Skew proximity functions

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§ 1. Skew proximity functions and proximity functions

A complex number is real if and only if it coincides with its conjugate: it is one of the aims of this note to give a similar characterization of proximity functions as introduced by B. BANASCHEWSKI and J. M. MARANDA [1].

We start with the following

Definition 1. *A skew proximity function on a set E is a mapping α from the set $\mathfrak{P}(E)$ of all subsets of E into the set $\Phi(E)$ of all filters on E satisfying the following conditions* [1]) *for $A, B, C \subseteq E$:*

(S0) $$\emptyset \in \alpha(\emptyset);$$

(S1) $$B \in \alpha(A) \Rightarrow B \supseteq A;$$

(S2) $$\alpha(A \cup B) = \alpha(A) \cap \alpha(B);$$

(S3) *For any $B \in \alpha(A)$ there exists a C such that $B \in \alpha(C)$ and $C \in \alpha(A)$.*

Lemma 1. $A \subseteq B \Rightarrow \alpha(A) \supseteq \alpha(B)$.

PROOF. $A \subseteq B \Leftrightarrow A \cup B = B$, and then by (S2) $\alpha(B) = \alpha(A) \cap \alpha(B)$, i. e. $\alpha(A) \supseteq \alpha(B)$.

Definition 2. *If α is a skew proximity function on the set E then the mapping $\bar{\alpha}$ from $\mathfrak{P}(E)$ into $\mathfrak{P}[\mathfrak{P}(E)]$ defined by the condition*

(C) $$B \in \bar{\alpha}(A) \Leftrightarrow E - A \in \alpha(E - B)$$

is said to be the conjugate of the skew proximity function α.

In order to justify this definition, we establish the following

Proposition 1. *The conjugate $\bar{\alpha}$ of a skew proximity function α on a set E is a skew proximity function on E.*

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[1]) I am indebted to Professor Á. CSÁSZÁR for the following example showing that condition (S0) is not redundant:

Let $\emptyset \neq T \subseteq E$, and $X \in \alpha(A) \Leftrightarrow A \cup T \subseteq X$. Then conditions (S1), (S2) and (S3) are fulfilled, but not (S0).

PROOF. First we show that $\bar{\alpha}(A)$ is a filter for any $A \subseteq E$: The class $\bar{\alpha}(A)$ is non-void for any $A \subseteq E$. Indeed, by (S0) we have $\alpha(\emptyset) = \mathfrak{P}(E)$, and so $E \in \bar{\alpha}(A) \Leftrightarrow \Leftrightarrow E - A \in \alpha(\emptyset)$ holds for any $A \subseteq E$. Moreover,

$$\left.\begin{array}{l} B \in \bar{\alpha}(A) \Leftrightarrow E - A \in \alpha(E - B) \\ C \supseteq B \Leftrightarrow E - B \supseteq E - C \Rightarrow \alpha(E - B) \subseteq \alpha(E - C) \end{array}\right\} \Rightarrow E - A \in \alpha(E - C) \Leftrightarrow C \in \bar{\alpha}(A).$$

On the other hand:

$$\left.\begin{array}{l} B \in \bar{\alpha}(A) \Leftrightarrow E - A \in \alpha(E - B) \\ C \in \bar{\alpha}(A) \Leftrightarrow E - A \in \alpha(E - C) \end{array}\right\} \Rightarrow E - A \in \alpha(E - B) \cap \alpha(E - C) =$$

$$= \alpha[(E - B) \cup (E - C)] = \alpha[E - (B \cap C)],$$

i. e.

$$E - A \in \alpha[E - (B \cap C)] \Leftrightarrow B \cap C \in \bar{\alpha}(A).$$

Let us now check the validity of the four conditions [2]) listed in Definition 1.:

$\overline{\text{(S0)}}$: $\emptyset \in \bar{\alpha}(\emptyset) \Leftrightarrow E \in \alpha(E)$.

$\overline{\text{(S1)}}$: $B \in \bar{\alpha}(A) \Leftrightarrow E - A \in \alpha(E - B) \Rightarrow E - A \supseteq E - B \Leftrightarrow B \supseteq A$.

$\overline{\text{(S2)}}$: Each of the following conditions is equivalent to the next one:

$$C \in \bar{\alpha}(A \cup B),$$

$$E - (A \cup B) \in \alpha(E - C),$$

$$(E - A) \cap (E - B) \in \alpha(E - C),$$

$$E - A \in \alpha(E - C) \ \& \ E - B \in \alpha(E - C),$$

$$C \in \bar{\alpha}(A) \ \& \ C \in \bar{\alpha}(B),$$

$$C \in \bar{\alpha}(A) \cap \bar{\alpha}(B).$$

$\overline{\text{(S3)}}$: Let $B \in \bar{\alpha}(A) \Leftrightarrow E - A \in \alpha(E - B)$. Then by (S3) there exists a set $E - C$ such that $E - A \in \alpha(E - C)$ and $E - C \in \alpha(E - B)$, i. e. $B \in \bar{\alpha}(C)$ and $C \in \bar{\alpha}(A)$. This completes the proof of Proposition 1.

As an immediate consequence of condition (C) we have

Proposition 2. $\bar{\bar{\alpha}} = \alpha$ *for any skew proximity function* α *on a set* E.

Now we are able to give the characterization of proximity functions in terms of skew proximity functions and their conjugates mentioned at the outset. Let us recall the definition of a proximity function:

Definition 3. *A proximity function on a set* E *is a mapping* $\alpha : \mathfrak{P}(E) \to \Phi(E)$ *satisfying the following conditions:*

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[2]) If (K) is a condition for α then the same condition for $\bar{\alpha}$ will as a rule be denoted by $(\overline{K})$.

(A1) $B \in \alpha(A) \Rightarrow B \supseteq A$;

(A2) $A \subseteq B \Rightarrow \alpha(A) \supseteq \alpha(B)$;

(A3) $B \in \alpha(A) \Rightarrow E - A \in \alpha(E - B)$;

(A4) For any $B \in \alpha(A)$ there exists a C such that $B \in \alpha(C)$ and $C \in \alpha(A)$.

A proximity function is always a skew proximity function. As a matter of fact, condition (S2) is an easy consequence of (A2) and (A3). (See [1], Proposition 4.) On the other hand, since a skew proximity function always satisfies conditions (A1), (A2) and (A4), [3]) it is a proximity function if and only if it satisfies also (A3). Thus a comparison of condition (C) and of (A3) yields the following

Proposition 3. *A skew proximity function α is a proximity function if and only if $\alpha = \bar{\alpha}$.*

§ 2. Some further results on skew proximity functions

We shall denote the set of all skew proximity functions on a set E by $\pi(E)$. This set $\pi(E)$ possesses a natural partial order defined by the condition "$\alpha(A) \subseteq \alpha'(A)$ for all $A \subseteq E$" which will be denoted by $\alpha \subseteq \alpha'$. (To be read: α is coarser than α', or α' is finer than α.)

The operation of forming the conjugate of a skew proximity function is isotone with respect to the partial order just introduced:

Proposition 4. $\alpha \subseteq \alpha' \Rightarrow \bar{\alpha} \subseteq \overline{\alpha'}$ for $\alpha, \alpha' \in \pi(E)$.

PROOF. $X \in \bar{\alpha}(A) \Leftrightarrow E - A \in \alpha(E - X) \Rightarrow E - A \in \alpha'(E - X) \Leftrightarrow X \in \overline{\alpha'}(A)$.

It was shown in [1] that conditions (A2), (A3) and (A4) in the definition of a proximity function can be replaced by the single condition

(A5) $$\alpha(A) \Delta [B] \Rightarrow \alpha(A) \Delta \alpha(B).$$ [4])

The following two propositions are concerned with the pair of conditions which in the case of skew proximity functions corresponds to (A5).

Proposition 5. *Any skew proximity function α on a set E satisfies the following two conditions:*

(D) $$\alpha(A) \Delta [B] \Rightarrow \alpha(A) \Delta \bar{\alpha}(B),$$

($\bar{\text{D}}$) $$\bar{\alpha}(A) \Delta [B] \Rightarrow \bar{\alpha}(A) \Delta \alpha(B).$$ [5])

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[3]) In order to give a „skew" generalization of proximity functions, the obvious thing to do would be to drop condition (A3) from Definition 3., and to define skew proximity functions by the remaining three conditions. Then, however, the proof of our Proposition 1. would break down at the place where we want to show that the intersection of two sets belonging to $\bar{\alpha}(A)$ belongs to $\bar{\alpha}(A)$.

[4]) $\mathfrak{a} \Delta \mathfrak{b}$ means that the filters $\mathfrak{a}$ and $\mathfrak{b}$ are incompatible, i. e. that together they generate the improper filter. Clearly, $\mathfrak{a} \Delta \mathfrak{b}$ iff some set from $\mathfrak{a}$ has void intersection with some set from $\mathfrak{b}$.

[5]) Instead of (D) and $(\bar{D})$ we could have written

$$\beta(A) \Delta [B] \Rightarrow \beta(A) \Delta \bar{\beta}(B) \quad \text{for} \quad \beta = \alpha, \bar{\alpha}.$$

PROOF. Let α be a skew proximity function and assume that $\alpha(A)\Delta[B]$, i. e. that $X\cap B=\emptyset$ for some $X\in\alpha(A)$. By (S3) we have $X\in\alpha(Y)$ and $Y\in\alpha(A)$ for some $Y\subseteqq E$. Now

$$\left.\begin{array}{l}X\in\alpha(Y)\Leftrightarrow E-Y\in\bar\alpha(E-X)\\ B\cap X=\emptyset\Rightarrow B\subseteqq E-X\Rightarrow\bar\alpha(B)\supseteqq\bar\alpha(E-X)\end{array}\right\}\Rightarrow E-Y\in\bar\alpha(B).$$

So we have $Y\in\alpha(A)$ and $E-Y\in\bar\alpha(B)$, i. e. $\alpha(A)$ and $\bar\alpha(B)$ are incompatible. This establishes (D), and $\overline{(\mathrm{D})}$ now follows by Propositions 1. and 2.

Conditions (D) and $\overline{(\mathrm{D})}$ together are able to replace part of the conditions in Definition 1. This is the essential content of the following

Proposition 6. *Let α be a mapping from $\mathfrak{P}(E)$ into $\Phi(E)$ satisfying conditions*

(S0) $$\emptyset\in\alpha(\emptyset);$$

(S1) $$B\in\alpha(A)\Rightarrow B\supseteqq A;$$

(S2′) $$\alpha(A)\cap\alpha(B)\subseteqq\alpha(A\cup B).$$

Let moreover the mapping $\bar\alpha\colon\mathfrak{P}(E)\to\mathfrak{P}[\mathfrak{P}(E)]$ be defined by condition

(C) $$B\in\bar\alpha(A)\Leftrightarrow E-A\in\alpha(E-B).$$

If conditions

(D) $$\alpha(A)\Delta[B]\Rightarrow\alpha(A)\Delta\bar\alpha(B),$$

and

$\overline{(\mathrm{D})}$ $$\bar\alpha(A)\Delta[B]\Rightarrow\bar\alpha(A)\Delta\alpha(B)$$

are satisfied then α and $\bar\alpha$ are skew proximity functions conjugate to each other.

PROOF. It will be sufficient to show that α satisfies (S2) and (S3).

First we establish Lemma 1.:

(L) $$A\subseteqq B\Rightarrow\alpha(A)\supseteqq\alpha(B).$$

If $A\subseteqq B$ and $X\in\alpha(B)$ then

$$\alpha(B)\Delta[E-X]\Rightarrow\alpha(B)\Delta\bar\alpha(E-X),$$

and so there exist disjoint sets $Y\in\alpha(B)$ and $Z\in\bar\alpha(E-X)$. From $Y\supseteqq B\supseteqq A$ one has

$$A\cap Z=\emptyset\Rightarrow\bar\alpha(E-X)\Delta[A]\Rightarrow\bar\alpha(E-X)\Delta\alpha(A).$$

Thus there exist disjoint sets $V\in\alpha(A)$ and $W\in\bar\alpha(E-X)$. We see that

$$\left.\begin{array}{l}V\in\alpha(A)\\ V\subseteqq E-W\end{array}\right\}\Rightarrow E-W\in\alpha(A),$$

while on the other hand [6])

$$W\in\bar\alpha(E-X)\Rightarrow W\supseteqq E-X\Rightarrow E-W\subseteqq X.$$

We get $X\in\alpha(A)$, and this establishes (L).

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[6]) Of course, (S1) and (C) together imply

$\overline{(\mathrm{S1})}$ $$B\in\bar\alpha(A)\Rightarrow B\supseteqq A.$$

By what we have just proved $\alpha(A\cup B)\subseteqq\alpha(A)\cap\alpha(B)$, and this, together with (S2') yields (S2).

In order to establish (S3) let $B\in\alpha(A)$, i. e. $\alpha(A)\Delta[E-B]$. Then by (D) $\alpha(A)\Delta\bar{\alpha}(E-B)$, i. e. there exist disjoint sets $X\in\alpha(A)$ and $Y\in\bar{\alpha}(E-B)$. We see that $Y\in\bar{\alpha}(E-B)\Rightarrow B\in\alpha(E-Y)$, and

$$X\cap Y=\emptyset\Rightarrow \left.\begin{array}{r}X\subseteqq E-Y\\ X\in\alpha(A)\end{array}\right\}\Rightarrow E-Y\in\alpha(A).$$

Thus (S3) holds and the proof of the proposition is complete.

§ 3. Skew proximity functions and topogenous structures

We have shown in an earlier paper that proximity functions are equivalent to symmetrical topogenous structures (see [3], Theorem 1.). This result leads one to guess that skew proximity functions will turn out to be equivalent to topogenous structures, and our next aim is to prove the correctness of this guess.

Let us first call in mind the definition of a topogenous structure (see [2]; cf. also [3], Definition 1.):

Definition 4. *A topogenous structure on a set E is a relation $<$ defined on $\mathfrak{P}(E)$, satisfying the following conditions*:

(01) $$\emptyset<\emptyset,\ E<E;$$

(02) $$A<B\Rightarrow A\subseteqq B;$$

(03) $$A\subseteqq A'<B'\subseteqq B\Rightarrow A<B;$$

(Q') $$A<B\ \&\ A'<B'\Rightarrow A\cap A'<B\cap B';$$

(Q'') $$A<B\ \&\ A'<B'\Rightarrow A\cup A'<B\cup B';$$

(7. 9) $$A<B\Rightarrow(\exists C)A<C<B.$$

The set of all topogenous structures on E possesses a natural partial order, namely the one induced by set-theoretical inclusion in $\mathfrak{P}(E)\times\mathfrak{P}(E)$:

$$<\subseteqq<_1\Leftrightarrow A<B \text{ implies } A<_1B \text{ for } A,B\subseteqq E.$$

The equivalence above mentioned can now be stated in a more formal way as follows:

Theorem 1. (1) *If $<$ is a topogenous structure on E then the function $\alpha_<$ defined on the subsets of E by*

$$\alpha_<(A)=\{X\,|\,A<X\}$$

is a skew proximity function on E.

(2) *If α is a skew proximity function on E then the relation $<_\alpha$ defined for subsets of E by*

$$A<_\alpha B\Leftrightarrow B\in\alpha(A)$$

is a topogenous structure on E.

(3) *The mappings $<\to\alpha_<$ and $\alpha\to<_\alpha$ are one-to-one correspondences, inverse to each other, between the sets of all topogenous structures and all skew proximity functions on E which preserve the respective partial orders.*

PROOF. (1) For any $A \subseteq E$, $\alpha_<(A) = \{X | A < X\}$ is a filter. This is an easy consequence of (01), (03) and of (Q′).

Moreover, this filter $\alpha_<(A)$ has the properties (S0)—(S3).

(S0): $\emptyset \in \alpha_<(\emptyset)$ follows from $\emptyset < \emptyset$.

(S1): $B \in \alpha_<(A) \Rightarrow B \supseteq A$ follows from (02).

(S2): $\alpha_<(A \cup B) = \alpha_<(A) \cap \alpha_<(B)$.

As a matter of fact, making use of (03) we get

$$X \in \alpha_<(A \cup B) \Leftrightarrow A \cup B < X \begin{cases} \Rightarrow A < X \Leftrightarrow X \in \alpha_<(A) \\ \Rightarrow B < X \Leftrightarrow X \in \alpha_<(B) \end{cases} \Rightarrow X \in \alpha_<(A) \cap \alpha_<(B).$$

This proves $\alpha_<(A \cup B) \subseteq \alpha_<(A) \cap \alpha_<(B)$. On the other hand, we get by (Q″)

$$X \in \alpha_<(A) \cap \alpha_<(B) \begin{cases} \Rightarrow A < X \\ \Rightarrow B < X \end{cases} \Rightarrow A \cup B < X \Leftrightarrow X \in \alpha_<(A \cup B), \quad \text{i. e.}$$

$$\alpha_<(A) \cap \alpha_<(B) \subseteq \alpha_<(A \cup B).$$

(S3): $B \in \alpha_<(A) \Rightarrow (\exists C) B \in \alpha_<(C) \ \& \ C \in \alpha_<(A)$.
This is an immediate consequence of (7. 9).

(2) The relation $A <_\alpha B \Leftrightarrow B \in \alpha(A)$ satisfies all the conditions listed in Definition 4.

(01): (S0) and $E \in \alpha(E)$ yield $\emptyset <_\alpha \emptyset$ and $E <_\alpha E$.

(02): $A <_\alpha B \Rightarrow A \subseteq B$ follows from (S1).

(03): $A \subseteq A' <_\alpha B' \subseteq B \Rightarrow A <_\alpha B$,

because we have the implication [7])

$$B \supseteq B' \in \alpha(A') \subseteq \alpha(A) \Rightarrow B \in \alpha(A).$$

(Q′): $$\left.\begin{array}{l} A <_\alpha B \Leftrightarrow B \in \alpha(A) \subseteq \alpha(A \cap A') \\ A' <_\alpha B' \Leftrightarrow B' \in \alpha(A') \subseteq \alpha(A \cap A') \end{array}\right\} \Rightarrow B \cap B' \in \alpha(A \cap A') \Leftrightarrow A \cap A' <_\alpha B \cap B'.$$

(Q″): $$\left.\begin{array}{l} A <_\alpha B \Leftrightarrow B \in \alpha(A) \\ A' <_\alpha B' \Leftrightarrow B' \in \alpha(A') \end{array}\right\} \Rightarrow B \cup B' \in \alpha(A) \cap \alpha(A') = \alpha(A \cup A') \Leftrightarrow$$
$$\Leftrightarrow A \cup A' <_\alpha B \cup B'.$$

(7. 9) $$A <_\alpha B \Rightarrow (\exists C) A <_\alpha C <_\alpha B.$$

This is an immediate consequence of (S3).

(3) The proof given in [3] fort Part (3) of Theorem 1. remains valid without change.

In [2] the complement $<^c$ of a semi-topogenous order $<$ was defined by $A <^c B \Leftrightarrow E - B < E - A$.

[7]) $\alpha(A') \subseteq \alpha(A)$ follows by Lemma 1.

If $<$ is a topogenous structure then so is $<^c$. This can be proved directly on the basis of Definition 4., but it is also an obvious consequence of the following

Proposition 7. *For any skew proximity function α on a set E the equality*

$$<_{\bar{\alpha}} = (<_\alpha)^c$$

holds.

PROOF.

$$\left.\begin{array}{l} A <_{\bar{\alpha}} B \Leftrightarrow B \in \bar{\alpha}(A) \quad\quad \Leftrightarrow \\ A(<_\alpha)^c B \Leftrightarrow E-B <_\alpha E-A \Leftrightarrow \end{array}\right\} E-A \in \alpha(E-B).$$

§ 4. Skew proximity relations

As is known from [1], proximity functions are equivalent to proximity relations — a fact to which they seem to owe their name. We are now going to give a similar characterization of skew proximity functions, with the help of "skew proximity relations" suitably defined:

Definition 5. *A skew proximity relation on a set E is a relation δ defined on $\mathfrak{P}(E)$, satisfying the following conditions (with $\bar{\delta}$ denoting the negation of δ):*

(P1) $$\begin{cases} A \cup B \bar{\delta} C \Leftrightarrow A \bar{\delta} C \;\&\; B \bar{\delta} C, \\ C \bar{\delta} A \cup B \Leftrightarrow C \bar{\delta} A \;\&\; C \bar{\delta} B; \end{cases}$$

(P2) *$x \delta x$ for $x \in E$;*

(P3) *$A \bar{\delta} \emptyset$ and $\emptyset \bar{\delta} A$ for $A \subseteq E$;*

(P4) *If $A \bar{\delta} B$ then there exist disjoint sets X and Y such that $A \bar{\delta} E-X$ and $E-Y \bar{\delta} B$.*

The set of all skew proximity relations on E possesses a natural partial order, namely the one induced by set-theoretical inclusion in $\mathfrak{P}(E) \times \mathfrak{P}(E)$:

$$\delta \subseteq \delta_1 \Leftrightarrow A \delta B \text{ implies } A \delta_1 B \text{ for } A, B \subseteq E.$$

The following two propositions will be needed in the sequel:

Proposition 8. *For any skew proximity relation δ on a set E*

$$(A \bar{\delta} B \;\&\; A_1 \subseteq A \;\&\; B_1 \subseteq B) \Rightarrow A_1 \bar{\delta} B_1.$$

PROOF.

$$A \bar{\delta} B \Leftrightarrow A_1 \cup A \bar{\delta} B \Rightarrow A_1 \bar{\delta} B \Leftrightarrow A_1 \bar{\delta} B \cup B_1 \Rightarrow A_1 \bar{\delta} B_1.$$

Proposition 9.

$$(A \delta B \;\&\; A \subseteq A_1 \;\&\; B \subseteq B_1) \Rightarrow A_1 \delta B_1.$$

Proof. $A_1 \bar{\delta} B_1$ would imply $A \bar{\delta} B$ by the preceding proposition.

The equivalence hinted at previously can now be stated and proved as follows:

Theorem 2. (1) *If α is a skew proximity function on E then the relation δ_α on $\mathfrak{P}(E)$ defined by*

$$A\delta_\alpha B \Leftrightarrow E-B\notin\alpha(A)$$

is a skew proximity relation on E.

(2) *If δ is a skew proximity relation on E then the mapping $\alpha_\delta\colon \mathfrak{P}(E)\to\mathfrak{P}[\mathfrak{P}(E)]$ defined by*

$$\alpha_\delta(A) = \{X \mid A\bar{\delta}E-X\}$$

is a skew proximity function on E.

(3) *The mappings $\alpha\to\delta_\alpha$ and $\delta\to\alpha_\delta$ are one-to-one correspondences, inverse to each other, between the sets of all skew proximity functions and all skew proximity relations on E which preserve the respective partial orders.*

Proof. (1) We must check the validity for δ_α of conditions (P1)—(P4) in Definition 5.

(P1): Each of the following conditions is equivalent to the next one:

$$A\cup B\,\bar{\delta}_\alpha C,$$

$$E-C\in\alpha(A\cup B) = \alpha(A)\cap\alpha(B),$$

$$E-C\in\alpha(A)\ \&\ E-C\in\alpha(B),$$

$$A\bar{\delta}_\alpha C\ \&\ B\bar{\delta}_\alpha C.$$

This establishes the first condition in (P1). As to the second condition:

$$C\bar{\delta}_\alpha A\cup B,$$

$$E-(A\cup B)\in\alpha(C),$$

$$(E-A)\cap(E-B)\in\alpha(C),$$

$$E-A\in\alpha(C)\,\&\,E-B\in\alpha(C),$$

$$C\bar{\delta}_\alpha A\ \&\ C\bar{\delta}_\alpha B.$$

(P2): $x\delta_\alpha x$ holds for $x\in E$, because $E-x\notin\alpha(x)$ by (S1).

(P3): $A\bar{\delta}_\alpha\emptyset \Leftrightarrow E\in\alpha(A)$ is clear, and $\emptyset\bar{\delta}_\alpha A \Leftrightarrow E-A\in\alpha(\emptyset)$ is true by (S0).

(P4): $A\bar{\delta}_\alpha B \Leftrightarrow E-B\in\alpha(A) \Rightarrow \alpha(A)\Delta[B] \Rightarrow \alpha(A)\Delta\bar{\alpha}(B)$.

The last condition means the existence of disjoint sets X and Y such that $X\in\alpha(A)$, $Y\in\bar{\alpha}(B) \Leftrightarrow E-B\in\alpha(E-Y)$, i. e. we have $A\bar{\delta}_\alpha E-X$ and $E-Y\ \bar{\delta}_\alpha B$.

(2) Let us show that α_δ satisfies the conditions of Definition 1.

$\alpha_\delta(A)$ is non-void for any $A\subseteq E$, since $E\in\alpha_\delta(A) \Leftrightarrow A\bar{\delta}\emptyset$ holds by (P3). Using Proposition 8. at the appropriate place, we get

$$\left.\begin{aligned} X\in\alpha_\delta(A) &\Leftrightarrow A\bar{\delta}E-X \\ Y\supseteq X &\Leftrightarrow E-Y\subseteq E-X \end{aligned}\right\} \Rightarrow A\bar{\delta}E-Y \Leftrightarrow Y\in\alpha_\delta(A).$$

Again, by (P1),

$$\left.\begin{aligned} X\in\alpha_\delta(A) &\Leftrightarrow A\bar\delta E-X \\ Y\in\alpha_\delta(A) &\Leftrightarrow A\bar\delta E-Y \end{aligned}\right\} \Rightarrow A\bar\delta(E-X)\cup(E-Y) \Leftrightarrow A\bar\delta E-(X\cap Y) \Leftrightarrow X\cap Y\in\alpha_\delta(A).$$

Hence $\alpha_\delta(A)$ is a filter.

(S0): $\emptyset\in\alpha_\delta(\emptyset) \Leftrightarrow \emptyset\bar\delta E$ holds by (P3).

(S1): $B\in\alpha_\delta(A) \Rightarrow B\supseteq A$.

Let indeed be $B\in\alpha_\delta(A) \Leftrightarrow A\bar\delta E-B$ and suppose $A\cap(E-B)\neq\emptyset$. Then

$$x\in A\cap(E-B)$$

for some $x\in E$, and by Proposition 10. $x\delta x \Rightarrow A\delta E-B$. This contradiction proves

$$A\cap(E-B)=\emptyset \Leftrightarrow B\supseteq A.$$

(S2): $\alpha_\delta(A\cup B) = \alpha_\delta(A)\cap\alpha_\delta(B)$.

Indeed,

$$X\in\alpha_\delta(A\cup B) \Leftrightarrow A\cup B\bar\delta E-X \Leftrightarrow$$

$$\Leftrightarrow A\bar\delta E-X \,\&\, B\bar\delta E-X \Leftrightarrow X\in\alpha_\delta(A)\,\&\,X\in\alpha_\delta(B).$$

(S3): $B\in\alpha_\delta(A) \Rightarrow (\exists C)\,B\in\alpha_\delta(C)\,\&\,C\in\alpha_\delta(A)$.

Let $B\in\alpha_\delta(A) \Leftrightarrow A\bar\delta E-B$. Then by (P4) there exist disjoint sets X and Y such that $A\bar\delta E-X$ and $E-Y\bar\delta E-B$, i. e. that $X\in\alpha_\delta(A)$ and $B\in\alpha_\delta(E-Y)$. Now $X\cap Y=\emptyset \Rightarrow$ $\Rightarrow X\subseteq E-Y$, so we have $E-Y\in\alpha_\delta(A)$ and (S3) holds with $C=E-Y$.

(3) The mappings $\alpha\to\delta_\alpha$ and $\delta\to\alpha_\delta$ being realized by the same "transition formulae" as in Proposition 7. of [1], the proof of Part (3) of that proposition applies without change. [8])

To a given skew proximity relation δ_α there corresponds in a natural way a "conjugate", namely the relation $\delta_{\bar\alpha}$, belonging to the conjugate skew proximity function. Between δ_α and $\delta_{\bar\alpha}$ a very simple connection exists:

Proposition 11. *For any skew proximity function α on a set E, $A\delta_{\bar\alpha}B \Leftrightarrow B\delta_\alpha A$.*

Proof.

$$A\bar\delta_{\bar\alpha}B \Leftrightarrow E-B\in\bar\alpha(A) \Leftrightarrow E-A\in\alpha(B) \Leftrightarrow B\bar\delta_\alpha A.$$

§ 5. An example

All our previous considerations derived their justification from the implicit assumption that the "skew" concepts introduced (i. e. those of skew proximity function, topogenous structure and skew proximity relation) are in fact different

[8]) Strictly speaking, the transition formulae of [1] do differ from ours inasmuch as they contain $E-X\bar\delta A$ where we have $A\bar\delta E-X$. This difference is however inessential, since δ in [1] is commutative: $A\delta B\Rightarrow B\delta A$.

from the corresponding "symmetrical" ones (proximity function, symmetrical topogenous structure and proximity relation respectively).

For completeness sake, let us discuss now a simple example proving the correctness of this assumption:

Let E be the real line, and for $A, B \subseteq E$ put

$$A\bar{\delta}B \Leftrightarrow (x \in A \,\&\, y \in B) \to x < y.$$

The relation δ so defined satisfies the conditions of Definition 5. For (P1), (P2) and (P3) this is immediately clear, while in order to see the validity of (P4) we can put $X = \{x | x \leqq a \text{ for some } a \in A\}$, and $Y = E - X$.

Thus δ is a skew proximity relation, but it is certainly not a proximity relation, because for non-void sets A and B, $A\bar{\delta}B$ and $B\bar{\delta}A$ are mutually exclusive conditions.

To the relation δ just defined there corresponds by Theorem 2. a skew proximity function α_δ, and then by Theorem 1. a topogenous structure $<_{\alpha_\delta}$. These yield examples of a skew proximity function which is not a proximity function, and of a topogenous structure which is not a symmetrical topogenous structure respectively.

As a matter of fact, the "transition formulae" used in our Theorem 2. are the same as those in Proposition 7. of [1]. So, if the skew proximity function α_δ were a proximity function, Proposition 7. of [1] would imply that δ is a proximity relation, in contradiction to the definition of δ.

Theorem 1. of the present note and Theorem 1. of [3] yield a similar conclusion for the topogenous structure $<_{\alpha_\delta}$.

Of course, it is also possible to check the properties of α_δ and of $<_{\alpha_\delta}$ directly, on the basis of their explicit characterizations:

$$\alpha_\delta(A) = \{X | A\bar{\delta}E - X\} = \{X | (x \in A \,\&\, y \in E - X) \to x < y\},$$

and

$$A <_{\alpha_\delta} B \Leftrightarrow B \in \alpha_\delta(A),$$

i. e.

$$A <_{\alpha_\delta} B \Leftrightarrow (x \in A \,\&\, y \in E - B) \to x < y.$$

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