On subgroups of primary abelian groups

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Problem 2 in [2] is the characterization of those subgroups of a divisible group which are intersections of divisible subgroups. B. CHARLES [1] and others have solved the problem. As one might expect, a slight modification of Charles' argument yields a characterization of those subgroups of an arbitrary abelian group which are intersections of neat subgroups — in particular, a proper subgroup H of a p-group G is the intersection of a family of neat subgroups of G if and only if $H[p]$ is a proper subgroup of $G[p]$. It is natural then that FUCHS (see [3] and [4]) raised the problem of characterizing those subgroups of an abelian group which are intersections of pure subgroups. This paper is devoted to the solution of this problem for the case of primary groups. Indeed, all groups in this paper will be assumed to be p-primary abelian groups. The notation and terminology of [2] will be followed. The result we wish to prove is the following:

Theorem. *If G is a p-group and H is a subgroup of G, then H is the intersection of a family of pure subgroups of G if and only if for each non-negative integer n, $(p^nG)[p]\subseteqq H$ implies $p^nG\subseteqq H$.*

Since every pure subgroup of G which contains $(p^nG)[p]$ must also contain p^nG, the necessity of the condition is obvious. To prove the sufficiency, we shall require the following lemmas, the proof of the first one of which is obvious.

Lemma 1. *If $[H_\lambda]$ is a family of subgroups of G each member of which contains the subgroup H and if for each λ and each $x\in H_\lambda$ such that $x\notin H$ and $px\in H$ there is a λ' such that $x\notin H_{\lambda'}$, then $H=\bigcap H_\lambda$.*

Lemma 2. *If A is a non-zero absolute direct summand of G and if H is a subgroup of G such that $H\cap A=0$, then H is the intersection of all the complementary summands of A which contain H.*

PROOF. Suppose that $G=K+A$ with $H\subseteqq K$. If $x\in K$, $x\notin H$ and $px\in H$, let $H'=\{H, x+z\}$ where z is a non-zero element of $A[p]$. Then $H'\cap A=0$ and if $K'\supseteqq H'$ is maximal in G with respect to intersecting A trivially, we have $G=K'+A$ with $x\notin K'$.

Lemma 3. *If H is a subgroup of G such that $H[p]$ is proper and dense (relative to the p-adic topology on G) in $G[p]$, then H is the intersection of all pure subgroups of G which contain H and have $H[p]$ as socle.*

PROOF. Let K be a pure subgroup of G such that $K[p]=H[p]$. Suppose $x \in K$, $x \notin H$ and $px \in H$. Set $H' = \{H, x+z\}$ where z is an element of $G[p]$ not in $H[p]$. Then $H'[p]=H[p]$ and if $K' \supseteqq H'$ is maximal in G with respect to having $H[p]$ as its socle, K' will be pure (see [5]) in G and $x \notin K'$.

We now turn to the proof of the sufficiency of the condition: $(p^n G)[p] \subseteqq H$ implies $p^n G \subseteqq H$. Suppose indeed that $p^n G \subseteqq H$ for some $n \geqq 1$. We may assume that $(p^{n-1}G)[p] \nsubseteqq H$. Therefore there is an element a in G of order p^n such that $\{a\} \cap H = 0$. $\{a+p^n G\}$ is then an absolute direct summand of $G/p^n G$ and, by Lemma 2, $H/p^n G$ is the intersection of all the complementary summands $K/p^n G$ with $H \subseteqq K$. Hence H itself is the intersection of the K's, all of which are readily seen to be pure in G.

Next suppose that $(p^n G)[p] \nsubseteqq H$ for all n. We distinguish two cases: (i) the reduced part of G is bounded and (ii) the reduced part of G is unbounded. In the first case, G has a non-zero absolute direct summand which intersects H trivially and we may apply Lemma 2. Suppose however that the reduced part of G is unbounded. Then let $A+B$ be a basic subgroup of G such that $A[p]$ is dense in $H[p]$. It is then easy to see that $H \cap B = 0$ and that we may assume that $H[p]+B[p]$ is proper in $G[p]$ — indeed, if it is not proper, then B is unbounded and may be replaced by a proper basic subgroup of itself. By Lemma 3, $H+B[p]$ is therefore the intersection of pure subgroups of G. Clearly then to complete the proof we need only show, if $x \in H+B[p]$ and $x \notin H$, that we can find a B' such that $A+B'$ is basic in G and $H[p]+B'[p]$ is a proper subgroup of $G[p]$ with $x \notin H+B'[p]$. This is easily done. Indeed, if $B[p] = \{b\}+Q$, where b is the $B[p]$-component of x, then we may take $B'[p] = \{b+z\}+Q$, where z is an element in $G[p]$ but not in $H+B[p]$ having height greater than that of b.

References

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