Inner automorphisms of universal algebras

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In this short note we introduce a notion of inner automorphism for universal algebras, which is a generalization of the corresponding notion referring to groups. This notion preserves some well-known properties of the usual one. Our terminology is essentially that of [1].

The following definition seems to be natural: a mapping is an inner automorphism if and only if it is an automorphism and a translation *) in the same time. Restricting, however, ourselves to groups, this notion is more general than the usual one; this is shown by the mapping $x \to 2x$ of the additive group of rationals. Further, we can have an automorphism, which is a translation, but its inverse is not a translation. Take, e. g., the set of natural numbers and a single unary operation on it, defined by the permutation $\varphi = (234) \dots (n^2+1, n^2+2, \dots, (n+1)^2) \dots$ The mapping φ is an automorphism and a translation, its inverse, however, is no translation. These deficiencies can be eliminated by defining inner automorphisms not for a single algebra, but for classes of algebras as follows:

Let $\mathfrak{A}$ be a primitive class ([1], p.114.) of universal algebras, let $A \in \mathfrak{A}$ and let α be an automorphism of A. Suppose we have in the class $\mathfrak{A}$ a principal derived operation (shortly operation, [1], p. 115.) μ depending on m variables with the properties:

I. There exist elements $a_2, \dots, a_m \in A$ such that for every $a \in A$ the equality $a\alpha = aa_2 \dots a_m\mu$ holds.

II. In each algebra B of $\mathfrak{A}$ the mapping $b \to bb_2 \dots b_m\mu$ $(b \in B)$ is for every choice of $b_2, \dots, b_m \in B$ an automorphism of B.

Such an automorphism α of A will be called an *inner automorphism*.

Examples. 1. For groups the introduced notion coincides with the usual notion of inner automorphism. Indeed, let G be a group, and suppose $g\alpha = h^{-1}gh$ for every $g \in G$, h being a fixed element of G. Then the operation $xy\mu = y^{-1}xy$ satisfies I, II. On the other hand, let β be an inner automorphism of G in the new sense. We must prove the existence of an element $h \in G$ such that for every $g \in G$ the equality $g\beta = h^{-1}gh$ holds. Set $g\beta = ga_2 \dots a_m\mu$ $(a_2, \dots, a_m \in G)$ with μ satisfying II too. We can suppose, without violating generality, that $xy_2 \dots y_m\mu = Y_0X_1Y_1 \dots X_nY_n$, where $X_i = x^{k_i}$ $(k_i \neq 0; i = 1, \dots, n)$, $Y_i = \prod_{j=1}^{t_i} y_{i_j}^{e_{i_j}}$ $(y_{i_j} \in \langle y_2, \dots, y_m \rangle;\ i = 0, \dots, n;$ $j = 1, \dots, t_i$; here $e_{i_j} \neq 0$, if $1 \leq i \leq n-1)$.

*) By translation we mean a derived operation with a single variable [1].

Now let $x, y_2, \ldots, y_m$ denote the distinct free generators of a free group F. Then according to II $\varphi: \xi \to \xi y_2 \ldots y_m \mu = Y_0 \Xi_1 Y_1 \ldots \Xi_n Y_n (\Xi_i = \xi^{k_i})$ is an automorphism of F. Especially, we have $x^{-1}\varphi = x^{-1} y_2 \ldots y_m \mu = Y_0 X_1^{-1} Y_1 \ldots X_n^{-1} Y_n = (x\varphi)^{-1} = (xy_2 \ldots y_m \mu)^{-1} = Y_n^{-1} X_n^{-1} \ldots Y_1^{-1} X_1^{-1} Y_0^{-1}$. Hence $Y_0 X_1^{-1} Y_1 \ldots X_n^{-1} Y_n Y_0 X_1 Y_1 \ldots \ldots X_n Y_n = 1$. Then we must have necessarily $Y_i = Y_{n-i}^{-1}$ $(i = 0, \ldots, n)$ and $X_i = X_{n-i+1}$ $(i = 1, \ldots, n)$. Moreover, $x^2\varphi = x^2 y_2 \ldots y_m \mu = Y_0 X_1^2 Y_1 \ldots X_n^2 Y_n = (x\varphi)^2 = (xy_2 \ldots y_m \mu)^2 = Y_0 X_1 Y_1 \ldots X_n Y_n Y_0 X_1 Y_1 \ldots X_n Y_n$. Since $Y_n Y_0 = 1$ and $X_1 X_n = x^{2k_1} \neq 1$, we may observe, that X_i (that is, a power of the element x) occurs on the right hand side $2n-1$ times, and n times on the left hand side. Therefore, $n = 2n-1$, whence $n = 1$. Hence $xy_2 \ldots y_m \mu = Y_1^{-1} X_1 Y_1$. By virtue of II, $\xi \to \xi 1 \ldots 1 \mu = \Xi_1 = \xi^{k_1}$ is also an automorphism of F; consequently, $k_1 = 1$, hence $\Xi_1 = \xi$ and thus $X_1 = x$. In this way,

$$xy_2 \ldots y_m \mu = \left(\prod_{j=1}^{t_1} y_{1_j}^{e_{1_j}} \right)^{-1} \cdot x \cdot \prod_{j=1}^{t_1} y_{1_j}^{e_{1_j}}. \tag{1}$$

Since F is a free group and $x, y_2, \ldots, y_m$ are its free generators, (1) is an identical equality for any group, thus for G too. Hence, for each $g \in G$ $g\beta = ga_2 \ldots a_m \mu = \left(\prod_{j=1}^{t_1} a_{1_j}^{e_{1_j}} \right)^{-1} \cdot g \cdot \prod_{j=1}^{t_1} a_{1_j}^{e_{1_j}}$, where $a_{1_j} = a_i$ $(j = 1, \ldots, t_1; i = 2, \ldots, m)$ if and only if $y_{1_j} = y_i$. We see, that the required h exists, namely $h = \prod_{j=1}^{t_1} a_{1_j}^{e_{1_j}}$.

2. In self-distributive quasigroups the right and left multiplications as well as the right and left cancellations are inner automorphisms [2].

3. In any vector space over a fixed skewfield S the multiplications with the elements of the centre of S are inner automorphisms.

Our observations concerning inner automorphisms are included in the following

Theorem. *The set $I(A)$ of all inner automorphisms of any algebra A in an arbitrary primitive class $\mathfrak{A}$ is a group. $I(A)$ is normal in the group of all automorphisms of A. If among the classes of a congruence of A there is only one subalgebra, then this latter is invariant under inner automorphisms.*

PROOF. Let α, β be inner automorphisms of A and denote by μ, ν the operations connected with these automorphisms. Let $a\alpha = aa_2 \ldots a_m \mu$, $a\beta = ab_2 \ldots b_n \nu$ $(a_2, \ldots, a_m, b_2, \ldots, b_n \in A)$ for each $a \in A$. Then one can see easily, that the mapping $\alpha\beta: a \to (aa_2 \ldots a_m \mu) b_2 \ldots b_n \nu$ is also an inner automorphism of A. We shall show, that α^{-1} is also an inner automorphism. Let M be a free algebra in the class $\mathfrak{A}$ with the free generators $x_1, \ldots, x_m$. According to II, the mapping $\bar{\alpha}: \xi \to \xi x_2 \ldots x_m \mu$ is an automorphism, hence there exists one and only one element $m \in M$ such that $m\bar{\alpha} = x_1$. Moreover, there exists in $\mathfrak{A}$ a principal derived operation $\bar{\mu}$, for which $m = x_1 \ldots x_m \bar{\mu}$. Thus we obtain

$$(x_1 \ldots x_m \bar{\mu}) x_2 \ldots x_m \mu = x_1. \tag{2}$$

Now let us consider the mapping $\alpha^*: a \to aa_2 \ldots a_m \bar{\mu}$ of A. Since (2) is an identical relation in $\mathfrak{A}$, we have $a(\alpha^*\alpha) = a$ for each $a \in A$, whence $\alpha^* = \alpha^{-1}$. Furthermore, we get similarly, that in any algebra B of $\mathfrak{A}$ for arbitrary $b_2, \ldots, b_m \in B$ the mapping $\beta^*: b \to bb_2 \ldots b_m \bar{\mu}$ is the inverse of the automorphism $b \to bb_2 \ldots b_m \mu$. Hence β^* is an automorphism. We have shown, that α^{-1} is also an inner automorphism. Thus $I(A)$ is a group.

Take now an arbitrary not necessarily inner automorphism φ of A and let $a \in A$. Then $a(\varphi^{-1}\alpha\varphi) = ((a\varphi^{-1})\alpha) = ((a\varphi^{-1})a_2 \dots a_m\mu)\varphi = a(a_2\varphi)\dots(a_m\varphi)\mu$, consequently $\varphi^{-1}\alpha\varphi$ is an inner automorphism of A. Therefore $I(A)$ is normal in the group of all automorphism of A.

To prove the third assertion of the Theorem it is sufficient to show, that any class of an arbitrary congruence θ of A maps onto an other class of θ under α. If $a \equiv a'(\theta)$, then $aa_2 \dots a_m\mu \equiv a'a_2 \dots a_m\mu(\theta)$, that is, $a\alpha \equiv a'\alpha(\theta)$. If, however, $a\alpha \equiv a'\alpha(\theta)$, then by a similar argumentation it follows, that $(a\alpha)\alpha^{-1} \equiv (a'\alpha)\alpha^{-1}(\theta)$, that is, $a \equiv a'(\theta)$. This completes the proof of the Theorem.

It would be of interest to investigate the question: for which primitive classes is the converse of the third assertion of the Theorem true? To put it otherwise, supposing that a subalgebra N is invariant under inner automorphisms, under which conditions does it follow that N is a class of a congruence θ and θ has no other class which would be a subalgebra? We have this latter case in the three examples above, but this does not hold, e. g., for the primitive class of semigroups, because they have only trivial inner automorphisms.

References

[1] A. G. KUROŠ, Vorlesungen über allgemeine Algebra, *Leipzig*, 1964.
[2] SH. K. STEIN, On the foundations of quasigroups, *Trans. Amer. Math. Soc.* **85** (1957), 228–256.

(Received February 24, 1965.)