = \{Y | X \subseteq Y\}$$

as its greatest, and

$$\varrho_m\colon \varrho_m(\emptyset) = \mathfrak{P}(E),\ \varrho_m(X) = \{E\} \text{ for } X \neq \emptyset$$

as its smallest element, while arbitrary joins and meets are given by

$$(\cup \varrho_\nu)(X) = \cup \varrho_\nu(X) \text{ and } (\cap \varrho_\nu)(X) = \cap \varrho_\nu(X).$$

(One easily sees that these equalities define in fact E-mappings.)

The concept of E-mapping is equivalent to the one of semi-topogenous order as introduced by Á. CSÁSZÁR in his book [1]:

Definition 2. *A semi-topogenous order on a set E is a relation $<$ defined on $\mathfrak{P}(E)$, satisfying the folloving conditions:*

(01) $$\emptyset < \emptyset, \quad E < E;$$

(02) $$A < B \Rightarrow A \subseteq B;$$

(03) $$A \subseteq A' < B' \subseteq B \Rightarrow A < B.$$

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[1]) I. e. a mapping by which there corresponds to each subset of the space E a class of subsets of E.

The set of all semi-topogenous orders on a set E possesses a natural partial order defined by

$$< \subseteq <_1 \text{ iff } A < B \Rightarrow A <_1 B \text{ for } A, B \subseteq E.$$

With respect to this partial order the set of all semi-topogenous orders on E constitutes a complete lattice having

$$<_M: A <_M B \Leftrightarrow A \subseteq B$$

as its greatest, and

$$<_m: A <_m B \Leftrightarrow A = \emptyset \text{ and/or } B = E$$

as its smallest element, while arbitrary joins and meets are defined by set-theoretical union and set-theoretical intersection respectively of the semi-topogenous orders concerned.[2])

The equivalence above mentioned can now be stated in a formal way as follows:

Proposition 1. (1) *If $<$ is a semi-topogenous order on E then the function $\varrho_<$ defined by*

$$\varrho_<(A) = \{X | A < X\}$$

is an E-mapping.

(2) *If ϱ is an E-mapping then the relation $<_\varrho$ defined by*

$$A <_\varrho B \Leftrightarrow B \in \varrho(A)$$

is a semi-topogenous order on E.

(3) *The mappings $< \to \varrho_<$ and $\varrho \to <_\varrho$ are one-to-one correspondences, inverse to each other, between the sets of all semi-topogenous orders and all E-mappings on E, which preserve the respective partial orders.*

We omit the easy proof which resembles the proof of Theorem 1. in [2].

In the remaining part of this note we are going to establish a few results concerning semi-topogenous orders which have originally been obtained for E-mappings. (See [3] and [4].)

We need first of all the following

Definition 3. *By the composition of two semi-topogenous orders $<_1$ and $<_2$ on a set E we mean the semi-topogenous order $<_1 \cdot <_2$ defined as follows:*

$$A <_1 \cdot <_2 B \Leftrightarrow A <_1 X <_2 B \text{ for some } X \subseteq E.$$

In order to justify this definition, we must verify of course that the relation $< \, = \, <_1 \cdot <_2$ satisfies conditions (01), (02) and (03), whenever $<_1$ and $<_2$ are semi-topogenous orders. The verification is straightforward.

The main properties of the operation of composition are summed up [3]) in the following

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[2]) These being considered as subsets of $\mathfrak{P}(E) \times \mathfrak{P}(E)$.

[3]) See [3], p. 1905., Proposition 1. and [4] Proposition 1. 1.

Semi-topogenous orders are of course relations, and their composition is a special case of the well-known operation of forming the product of relations. The contents of Proposition 2. is accordingly not essentially new. Still, for completeness' sake, we give an outline of the proof.

Proposition 2. (1) *The composition of semi-topogenous orders is associative.*

(2) *For any two semi-topogenous orders* $<_1$ *and* $<_2$ *we have*

$$<_1 \cdot <_2 \subseteqq <_1 \text{ and } <_1 \cdot <_2 \subseteqq <_2.$$

(3) *Any semi-topogenous order* $<$ *satisfies the condition* $<^2 \subseteqq <$. *The equality* $<^2 = <$ *holds if and only if* $A < B$ *always implies the existence of a set* C *such that* $A < C < B$.

(4) *For two arbitrary families* $\{<_i | i \in I\}$ *and* $\{<_j | j \in J\}$ *of semi-topogenous orders*

$$[\bigcup_{i \in I} <_i] \cdot [\bigcup_{j \in J} <_j] = \bigcup_{(i, j) \in I \times J} <_i \cdot <_j.$$

(5) *For any semi-topogenous order* $<$ *on* E:

$$<_M \cdot < \; = \; < \cdot <_M \; = \; < \text{ and } <_m \cdot < \; = \; < \cdot <_m \; = \; <_m.$$

(6) *For any semi-topogenous order* $<_1$ *on* E, *the mappings* $< \; \to \; <_1 \cdot <$ *and* $< \; \to \; < \cdot <_1$ *are isotone, i. e.*

$$<_\alpha \subseteqq <_\beta \Rightarrow <_1 \cdot <_\alpha \subseteqq <_1 \cdot <_\beta,$$

and

$$<_\alpha \subseteqq <_\beta \Rightarrow <_\alpha \cdot <_1 \subseteqq <_\beta \cdot <_1.$$

Proof. (1) Clear.

(2) $A <_1 <_2 B$ means that $A <_1 C <_2 B$ for some C, and this implies both $A <_1 B$ and $A <_2 B$ by (03).

(3) $<^2 \subseteqq <$ results by (2), while the condition stated is the one for $< \subseteqq <^2$.

(4) Each of the following statements is equivalent to the next one:

$$A[\bigcup_{i \in I} <_i] \cdot [\bigcup_{j \in J} <_j] B,$$

$$A[\bigcup_{i \in I} <_i] \, C [\bigcup_{j \in J} <_j] B \text{ for some } C,$$

$$A <_i C <_j B \text{ for some } C \text{ and for some } i \in I \text{ and } j \in J,$$

$$A <_i <_j B \text{ for some } (i, j) \in I \times J,$$

$$A[\bigcup_{(i, j) \in I \times J} <_i <_j] B.$$

(5) $A < B$ i. e. $A < B \subseteqq B$ implies $A < \cdot <_M B$, and $A \subseteqq A < B$ implies $A <_M < B$. Thus we have $< \subseteqq < \cdot <_M$ and $< \subseteqq <_M \cdot <$, while the reverse inclusions follow by (2).

Again by (2) $<_m \cdot < \subseteqq <_m$ and $< \cdot <_m \subseteqq <_m$, and since $<_m$ is the smallest semi-topogenous order, equality must hold.

(6) If $A <_\alpha B$ implies $A <_\beta B$ then $A <_1 <_\alpha B$ i. e. $A <_1 C <_\alpha B$ for some C clearly implies $A <_1 C <_\beta B$ i. e. $A <_1 <_\beta B$. The other relation of isotony can be proved similarly.

Definition 4. *By a system (of subsets of* E*) we mean a class of subsets of* E *containing the void set and the set* E.

As is known from [1], any system σ generates a semi-topogenous order $<$ in the following way:

$$A < B \Leftrightarrow A \subseteq S \subseteq B \text{ for some } S \in \sigma.$$

On the other hand, not every semi-topogenous order is generated by some system. (See [1], (2. 3).)

A necessary and sufficient condition [4]) for a semi-topogenous order to be generated by a system is contained in the following

Proposition 3. (1) *For any semi-topogenous order $<$ on a set E the class*

$$\sigma_< = \{S | S < S\}$$

of subsets of E is a system which generates a semi-topogenous order $<_1$ satisfying $<_1 \subseteq <$.

(2) *The semi-topogenous order $<$ is generated by some system if and only if $<_1 = <$, and then $\sigma = \sigma_<$.*

Proof. (1) $\emptyset \in \sigma_<$ and $E \in \sigma_<$ by (01). Moreover, $A <_1 B$ i. e. $A \subseteq S \subseteq B$ with some $S < S$ implies $A < B$ by (03).

(2) If a semi-topogenous order $<$ is generated by a system σ then this system is uniquely determined [5]): $\sigma = \{S | S < S\} = \sigma_<$, and so $< = <_1$. On the other hand, if $< = <_1$ then $<$ is generated by $\sigma_<$.

Proposition 4.[6]) *Any semi-topogenous order $<$ generated by a system is idempotent: $< = <^2$.*

Proof. We already know that $<^2 = < \cdot < \subseteq <$.

On the other hand $A < B$ i. e. $A \subseteq S \subseteq B$ $(S \in \sigma)$ clearly implies $A < S < B$, i. e. $A <^2 B$.

A semi-topogenous order $<$ is said to be *perfect*, if it satisfies condition

$$\text{(P)} \qquad A_i < B_i \ (i \in I) \Rightarrow \bigcup_{i \in I} A_i < \bigcup_{i \in I} B_i.$$

(See [1], Chapter 4.)

Perfect semi-topogenous orders are capable of the following characterization:

Proposition 5. *A semi-topogenous order $<$ on a set E is perfect if and only if for $X < Y$ there exists a maximal superset X_M of X for which $X_M < Y$ still holds.*

Proof. The condition is necessary. Indeed, for $X < Y$ let

$$\mathfrak{A} = \{A | X \subseteq A \ \& \ A < Y\}.$$

Clearly $X \in \mathfrak{A}$ and $\cup \{A | A \in \mathfrak{A}\}$ is the X_M required.

The condition is sufficient: If $A_i < B_i$ for $i \in I$ then applying (03) twice we get $\cap \{A_i | i \in I\} < \cup \{B_i | i \in I\}$. Let now be X_M the maximal superset of $\cap \{A_i | i \in I\}$

[4]) See [3], p. 1905., Proposition 2. and [4] Proposition 1. 2.

[5]) See [1], (2. 1).

[6]) See [3], p. 1905., Proposition 3. and [4], Proposition 1. 3.

satisfying $X_M < \cup \{B_i | i \in I\}$. By the maximality of X_M and by $A_i < \cup \{B_i | i \in I\}$ we get $A_i \subseteq X_M$ for $i \in I$, and therefore

$$\bigcup_{i \in I} A_i \subseteq X_M < \bigcup_{i \in I} B_i \Rightarrow \bigcup_{i \in I} A_i < \bigcup_{i \in I} B_i.$$

We call a semi-topogenous order $<$ coperfect, if its complement $<^c$ (defined by $A <^c B \Leftrightarrow E - B < E - A$) is perfect. Clearly, $<$ is coperfect if and only if

$$A_i < B_i (i \in I) \Rightarrow \bigcap_{i \in I} A_i < \bigcap_{i \in I} B_i.$$

Proposition 5., characterizing the complements of coperfect semi-topogenous orders, yields the following

Proposition 6. *A semi-topogenous order $<$ on a set E is coperfect if and only if for $X < Y$ there exists a minimal subset Y_m of Y for which $X < Y_m$ still holds.*

It will be worth while to put down also

Proposition 7. *A semi-topogenous order which is perfect or coperfect, is idempotent if and only if it is generated by a system.*

PROOF. Let $<$ be a perfect idempotent semi-topogenous order. If $X < Y$, there is a maximal X_M with $X \subseteq X_M < Y$. Since $<$ is idempotent, there exists a set Z such that $X_M < Z < Y$. By the maximality of X_M, $X_M = Z$ follows. Thus $X_M < X_M$, i. e. $X_M \in \sigma_<$. We see that $X < Y$ implies $X \subseteq X_M \subseteq Y$ for some $X_M \in \sigma_<$, i. e. that $X < Y \Rightarrow$ $\Rightarrow X <_1 Y$. Now, $< \subseteq <_1$ yields by Proposition 3. $< = <_1$.

Again, let $<$ be a coperfect idempotent semi-topogenous order. If $X < Y$, there is a minimal Y_m with $X < Y_m \subseteq Y$. Since $<$ is idempotent, there exists a set Z such that $X < Z < Y_m$. By the minimality of Y_m, $Z = Y_m$ follows. Thus $Y_m < Y_m$, i. e. $Y_m \in \sigma_<$. We see that $X < Y$ implies $X \subseteq Y_m \subseteq Y$ for some $Y_m \in \sigma_<$, and $< = <_1$ follows.

So far we have proved [7]) that if a semi-topogenous order which is perfect or coperfect satisfies the condition of idempotency, then it is generated by a system. The converse statement follows by Proposition 4.

Remarks. 1. If the semi-topogenous order $<$ is symmetrical: $< = <^c$, i. e. $A < B \Leftrightarrow E - B < E - A$, then the conditions formulated in Propositions 5. and 6. respectively, are of course equivalent (and $<$ is a biperfect topogenous order).

2. If the set E is infinite then there exists at least one idempotent semi-topogenous order which is not generated by any system. Indeed, let $<$ be the semi-topogenous order defined as follows:

$$X < Y \text{ if and only if } X = \emptyset, \text{ or } Y = E$$

or else $X \subseteq Y$ and $Y - X$ is infinite.

One easily sees that this semi-topogenous order is idempotent, and also that it is not generated by a system: we cannot have $S < S$ for $\emptyset \neq S \neq E$.

[7]) Compare with this the result embodied in formulae (7. 17)—(7. 21) of [1]. — Idempotent perfect or coperfect semi-topogenous orders cannot in general be considered as syntopogenous structures, because they are only semi-topogenous and not topogenous orders.

3. If the set E has at least three elements then there is at least one semi-topogenous order $<$ on E, non-idempotent and not generated by a system: The relation $<$ defined by

$$\emptyset < X \text{ for } X \subseteq E,\ E < E \text{ and } X < Y \Leftrightarrow X \subset Y \text{ for } \emptyset \neq X \neq E$$

is a non-idempotent semi-topogenous order.

References

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