A note on translative mappings*)

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In [1] M. HOSSZÚ has defined that a system of mappings $F_u: x \to F_u(x)$ of a set X into itself, where u ranges over elements of a set U is called *commutable* if

(1) $$\forall u, v \in U, x \in X, F_u F_v(x) = F_v F_u(x)$$

holds. A system of mappings $F_u: x \to F_u(x)$ is *transitive* if

$$\forall x \in X, F_U(x) = X$$

holds, where the set of elements $F_u(x)$ $(u \in U)$ is denoted by $F_U(x)$. We, in this paper, define that if X is a groupoid with respect to operation of multiplication then a system of mappings $F_u: x \to F_u(x)$ of X into X is called *right translative* or *left translative* or *translative* respectively according as

(2) $$\forall x, y \in X, u \in U$$

$$F_u(xy) = x F_u(y)$$

or

$$F_u(xy) = F_u(x) y$$

or

$$F_u(xy) = F_u(x) y = x F_u(y)$$

holds.

Theorem 1. *A given mapping F_u of a translative system of mappings over a semigroup X is $1-1$ if the (rt. or lt.) cancellation law holds with respect to every element of $F_u(X)$.*

PROOF. We shall prove the theorem supposing that the right cancellation law holds w. r. to every element of $F_u(X)$. Let for $x, y \in X$

$$F_u(x) = F_u(y).$$

Then

$$F_u(xy) = F_u(x) y = F_u(y) y = F_u(y^2).$$

Also

$$F_u(xy) = x F_u(y) = x F_u(x) = F_u(x^2).$$

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Hence,

$$F_u(x^2) = F_u(y^2) \Rightarrow xF_u(x) = yF_u(y) \Rightarrow xF_u(x) = yF_u(y) \Rightarrow x = y$$

This completes the proof.

Theorem 2. *If X is a semigroup such that $yX = X$ for some $y \in X$ then:*

(a) *Every mapping F_u of a transitive system of translative mappings over X maps X onto itself.*

(b) *Every translative mapping F_u of X into X is $1-1$ if left cancellation w. r. to $F_u(y)$ holds.*

PROOF. (a) We have

$$F_u(X) = F_u(F_U(x)) = F_u(y\,F_U(x)) = F_u(F_U(y)x) =$$
$$= F_U(y)\,F_u(x) = F_U(y\,F_u(x)) = X.$$

(b) Let $x_1, y_1 \in X$. We have

$$F_u(x_1) = F_u(y_1) \Rightarrow F_u(yx') = F_u(yx'') \text{ where } x', x'' \in X \Rightarrow$$
$$\Rightarrow F_u(y)x' = F_u(y)x'' \Rightarrow x' = x'' \Rightarrow yx' = yx'' \Rightarrow x_1 = y_1.$$

Hence the proof is complete.

Theorem 3. *If X is a semigroup with identity e then:*

(a) *Every system of translative mappings over X is commutable.*

(b) *If there exists a transitive system of translative mappings over X then X i abelian.*

(c) *The system of all translative mappings over X is an abelian semigroup. If there exists a translative mapping F_{u_0} such that $F_{u_0}(e) = e$ then it coincides with the identity mapping.*

PROOF. (a) $\forall x \in X, u, v \in U,$

$$F_uF_v(x) = F_uF_v(ex) = F_u(F_v(e)x) =$$
$$= F_v(e)F_u(x) = F_v(e\,F_u(x)) = F_vF_u(x).$$

(b) Let $x, y \in X$ be arbitrary, then if $u \in U$ s. t.

$$F_u(e) = y$$

(such a u always exists by supposition of transitivity), then we have

$$F_u(x) = F_u(ex) = F_u(xe) \Rightarrow F_u(e)x = x\,F_u(e) \Rightarrow yx = xy$$

(c) $\forall x, y \in X, \quad u, v \in U$

$$F_uF_v(xy) = F_u(F_v(x)y) = (F_uF_v(x))y.$$

Also,

$$F_uF_v(xy) = F_u(x\,F_v(y)) = x(F_uF_v(y)).$$

Hence F_uF_v is a translative mapping over X.

Thus, in view of (a), the system of all translative mappings over X is an abelian semigroup.

Finally, for any $x \in X$,

$$F_{u_0}(x) = F_{u_0}(ex) = F_{u_0}(e)x = ex = x.$$

Hence F_{u_0} is the identity mapping for elements of X.

Thus the proof is complete.

Corollary 1. If the system of all translative mappings over a semigroup X with identity is transitive, it is an abelian semigroup with identity where the identity is the identity mapping.

Remark. We can very easily varify, as in the above theorem, that if the system of all (rt. or lt.) translative mappings over X is transitive, it is a semigroup with identity.

Theorem 4. *Every (rt. or lt.) translative mapping over a group G is a (rt. or lt.) multiplication mapping determined by an element of G and conversely.*

PROOF. If F_u is a left translative mapping over G, then for any element $g \in G$, we have

$$F_u(g) = F_u(eg) = F_u(e)g = g_1 g$$

where e is the identity in G and $F_u(e) = g_1$.

Hence F_u is the left multiplication of G determined by g. Similarly, we can see that a rt. translative mapping over G is a rt. multiplication of G. The converse is trivial.

Hence the theorem is proved.

We note that in a group G all rt. or lt. multiplications form a transitive system of rt. or lt. translative mappings over G respectively, and hence, in an abelian group, they give rise to a transitive system of translative mappings. Further, the theorem in [1] yields the following important consequence in view of result (a) of theorem 3:

Theorem 5. *Every transitive system of translative mappings over a multiplicative semigroup X with identity has the form*

$$x \to F_u(x) = x + \Phi(u),$$

where $+$ is an abelian group operation on X and $\Phi\colon U \to X$ is a mapping of U onto X.

We can easily verify that a translative mapping F_u over a semigroup X satisfies the following properties:

1. $\forall x \in X,\ F_u F_u(x^2) = F_u(x) F_u(x)$.
2. For $x, y \in X$, $F_u(x) = F_u(y)$ implies $F_u(x^i) = F_u(y^i)$ for all positive integers i.

This study is useful in connection with the study of commutable mappings over a semigroup with identity and properties of mappings in algebraic structures.

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References

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