Publ. Math. Debrecen
48 / 3-4 (1996), 327–338

Jordan $*$-derivations with respect to the Jordan product

By PÉTER BATTYÁNYI (Debrecen)

Abstract. In this note we give a description of Jordan $*$-derivations on standard operator algebras with respect to the Jordan product defined by $A \circ B = \frac{1}{2}(AB + BA)$. That is, we characterize the additive solutions of the functional equation $E(T \circ T) = T \circ E(T) + E(T) \circ T^*$ $(T \in \mathcal{A})$, where $\mathcal{A} \subset \mathcal{B}(H)$ is a standard operator algebra.

An additive mapping J on a $*$-ring $\mathcal{R}$ is called a Jordan $*$-derivation if it satisfies

$$J(a^2) = aJ(a) + J(a)a^* \qquad (a \in \mathcal{R}).$$

The theory of Jordan $*$-derivations originates from the result that the structure of these mappings is in firm connection with the problem of the representability of quadratic forms by sesquilinear ones on modules over $*$-rings (cf.[9,11]). Though the notion of Jordan $*$-derivations is relatively new, there are already a considerable number of results concerned with these mappings (see e.g. [1–5,7,8]). The result which is in the closest connection with the subject of this note is that of Šemrl [10], who proved that on standard operator algebras Jordan $*$-derivations are of the form

$$J(T) = TA - AT^* \qquad (T \in \mathcal{B}(H))$$

for some operator $A \in \mathcal{B}(H)$.

In the last decades the importance of non-associative rings and algebras has been growing rapidly (cf. [6]). Owing to this fact it seems natural to examine Jordan $*$-derivations of these kind of structures. Zalar [12] was the first to begin investigations in this direction discussing Jordan

Mathematics Subject Classification: Primary 47B47, 39B42.
Key words and phrases: Jordan $*$-derivation, Jordan product.

$*$-derivations on Cayley-Dickson algebra. In the present paper we are going to deal with Jordan $*$-derivations on operator algebras with respect to the well-known Jordan product $A \circ B = \frac{1}{2}(AB + BA)$ $(A, B \in \mathcal{B}(H))$, which operation plays a fundamental role in many areas of functional analysis and algebra. Similarly to the above statement of Šemrl, we give a full description of Jordan $*$-derivations on standard operator algebras with respect to the commutative structure of $\mathcal{B}(H)$ induced by the Jordan product.

Since the function in the equation of the following theorem is an operator function, the methods applied in the proof are partly different from those used in the theory of functional equations concerning real or complex functions.

Theorem. *Let H be a real or complex Hilbert space with $\dim H \geq 3$. Let us suppose that $\mathcal{A} \subset \mathcal{B}(H)$ is a standard operator algebra and $E : \mathcal{A} \to \mathcal{B}(H)$ is an additive function. Then E satisfies the equality*

$$E(T \circ T) = T \circ E(T) + E(T) \circ T^* \qquad (T \in \mathcal{A})$$

if and only if there is a scalar $\lambda \in \mathbb{K}$ such that

$$E(T) = \lambda(T - T^*) \qquad (T \in \mathcal{A}).$$

Proof. Let H be a Hilbert space of dimension at least three over the field $\mathbb{K}$, where $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$ respectively and, moreover let $\mathcal{A} \subset \mathcal{B}(H)$ be a standard operator algebra. Suppose that $E : \mathcal{A} \to \mathcal{B}(H)$ is a Jordan $*$-derivation with respect to the Jordan product. The defining equation of E concerning the usual operator product takes the following form

$$(1) \quad E(T^2) = \tfrac{1}{2}(E(T)T + TE(T) + T^*E(T) + E(T)T^*) \quad (T \in \mathcal{B}(H)).$$

Substituting $T + S$ into equation (1) and subtracting from the result the equalities obtained by writing T and S into (1) we have

$$(2) \quad \begin{aligned} E(TS + ST) &= \tfrac{1}{2}(E(T)S + SE(T) + E(T)S^* + S^*E(T)) \\ &\quad + \tfrac{1}{2}(E(S)T + TE(S) + E(S)T^* + T^*E(S)). \end{aligned}$$

In the first part of the proof we show that there exists a $\lambda \in \mathbb{K}$ such that $E(T) = \lambda(T - T^*)$ holds for every $T \in \mathcal{F}(H)$, where $\mathcal{F}(H) \subset \mathcal{B}(H)$ stands for the algebra of finite rank operators. After this the statement will be obtained easily.

In order to prove the theorem for finite rank operators, by the additivity of E it is enough to restrict our attention to rank-one operators.

Let a and b be orthogonal unit vectors. We are going to demonstrate that there exists a $\lambda_{ab} \in \mathbb{K}$ of which the value depends on the choice of the vectors a, b and for which $E(a \otimes b) = \lambda_{ab}(a \otimes b - b \otimes a)$ is valid. Let us prove first that there is a $\lambda_{ab} \in \mathbb{K}$ for which $E(a \otimes b)b = \lambda_{ab}a$ and $E(a \otimes b)a = -\lambda_{ab}b$ hold. Let c be a vector orthogonal to both a and b. Substitute $T = a \otimes b$ into equation (1)

$$\begin{aligned} 0 = E((a \otimes b)^2) = {} & \tfrac{1}{2}(a \otimes bE(a \otimes b) + E(a \otimes b)a \otimes b) \\ & + \tfrac{1}{2}(b \otimes aE(a \otimes b) + E(a \otimes b)b \otimes a). \end{aligned} \tag{3}$$

Take the operators on both sides of this equality at a and then forming inner product with c we have

$$\langle E(a \otimes b)b, c\rangle = 0. \tag{4}$$

Similarly, taking equation (3) at b and then forming inner product with c

$$\langle E(a \otimes b)a, c\rangle = 0. \tag{5}$$

Also, considering the values of both sides of (3) at a and then creating inner product with a we arrive at

$$0 = \langle E(a \otimes b)a, b\rangle + \langle E(a \otimes b)b, a\rangle,$$

that is,

$$\langle E(a \otimes b)a, b\rangle = -\langle E(a \otimes b)b, a\rangle. \tag{6}$$

Considering (2) with the substitutions $T = a \otimes c$, $S = c \otimes b$ we have

$$\begin{aligned} & E(a \otimes b) = E(a \otimes c \cdot c \otimes b + c \otimes b \cdot a \otimes c) \\ & = \tfrac{1}{2}(E(a \otimes c)c \otimes b + c \otimes bE(a \otimes c) + E(a \otimes c)b \otimes c + b \otimes cE(a \otimes c)) \\ & +\tfrac{1}{2}(E(c \otimes b)a \otimes c + a \otimes cE(c \otimes b) + E(c \otimes b)c \otimes a + c \otimes aE(c \otimes b)). \end{aligned} \tag{7}$$

Let us constitute the inner product of (7) at a with a. In this case we obtain

$$\langle E(a \otimes b)a, a\rangle = \tfrac{1}{2}(\langle E(c \otimes b)a, c\rangle + \langle E(c \otimes b)c, a\rangle). \tag{8}$$

Interchanging a and c in relation (5)

$$\langle E(c \otimes b)c, a\rangle = 0. \tag{9}$$

Substitute $c \otimes b$ for $a \otimes b$ in (3) and take the inner product of this equation at a with b. We obtain the equality

$$\langle E(c \otimes b)a, c\rangle = 0. \tag{10}$$

Thus, returning to (8), from (9) and (10) it follows that

$$\langle E(a\otimes b)a,a\rangle=0 \tag{11}$$

holds.

In a similar way we also have

$$\langle E(a\otimes b)b,b\rangle=0. \tag{12}$$

For this purpose consider the operators in (7) at b and form inner product with b, too.

$$\langle E(a\otimes b)b,b\rangle=\tfrac{1}{2}(\langle E(a\otimes c)c,b\rangle+\langle E(a\otimes c)b,c\rangle) \tag{13}$$

If we write b into (4) in place of c and the other way round, we arrive at

$$\langle E(a\otimes c)c,b\rangle=0. \tag{14}$$

Replacing b with c in (3), taking this equation at b then forming inner product with a we have

$$\langle E(a\otimes c)b,c\rangle=0. \tag{15}$$

Hence, on the account of (14) and (15), $\langle E(a\otimes b)b,b\rangle=0$ is valid in (13). This implies, together with (4), (5), (6) and (11), (12) that $E(a\otimes b)b=\lambda_{ab}a$ and $E(a\otimes b)a=-\lambda_{ab}b$ hold for some $\lambda_{ab}\in\mathbb{K}$.

Let us prove now that if a, b and c are arbitrary, pairwise orthogonal unit vectors, then $E(a\otimes b)c=0$ holds true. Taking equality (7) at c and forming inner products with a then with b we obtain the equations

$$\langle E(a\otimes b)c,a\rangle=\tfrac{1}{2}(\langle E(a\otimes c)b,a\rangle+\langle E(c\otimes b)a,a\rangle+\langle E(c\otimes b)c,c\rangle) \tag{16}$$

and

$$\langle E(a\otimes b)c,b\rangle=\tfrac{1}{2}(\langle E(a\otimes c)b,b\rangle+\langle E(a\otimes c)c,c\rangle+\langle E(c\otimes b)a,b\rangle). \tag{17}$$

Perform in (10) the following simultaneous substitutions: let us replace a with b, b with c and finally c with a. Hence we have

$$\langle E(a\otimes c)b,a\rangle=0. \tag{18}$$

Let us form the inner product of the value of (7) at c with c

$$\begin{aligned}\langle E(a\otimes b)c,c\rangle&=\tfrac{1}{2}(\langle E(a\otimes c)c,b\rangle+\langle E(a\otimes c)b,c\rangle)\\&\quad+\tfrac{1}{2}(\langle E(c\otimes b)a,c\rangle+\langle E(c\otimes b)c,a\rangle).\end{aligned} \tag{19}$$

Comparing the relations (9), (10) and (14), (15) we acquire

$$\langle E(a \otimes b)c, c\rangle = 0. \tag{20}$$

Changing the roles of a and b in the above line we arrive at

$$\langle E(c \otimes b)a, a\rangle = 0. \tag{21}$$

If we rewrite (11) with c instead of a we have the equation

$$\langle E(c \otimes b)c, c\rangle = 0. \tag{22}$$

Thus, turning back to (16), we can see that (18), (21) and (22) yield together

$$\langle E(a \otimes b)c, a\rangle = 0.$$

We can also obtain from (17) the result $\langle E(a \otimes b)c, b\rangle = 0$ in a similar fashion. Indeed, substitute in (20) c for b and vice versa. We then have

$$\langle E(a \otimes c)b, b\rangle = 0. \tag{23}$$

If we replace b by c in (12), we arrive at

$$\langle E(a \otimes c)c, c\rangle = 0. \tag{24}$$

Substitute c for a, b for c and a for b in (15) to obtain

$$\langle E(c \otimes b)a, b\rangle = 0. \tag{25}$$

By (23), (24) and (25) we can deduce that in (17)

$$\langle E(a \otimes b)c, b\rangle = 0$$

holds.

If H is at least four dimensional, in order to prove the relation $E(a \otimes b)c = 0$ we have to show $E(a \otimes b)c, d\rangle = 0$ as well, where d is an arbitrary vector being orthogonal to a, b and c alike. Prove first that $\langle E(a \otimes b)c, d\rangle = -\langle E(b \otimes a)c, d\rangle$ is true with a, b, c, d chosen as above.

Apply (2) to $T = c \otimes a + a \otimes c$ and $S = b \otimes c + c \otimes b$, and interchange a and b in (7). Taking the so obtained equations as well as (7) at c and then forming inner products with d we have the following equalities

$$\begin{aligned} &\langle E(a \otimes b + b \otimes a)c, d\rangle \\ &= \langle E(c \otimes a + a \otimes c)b, d\rangle + \langle E(b \otimes c + c \otimes b)a, d\rangle, \end{aligned} \tag{26}$$

$$\langle E(a \otimes b)c, d\rangle = \tfrac{1}{2}(\langle E(a \otimes c)b, d\rangle + \langle E(c \otimes b)a, d\rangle), \tag{27}$$

$$\langle E(b \otimes a)c, d\rangle = \tfrac{1}{2}(\langle E(c \otimes a)b, d\rangle + \langle E(b \otimes c)a, d\rangle). \tag{28}$$

Subtracting (26) from the duplicated sum of (27) and (28) we have

$$\langle E(a \otimes b + b \otimes a)c, d\rangle = 0. \tag{29}$$

After this let us show that $\langle E(a \otimes b)c, d\rangle = 0$ is valid. Let us replace in (27) a by b, b by c, c by a and eventually leave d fixed. Applying these kind of substitutions once more to the equation obtained in this way from (27) we arrive at the equalities

$$\langle E(a \otimes b)c, d\rangle = \tfrac{1}{2}(\langle E(a \otimes c)b, d\rangle + \langle E(c \otimes b)a, d\rangle), \tag{27}$$

$$\langle E(b \otimes c)a, d\rangle = \tfrac{1}{2}(\langle E(b \otimes a)c, d\rangle + \langle E(a \otimes c)b, d\rangle), \tag{30}$$

$$\langle E(c \otimes a)b, d\rangle = \tfrac{1}{2}(\langle E(c \otimes b)a, d\rangle + \langle E(b \otimes a)c, d\rangle). \tag{31}$$

Making use of (29) and the similar relations working for the operators $b \otimes c$ and $c \otimes a$, namely of the equalities

$$\langle E(b \otimes c)a, d\rangle = -\langle E(c \otimes b)a, d\rangle \quad \text{and} \quad \langle E(c \otimes a)b, d\rangle = -\langle E(a \otimes c)b, d\rangle,$$

we can solve the linear equation system constituted by (27), (30), (31), which results in

$$\langle E(a \otimes b)c, d\rangle = \langle E(b \otimes c)a, d\rangle = \langle E(c \otimes a)b, d\rangle = 0.$$

Summing up, we have proved the relations

$$\begin{aligned} &\langle E(a \otimes b)c, a\rangle = 0, \qquad &&\langle E(a \otimes b)c, b\rangle = 0,\\ &\langle E(a \otimes b)c, c\rangle = 0, \qquad &&\langle E(a \otimes b)c, d\rangle = 0, \end{aligned}$$

which yields that $E(a \otimes b)c = 0$ really holds for any unit vector $c \in \mathcal{M}(a, b)^{\perp}$, where $\mathcal{M}(a, b)$ denotes the subspace spanned by a and b.

It follows from what we have proved so far that for every pair of orthogonal unit vectors a,b there exists a $\lambda_{ab} \in \mathbb{K}$ such that

$$E(a \otimes b) = \lambda_{ab}(a \otimes b - b \otimes a) \tag{32}$$

is satisfied, where the value of λ_{ab} may depend on the choice of a and b.

In what follows we are going to prove that if a and b are unit vectors, $\lambda_{ab} \in \mathbb{K}$ is the scalar obtained in (32) and $\alpha \in \mathbb{K}$ is arbitrary, we have

$$E(\alpha T) = \lambda_{ab}(\alpha T - (\alpha T)^*), \tag{33}$$

where T denotes the operator $a \otimes a$ or $a \otimes b$, respectively.

Let us fix two orthogonal unit vectors $a, b \in H$, and let λ stand for the scalar $\lambda_{ab} \in \mathbb{K}$ corresponded to a and b by (32). We shall prove that for the underlying vectors a, b

$$E(a \otimes b) = -E(b \otimes a) \tag{34}$$

holds. For this purpose, by (32), it is enough to verify the equality

$$\langle E(a \otimes b)a, b\rangle = -\langle E(b \otimes a)a, b\rangle.$$

Let $c \perp a, b$ be a unit vector. Consider (2) again with the substitutions $T = c \otimes a + a \otimes c$ and $S = b \otimes c + c \otimes b$. Taking this equation, together with (7) and the equality obtained from (7) by interchanging a and b, and forming inner products with b we arrive at

$$\begin{aligned} &\langle E(a \otimes b + b \otimes a)a, b\rangle \\ &= \langle E(c \otimes a + a \otimes c)a, c\rangle + \langle E(b \otimes c + c \otimes b)c, b\rangle, \end{aligned} \tag{35}$$

$$\langle E(a \otimes b)a, b\rangle = \tfrac{1}{2}(\langle E(a \otimes c)a, c\rangle + \langle E(c \otimes b)c, b\rangle), \tag{36}$$

$$\langle E(b \otimes a)a, b\rangle = \tfrac{1}{2}(\langle E(c \otimes a)a, c\rangle + \langle E(b \otimes c)c, b\rangle). \tag{37}$$

Multiplying the sum of (36) and (37) by two and subtracting (35) from the result we arrive at

$$\langle E(a \otimes b + b \otimes a)a, b\rangle = 0,$$

which supplies relation (34), as needed.

Let us prove that

$$E(a \otimes a) = E(b \otimes b) = 0 \tag{38}$$

also holds. By the equations $E(a \otimes a + b \otimes b) = E((a \otimes b + b \otimes a)^2)$ it follows from (1), taking (34) into account, that

$$E(a \otimes a) + E(b \otimes b) = 0. \tag{39}$$

Let $c \perp a, b$ be a vector with norm one. In this case, similarly to (39), the equalities

$$\begin{aligned} E(a \otimes a) + E(b \otimes b) &= 0, \\ E(b \otimes b) + E(c \otimes c) &= 0, \\ E(c \otimes c) + E(a \otimes a) &= 0 \end{aligned}$$

are also valid, which yields (38).

At this point the proof is separated into two branches according as H is a real or a complex Hilbert space.

Supposing that H is real let us consider the self-adjoint, negative semi-definit operator

$$S = \alpha(a \otimes a + b \otimes b),$$

where $\alpha < 0$ is an arbitrary real number. If we define T as

$$T = \sqrt{(-\alpha)}(a \otimes b - b \otimes a),$$

then $T^* = -T$ and moreover $S = T^2$ holds true. Applying (1) to $S = T^2$ we have

$$E(S) = E(T^2) = 0. \tag{40}$$

Substituting $T = a \otimes a$, $S = \alpha(a \otimes a + b \otimes b)$ into (2), by (38) and (40) we arrive at

$$E(\alpha a \otimes a) = 0. \tag{41}$$

Furthermore equation (2) taken at $T = a \otimes b$, $S = \alpha b \otimes b$ together with (32) and (41) results in

$$\begin{aligned} E(\alpha a \otimes b) &= \alpha b \otimes b E(a \otimes b) + E(a \otimes b)\alpha b \otimes b \\ &= \lambda(\alpha a \otimes b - \alpha b \otimes a). \end{aligned} \tag{42}$$

Since E is additive, we have (33) for every $\alpha \in \mathbb{R}$.

Assume now that H is complex. Let us suppose that a and b are chosen as above. First of all determine $E(ia \otimes a)$. By (38) the following equality is true

$$\begin{aligned} 2E(ia \otimes a) &= E((ia \otimes a)a \otimes a + a \otimes a(ia \otimes a)) \\ &= a \otimes aE(ia \otimes a) + E(ia \otimes a)a \otimes a. \end{aligned}$$

Taking this equation at y, where $y \perp a$ is arbitrary, and then at a, we can deduce by forming inner products with appropriate vectors that

$$E(ia \otimes a) = \lambda_a(ia \otimes a - (ia \otimes a)^*) \tag{43}$$

holds for some $\lambda_a \in \mathbb{C}$, where λ_a may depend on the choice of a.

Applying (2) to $T = ia \otimes a$ and $S = -ia \otimes b$, taking into account that T is skew-symmetric and (43) holds true, we have

$$E(a \otimes b) = \lambda_a(a \otimes b - b \otimes a).$$

It follows from (32) that $\lambda_a = \lambda$. With this in hand, by the additivity of E, the relation

$$E(\alpha a \otimes a) = \lambda(\alpha a \otimes a - (\alpha a \otimes a)^*) \qquad (\alpha \in \mathbb{C}) \tag{44}$$

can be obtained easily. If we argue in the same way as in the case of (42), we also have

$$E(\alpha a \otimes b) = \lambda(\alpha a \otimes b - (\alpha a \otimes b)^*), \tag{45}$$

where $\alpha \in \mathbb{C}$ is arbitrary.

In what follows we can treat the real and complex cases jointly again. We claim that the scalar λ is in fact independent of the choice of a and b.

Prior to this, let us demonstrate that for a, b fixed as above and for a unit vector c chosen to be orthogonal to a and b

$$\lambda = \lambda_{ab} = \lambda_{ac} \tag{46}$$

holds. The course of the proof is analogous to that of the assertions from (26) to (31).

First of all, on account of (32) and (34) for any two orthogonal unit vectors d and e the relation

$$\langle E(d \otimes e)e, d\rangle = \langle E(e \otimes d)d, e\rangle \tag{47}$$

is valid.

Let us consider the inner product of the value of (7) at b with a. Change a for b, b for c and finally c for a in this equation. Carry out these substitutions one more time for the equation obtained in this way, we thus arrive at the equalities

$$\begin{aligned}
\langle E(a \otimes b)b, a\rangle &= \tfrac{1}{2}(\langle E(a \otimes c)c, a\rangle + \langle E(c \otimes b)b, c\rangle) \\
\langle E(b \otimes c)c, b\rangle &= \tfrac{1}{2}(\langle E(b \otimes a)a, b\rangle + \langle E(a \otimes c)c, a\rangle) \\
\langle E(c \otimes a)a, c\rangle &= \tfrac{1}{2}(\langle E(c \otimes b)b, c\rangle + \langle E(b \otimes a)a, b\rangle).
\end{aligned}$$

This linear equation system, taking into consideration (47), yields

$$\lambda_{ab} = \langle E(a \otimes b)b, a\rangle = \langle E(b \otimes c)c, b\rangle = \langle E(a \otimes c)c, a\rangle = \lambda_{ac},$$

as required.

Now we can prove the above claim asserting that λ is independent of a and b. Let c be a unit vector. There exist a unit vector $d \in \mathcal{M}(a, b)^\perp$ and scalars $\mu, \nu, \rho \in \mathbb{K}$ such that

$$c = \mu a + \nu b + \rho d.$$

Applying (33), (46) we obtain

$$\begin{aligned} (48) \qquad & E(a \otimes c) = E(\overline{\mu} a \otimes a + \overline{\nu} a \otimes b + \overline{\rho} a \otimes d) \\ &= \lambda_{ab}(\overline{\mu} a \otimes a - \mu a \otimes a) + \lambda_{ab}(\overline{\nu} a \otimes b - \nu b \otimes a) + \lambda_{ad}(\overline{\rho} a \otimes d - \rho d \otimes a) \\ &= \lambda_{ab}(a \otimes (\mu a + \nu b + \rho d) - (\mu a + \nu b + \rho d) \otimes a) = \lambda_{ab}(a \otimes c - c \otimes a). \end{aligned}$$

Hereupon, let c and d be orthogonal unit vectors. Substituting $T = c \otimes a$ and $S = a \otimes d$ into (2) and taking into account (34) and (48) we come to the conclusion that

$$E(c \otimes d) = \lambda(c \otimes d - d \otimes c) \tag{49}$$

holds true. Obviously, (33) is valid with respect to c and d also, hence there is a $\lambda \in \mathbb{K}$ such that $E(T) = \lambda(T - T^*)$ for every operator of the form $T = \alpha c \otimes d$ and $T = \alpha c \otimes c$, where $\alpha \in \mathbb{K}$ and $c, d \in H$ are orthogonal unit vectors.

In possession of these results we can prove the theorem for an arbitrary finite rank operator as well. Let $S \in \mathcal{F}(H)$ and denote by $\{e_1, \ldots, e_n\}$ an orthonormal basis of the subspace $\mathcal{M}(\operatorname{rng} S \cup \operatorname{rng} S^*)$. In this case

$$S = \sum_{i=1}^{n} e_i \otimes S^* e_i = \sum_{i=1}^{n} e_i \otimes \sum_{k=1}^{n} \alpha_k^i e_k = \sum_{i,k=1}^{n} \overline{\alpha}_k^i e_i \otimes e_k \tag{50}$$

holds. By the additivity of E the claim follows from the last term of (50) in considerations with the previous remark as well.

Eventually, let us define the function $F : \mathcal{A} \to \mathcal{B}(H)$ in the following way

$$F(T) = E(T) - \lambda(T - T^*) \qquad (T \in \mathcal{A}).$$

By what has been proved before, $F(T) = 0$ in case of $T \in \mathcal{F}(H)$. Let $T \in \mathcal{A}$ and assume that there exists $x \in H$ such that $F(T)x \neq 0$. Let $P \in \mathcal{F}(H)$ be the projection for which $\operatorname{rng} P = \mathcal{M}(x, Tx, T^*x, E(T)x)$. Since, as it can be seen easily, F is a Jordan $*$-derivation, then by (2) we obtain

$$0 = F(TP + PT)x = (F(T)P + PF(T))x = 2F(T)x \neq 0,$$

which is a contradiction.

To end up with, we would mention that the mappings $T \mapsto \lambda(T - T^*)$ $(\lambda \in \mathbb{K})$ clearly satisfy the equation in the statement of the theorem.

By this the proof is completed. $\square$

Remark. In what follows we are going to give a sketch of the cases $\dim H \leq 2$.

If H is a complex Hilbert space with $\dim H \leq 2$, then a considerably shorter computation than the above one, making use of the existence of the operator iI, leads to the result that the solutions of our equality are supplied by the functions

$$E(T) = \lambda(T - T^*) \qquad (T \in \mathcal{A}),$$

where $\lambda \in \mathbb{C}$ is an appropriate scalar.

The situation is completely different in the real case. If H is real and $\dim H = 2$, then the solutions of the equation are of the form

$$E(T) = TA - AT^* \qquad (T \in \mathcal{A}),$$

where A is a symmetric operator. The proof is based on an argument similar to that of the theorem. Considerations of space forbid us, however, to enter into the details of the proof.

In the end, in the case when H is real and $\dim H = 1$ by the commutativity the Jordan product coincides with the usual product and the operation of adjoining is now the identity. It is easy to see that in this case we have actually to consider the additive functions $E : \mathbb{R} \to \mathbb{R}$ which satisfy

$$E(x^2) = xE(x) + E(x)x \qquad (x \in \mathbb{R}).$$

Linearizing this equality, that is, substituting $x + y$ for x, we have

$$E(xy) = xE(y) + E(x)y \qquad (x, y \in \mathbb{R}).$$

Thus, as a solution, we have obtained the set of all additive derivations on $\mathbb{R}$. Since, as it is well-known, the only continuous one among these mappings is the identically zero function, in this case the solutions of the equality cannot be given explicitly as in the previous cases.

Acknowledgements. I would like to thank Prof. Lajos Molnár for suggesting this subject of investigation to me. I would like to express my thanks also to the referee, whose valuable idea helped me with making significant reductions in the proof.

References

[1] M. Brešar and B. Zalar, On the structure of Jordan $*$-derivations, *Colloquium Math.* **63** (1992), 163–171.
[2] L. Molnár, The range of a Jordan $*$-derivation, *Math. Japon.* (*to appear*).
[3] L. Molnár, On the range of a normal Jordan $*$-derivation, *Comment. Math. Univ. Carolinae* **35** (1994), 691–695.
[4] L. Molnár, Jordan $*$-derivation pairs on a complex $*$-algebra, (*preprint*).
[5] L. Molnár, A condition for a subspace of $\mathcal{B}(H)$ to be an ideal, *Linear Algebra and Appl.* (*to appear*).
[6] R. D. Schafer, An Introduction to Nonassociative Algebras, *Academic Press*, 1966.
[7] P. Šemrl, On Jordan $*$-derivations and an application, *Colloquium Math.* **59** (1990), 241–251.
[8] P. Šemrl, Quadratic functionals and Jordan $*$-derivations, *Studia Math.* **97** (1991), 157–165.
[9] P. Šemrl, Quadratic and quasi-quadratic functionals, *Proc. Amer. Math. Soc.* **119** (1993), 1105–1113.
[10] P. Šemrl, Jordan $*$-derivations of standard operator algebras, *Proc. Amer. Math. Soc.* **120** (1994), 515–518.
[11] B. Zalar, Jordan $*$-derivation pairs and quadratic functionals on modules over $*$-rings, (*preprint*).
[12] B. Zalar, Jordan $*$-derivations and quadratic functionals on octonion algebras, *Comm. Algebra* **22** (1994), 2845–2859.

PÉTER BATTYÁNYI
SZEGFŰ U. 5. III/10.
H–4027 DEBRECEN
HUNGARY

(Received May 5, 1995; revised November 14, 1995)