On partially ordered vector spaces with the Riesz interpolation property

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Dedicated to Professor Paul Szász on his 65th birthday

Recently the author has proved ([5]) some results on partially ordered groups G satisfying the Riesz interpolation property: if $a_i \leqq b_j$ in G $(i=1, 2; j=1, 2)$ then there exists some $c \in G$ such that $a_i \leqq c \leqq b_j$ for $i=1, 2; j=1, 2$. It turned out that in the theory of Riesz groups a distinguished role is played by the subdirectly irreducible ones which can be characterized by the property that no two elements have a g. l. b. (or l. u. b.) unless one of them is greater than or equal to the other. These groups have been called (for want of a better name) *antilattices*, a typical example being the group of all polynomials in the unit interval or the group of all trigonometric polynomials in the interval $[0, 2\pi]$, with real coefficients of course. As pointed out, in antilattices a natural topology can be introduced by taking the open intervals as a basis of open neighbourhoods, and then every antilattice becomes a topological group (under a mild condition ensuring that it is a Hausdorff space). A necessary and sufficient condition can be given to guarantee that the topological completion of a commutative antilattice be a lattice-ordered group; for this and related results we refer to [6].

Our main objective here is to consider antilattices which are at the same time vector spaces over the real numbers. We shall see that in this case some results can be stated which resemble those on vector lattices. [1]) The bounded and continuous linear maps between antilattices will be considered briefly, and it will be shown that bounded and continuous linear functionals on o-simple antilattices have the same meaning. A certain amount of simplification arises if we assume that multiplication by scalars acts continuously, i. e. if the antilattice is a topological vector space over the reals. Then the antilattice A must be o-simple, and it is easy to see that a real-valued norm can be introduced, and A possesses enough continuous linear functionals to distinguish points. Moreover, A is then isomorphic to a group of real-valued continuous functions on a compact Hausdorff space Ω where for the functions f and g we put $f < g$ if and only if $f(\sigma) < g(\sigma)$ for all $\sigma \in \Omega$. It is easy to derive an approximation theorem of Stone—Weierstrass type. Finally, we characterize the dual spaces of antilattices (with continuous scalar multiplication) as abstract Lebesgue spaces.

[1]) For vector lattices we refer to [1].

A few remarks on terminology may be inserted here. [2])

Group will always mean an abelian group written additively. A group G is called a partially ordered group if it is a partially ordered set under a binary relation $\leqq$ such that $a \leqq b$ implies $a+c \leqq b+c$ for all $c \in G$. If in addition G is a vector space over the real number field R such that $a \leqq b$ implies $\lambda a \leqq \lambda b$ for all $\lambda > 0$ in R, then G is called a partially ordered vector space. The set P of all $x \in G$ with $x \geqq 0$ is the positivity domain of G which completely determines the partial order of G; namely, $a \leqq b$ if and only if $b-a \in P$. G is lattice-ordered if $\leqq$ makes G into a lattice. G is directed if to each pair $a, b \in G$ there exists a $c \in G$ satisfying $a \leqq c$ and $b \leqq c$. A directed group with the Riesz interpolation property mentioned above is said to be a Riesz group. There are several properties equivalent to the Riesz interpolation property; for instance: the closed intervals are additive: $[a, b]+[c, d] = [a+c, b+d]$ for all $a, b, c, d \in G$. An antilattice is a Riesz group such that the g. l. b. of a and b never exists unless it equals a or b.

An element $z \in G$ is a pseudo-zero if $z \neq 0$ and $z+P^* = P^*$; and $w \in G$ is pseudo-positive if $w \notin P$ and $w+P^* \subseteq P^*$ where P^* denotes the positivity domain P with 0 omitted. (Cf. [5].)

A partially ordered group (vector space) containing no non-trivial convex, directed subgroup (subspace) is called o-simple. If G_σ $(\sigma \in I)$ is a family of partially ordered groups, then their mild cartesian product is the group of all vectors $\langle \dots, g_\sigma, \dots \rangle$ with σ th component g_σ in G_σ such that $\langle \dots, g_\sigma, \dots \rangle < \langle \dots, g'_\sigma, \dots \rangle$ if and only if $g_\sigma < g'_\sigma$ for all $\sigma \in I$. If we consider a subdirect product of the G_σ with this "mild" definition of ordering, then we speak of a mild subdirect product. Finally, an o-isomorphism is an algebraic isomorphism preserving order relation in both directions (i. e. it is isotone in both directions).

§ 1. Vector antilattices

Let A be a partially ordered vector space over the real number field R such that it is an antilattice and contains no pseudo-zeros. For brevity, we shall refer to A as a *vector antilattice*. We have the following results which can be proved by slight modifications of the proofs in [5] and [6].

(1) If the open intervals $(-a, a)$ with $a > 0$ in A are taken as a basis of open neighbourhoods around 0, then A becomes a non-discrete topological group. A will always be considered as furnished with this topology.

(2) If C is a maximal trivially ordered subspace of A, then A/C is fully ordered under the induced ordering.

(3) A is o-isomorphic to a mild subdirect product of the fully ordered vector spaces A/C where C runs over all maximal trivially ordered subspaces.

(4) The topological completion of A is a vector lattice if and only if A is an approximation antilattice in the sense that, given $a, b \in A$ and $u > 0$ in A, there exists a $c \in A$ such that $c \leqq a, b$ and if $x \leqq a, b$ then $x < c+u$.

[2]) In the main we follow the terminology of [4]. For the definitions not reproduced here see [4].

(5) If L is a vector lattice and if T is a filter in the positivity domain of L such that (i) $a \in (-t, t)$ for all $t \in T$ implies $a=0$, and (ii) $t \in T$, $\lambda > 0$ implies $\lambda t \in T$, then keeping L as a vector space over R and weakening the order of L by declaring only 0 and the elements of T to be positive, we obtain an approximation antilattice A. The topological completion of A contains an o-isomorphic copy of L.

(6) Every dense subspace of A is again a vector antilattice.

(7) The mild cartesian product of vector antilattices is again a vector antilattice.

We list some more properties of vector antilattices.

(8) If K is a trivially ordered subspace of A, then its closure $\bar{K}$ is likewise trivially ordered. Thus *maximal trivially ordered subspaces of A are closed.*

To prove this, assume $a, b \in \bar{K}$ and $a < b$. Choose $u > 0$ in A such that $2u < b-a$. There exist $c, d \in K$ satisfying $c \in (a-u, a+u)$, $d \in (b-u, b+u)$. Hence $c < a+u < b-u < d$ would be a contradiction.

(9) If K is a non-trivially ordered convex subspace of A, then K is open. If A is an approximation antilattice, then A/K is a vector lattice.

Let $u > 0$ belong to K. Given $a \in K$, $(a-u, a+u) \subset K$ too whence K is open. If to $a, b \in A$, $c \in A$ is chosen as described in (4), then we have $aK \wedge bK = cK$ for the cosets in A/K.

(10) *If A contains no pseudo-positive elements, then its topological dimension is 0 or 1.*

Let $a \in A$ belong to the closure of the open set $(-u, u)$ where $u > 0$. Thus for every $v > 0$ in A, $(a-v, a+v)$ intersects $(-u, u)$. On account of the Riesz interpolation property, this is equivalent to the fulfilment of the inequalities $a-v < u$ and $-u < a+v$. In other words, both $-a+u$ and $a+u$ have the property that adding an arbitrary element $v > 0$ to them makes them positive. By hypothesis, $-a+u \geqq 0$ and $a+u \geqq 0$, whence $-u \leqq a \leqq u$. Thus the closure of $(-u, u)$ is the closed interval $[-u, u]$. We claim that the boundary of $[-u, u]$ consists of $-u, u$ only. For if $a \in [-u, u]$ and $a \neq -u, u$, then $-u < -a, a$, and by the antilattice property there is some $v > 0$ in A such that $-u+v < -a, a$. Therefore $(a-v, a+v) \subset (-u, u)$, and a is an interior point. We arrive at the desired conclusion that the topological dimension of A does not exceed 1.

Our proof shows that in general the closure of $(-u, u)$ is the set $(-u+Q) \cap (u-Q)$ where Q denotes the set of all positive and pseudo-positive elements. We also have:

(11) *Q is the closure of P^*.*

§ 2. Continuous linear maps

Let A and B denote vector antilattices in the sense of the preceding section. We are going to consider the continuous and the bounded linear maps from A into B.

Recall that a continuous linear map $f: A \to B$ is a function of A into B such that (i) $f(a_1+a_2) = f(a_1)+f(a_2)$ for all $a_i \in A$, (ii) $f(\lambda a) = \lambda f(a)$ for all $a \in A, \lambda \in R$, (iii) f is continuous in the open interval topology.

Theorem 1. *A linear map $f: A \to B$ is continuous if and only if to every $b>0$ in B there is an $a>0$ such that*

$$0 \leqq c < a \quad \textit{implies} \quad f(c) \in (-b, b).$$

If f is continuous, then some neighbourhood $(-a, a)$ of 0 is contained in [3] $f^{-1}(-b, b)$. Hence the stated condition is necessary. Conversely, if f satisfies the condition and if given $b>0$ in B, then let $d \in f^{-1}(-b, b)$ be arbitrary. Since B is an antilattice and $0 \in (-b-f(d), b-f(d)) \cap (-b+f(d), b+f(d))$, there is an open interval $(-b_1, b_1)$ with $b_1>0$ in the intersection. Choose $a \in A$ so as to satisfy that $0 \leqq c < a$ implies $f(c) \in (-\frac{1}{2}b_1, \frac{1}{2}b_1)$. Then $f(-a, a) \subset (-b_1, b_1)$, and it follows $f(d-a, d+a) \subset (-b, b)$, i. e. $f^{-1}(-b, b)$ is open.

It follows at once:

Corollary 2. *An isotone (antitone) linear map $f: A \to B$ is continuous if and only if to every $b>0$ in B there is an $a>0$ $(a<0)$ in A such that $0 \leqq f(a) < b$.*

In particular, an o-epimorphism is always continuous.

Corollary 3. *An isotone or antitone linear map into an o-simple vector antilattice B is necessarily continuous.*

For, if given $b>0$ in B, then some $a>0$ (or $a<0$) satisfies $0 \leqq f(a)$. If $\lambda>0$ is chosen such that $\lambda f(a) < b$, then $\lambda a>0$ (or $\lambda a<0$) satisfies $0 \leqq f(\lambda a) < b$.

Let f and g be linear maps from A into B. If λf and $f+g$ are defined as usual, then the set of all linear maps from A into B is made into a vector space over the reals. Let $L(A, B)$ denote this space.

We shall use the convention: linear maps into R will be called linear functionals.

Define the linear map $f: A \to B$ *positive*, written: $f \geqq 0$ (where 0 denotes the zero map of A into B) if $a \geqq 0$ implies $f(a) \geqq 0$; that is to say, f is positive if and only if it is isotone. Accordingly, we set $f \leqq g$ to mean that $g-f$ is positive. Under this definition $L(A, B)$ is a partially ordered vector space over R. Call the linear map $f: A \to B$ *bounded* if there exist $g, h \in L(A, B)$ satisfying $g \leqq f, 0$ and $f, 0 \leqq h$. Obviously, the sum, difference and scalar multiples of bounded linear maps are bounded, and so are the positive linear maps. The bounded linear maps from A into B form a subspace $L_b(A, B)$ of $L(A, B)$, actually the one generated by the positive linear maps.

If f and g are continuous linear maps from A into B, then so are $f+g$ and λg for all $\lambda \in R$. Hence the continuous linear maps from A into B form a subspace $L_c(A, B)$ of $L(A, B)$. The following result tells us the connection between bounded and continuous linear maps.

Theorem 4. *If A, B are vector antilattices and B is o-simple, then bounded linear maps are continuous. The bounded and continuous linear functionals on an o-simple vector antilattice A are the same.*

By virtue of Corollary 3, isotone linear maps of A into B are continuous, and so are their differences, i. e. the bounded linear maps. The second statement is a consequence of the following well-known result of Riesz [11].

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[3]) We write simply $f(-b, b)$ for $f((-b, b))$.

Let G be a partially ordered vector space over R with the Riesz interpolation property. The relatively bounded [4]) linear functionals of G form a complete vector lattice L where the lattice-operations $\vee$ and $\wedge$ are as follows:

$$(f\vee g)(a) = \sup\{f(b)+g(c) \text{ with } 0\leqq b, 0\leqq c, b+c = a\},$$

$$(f\wedge g)(a) = \inf\{f(b)+g(c) \text{ with } 0\leqq b, 0\leqq c, b+c = a\}$$

for $a\geqq 0$ in G. [5])

In order to complete the proof of Theorem 4. let $f: A\to R$ be a continuous linear functional and $\varepsilon>0$ a real number. If $(-b, b)\subset f^{-1}(-\varepsilon, \varepsilon)$ and $\lambda>0$ satisfies $[-a, a]\subset(-\lambda b, \lambda b)$, then $f[-a, a]\subset(-\lambda\varepsilon, \lambda\varepsilon)$ and f is relatively bounded. By the cited result, both $f\vee 0$ and $f\wedge 0$ exist. Hence f is bounded and the proof is completed.

§ 3. Antilattices which are topological vector spaces

Henceforth we shall confine our attention to the case when the vector antilattice A is at the same time a topological vector space over the reals, that is, the map $(\lambda, a)\to\lambda a$ of $R\times A$ into A is continuous [6]).

Theorem 5. *A vector antilattice A has the property that* $(\lambda, a)\to\lambda a$ *is continuous if and only if A is o-simple.*

Assume that A satisfies the continuity condition, and let $v>0$ in A. Then $V=(-v, v)$ is an open neighbourhood of 0. By hypothesis, to $a\in A$ there exists a real $\varepsilon>0$ such that $|\mu|<\varepsilon$ implies $\mu a\in V$. Hence $-v<\mu a<v$, and thus the convex subgroup generated by v exhausts the whole of A. Conversely, let A be o-simple. Observe that the neighbourhoods $(-v, v)$ are star-like, i. e. $a\in(-v, v)$ and $|\lambda|<1$ imply $\lambda a\in(-v, v)$. If given λa and $v>0$ in A, we can find $\varepsilon>0$ such that $\eta a<\frac{1}{2}v$ for $|\eta|\leqq\varepsilon$, and $u>0$ such that $(\lambda+\varepsilon)u<\frac{1}{2}v$. Then (e. g. for $\lambda\geqq 0$) $\lambda a-v< <(\lambda-\varepsilon)(a-u)<(\lambda+\varepsilon)(a+u)<\lambda a+v$ implies the continuity of multiplication by scalars.

A vector antilattice in which multiplication by scalars is continuous will be called a *topological vector antilattice*.

Theorem 6. *A topological vector antilattice is normable and admits enough continuous linear functionals to separate points.*

Recall that according to a well-known result by KOLMOGOROV ([8]), a topological vector space is normable (i. e. it admits a norm such that the induced metric determines the topology) if and only if it is locally convex and has a bounded open neighbourhood system of 0. These conditions are satisfied in our present case,

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[4]) Relatively bounded means that for all $a>0$ in A, the set $\{f(b)$ with $b\in[-a, a]\}$ is a bounded set of real numbers.

[5]) For an arbitrary $a\in G$ the definition of $(f\vee g)(a)$ is immediate, since every a is the difference of two positive elements of G. — For other proofs see [3] or [7].

[6]) For topological vector spaces we refer to [2] or [3].

hence with each $x \in A$ we can associate a non-negative real number $\|x\|$ given by the Minkowski functional

$$(*) \qquad \|x\| = \inf\{\lambda | \lambda^{-1}x \in U, \lambda > 0\}$$

for some fixed neighbourhood $U = (-u, u)$ of 0. Note that this norm possesses the property: $-x < y < x$ implies $\|y\| \leqq \|x\|$.

In order to verify the second statement of Theorem 6, we can make use of the following consequence of Eidelheit's separation theorem: [7]) if K is a closed convex set in a locally convex linear space L and if $x \notin K$, then there exists a continuous linear functional f of L such that $f(x) > \sup\{f(y) | y \in K\}$. In our present case we take $K = Q$ or $-Q$ according as $x \neq 0$ is not in Q or not in $-Q$. Then K is closed (cf. (11) in § 1), and $0 \in K$ implies that our sup is $\geqq 0$ whence $f(x) > 0$. Therefore the continuous linear functionals separate the elements of A, indeed.

Next we prove the following result which gives a fairly good description of topological vector antilattices.

Theorem 7. *Every topological vector antilattice A is algebraically, order-theoretically and topologically isomorphic to a group of real-valued continuous functions on a compact Hausdorff space Ω where ordering is mild. If in A the norm is defined by $(*)$, then there exists an isomorphism of A with a group of continuous functions on Ω which preserves the norm* [8]).

Let C be a maximal trivially ordered subspace of A. From (2) we know that A/C is fully ordered. We claim that A/C is a vector space o-isomorphic to the real numbers. For if $u + C$ is an arbitrary positive coset, then $\lambda(u + C)$ becomes greater than any other coset $v + C$ if λ is sufficiently large, and thus A/C is archimedean. Because of (8), the canonical map $A \to A/C$ is a continuous linear functional of A. If we let C_ϱ run over all maximal trivially ordered subspaces of A, then by (3) the totality of canonical maps $A \to A/C_\varrho$ yields an o-isomorphism π of A with a mild subdirect product of the 1-dimensional real vector spaces A/C_ϱ. Hence π is an o-isomorphism with a group of functions.

We topologize the set of the C_ϱ, or the set Ω of the indices ϱ of the C_ϱ as follows. Given a real number $\delta > 0$ and a finite number of elements $a_1, \ldots, a_n$ of A, let the $(\delta, a_1, \ldots, a_n)$-neighbourhood of $\varrho \in \Omega$ be defined as

$$(**) \qquad V_{\delta, a_1, \ldots, a_n}(\varrho) = \{\sigma \in \Omega \text{ with } |\bar{a}_i(\varrho) - \bar{a}_i(\sigma)| < \delta, \ \ i = 1, \ldots, n\},$$

where $\bar{a}_i(\sigma)$ denotes the "value" of the function $\bar{a}_i$ corresponding to a_i under π, or otherwise expressed, $\bar{a}_i(\sigma)$ is the image of a_i under the canonical map: $A \to A/C_\sigma$. Clearly, $(**)$ defines the coarsest topology that makes all functions $\bar{a}$ corresponding to $a \in A$ continuous. It is easily seen that this is a Hausdorff topology and, as shown by GELFAND, [9]) this topology makes Ω into a compact space. (It is to be

[7]) See [3], p. 22. Here convexity is to be taken in the sense of the theory of topological vector spaces.

[8]) Thus it is an isometry. Recall that the norm of a real-valued function is the sup of its absolute value.

[9]) See e. g. NEUMARK [9]. Observe also that $(**)$ defines the unique topology under which Ω is a compact Hausdorff space and the functions corresponding to elements of A are continuous.

noted that the kernel of an isotone linear functional of A is a maximal trivially ordered subspace of A.)

If we have chosen $U = (-u, u)$ as a neighbourhood of 0 to define the norm $\|x\|$, then clearly $\|u\| = 1$. The vector space isomorphism between A/C and R can be assumed to be a normed space isomorphism if A/C is considered as a normed space under the induced norm $\|x+C\| = \inf\{\|x+c\|, c \in C\}$. Now $u+C$ does not intersect the interval $(-u, u)$, since $u+C$ contains no negative elements, and therefore $\|u+C\| = \|u\|$. Thus $u+C$ corresponds to the real number 1, and the arising isomorphism π mapping A into the group of continuous functions over Ω will map u upon the constant 1 function $\bar{u}$. Since π is an o-isomorphism and the norm on the space of continuous functions can be defined in terms of constant functions and order in the same way as the norm was defined by $(*)$, we conclude that π must be an isometry.

Note that Theorem 7 holds verbatim if A is only assumed to be an o-simple antilattice which is divisible and has isolated order.

Observe also that our result contains as a special case the Ky Fan—Fleischer theorem on the representation of commutative lattice-ordered groups as groups of continuous functions [10]. In fact, if G is an archimedean lattice-ordered group containing a subgroup H o-isomorphic to the group of real numbers such that the elements >0 of H are strong units, then the filter generated by the strictly positive elements of H gives rise to an antilattice A on G which is o-simple and hence the preceding theorem holds true for A. The transition from A to G is easily performed by simply declaring also the pseudo-positive elements to be positive.

Next we are concerned with a result of Stone—Weierstrass type.

Theorem 8. *Let A be a topological vector antilattice which is an approximation antilattice. If Ω is the compact space of the maximal trivially ordered subspaces of A, then the topological completion of A is the vector lattice $C(\Omega)$ of all continuous functions on Ω.*

We begin with observing that [11] A contains the constant functions, and the definition of Ω implies that A possesses sufficiently many functions to distinguish the points of Ω. Hence we conclude [12] that $C(\Omega)$ is the smallest closed vector sublattice, containing A, of $C(\Omega)$. Since lattice-ordered groups are distributive as lattices, the vector sublattice generated by A in $C(\Omega)$ consists of functions of the form

$$\bigvee_i (f_{i1} \wedge \ldots \wedge f_{ik_i}) \qquad (f_{ik} \in A)$$

where $\vee$ and $\wedge$ mean point-wise maximum and minimum, respectively. In view of the approximation property, these functions can be approximated by functions $f \in A$ arbitrarily. Hence the stated conclusion.

Our Theorem 8 generalizes the classical approximation theorem by Weierstrass stating that polynomials and trigonometric polynomials approximate the continuous

[10]) See e. g. RIBENBOIM [10].

[11]) We may without fear of ambiguity identify A with $\pi(A)$.

[12]) See [3], p. 103.

functions on a finite interval. This generalization shows that from the point of view of approximation it is more natural to consider a polynomial positive only if it is >0 everywhere. (In fact, an approximation with a deviation $<\varepsilon$ makes this tacit assumption.)

§ 4. The dual spaces

Next we shall be concerned with the dual spaces of topological vector antilattices. These are the vector spaces A^* of all continuous linear functionals f of the topological vector antilattices A. Since the A are normed vector spaces, so are A^* with the norms

$$\|f\| = \sup\{|f(x)| \text{ for } \|x\| \leqq 1\}.$$

This norm determines the same topology on A^* as the so-called polar sets of the intervals $(-a, a)$.

Before stating our theorem on the dual spaces, recall that an abstract Lebesgue space is a normed vector lattice L such that (i) L is a real Banach space, (ii) $f \wedge g = 0$ implies $\|f+g\| = \|f-g\|$, (iii) $f, g \geqq 0$ implies $\|f+g\| = \|f\|+\|g\|$. Note that (ii) is equivalent to: (ii′) f and $|f| = f \vee -f$ have the same norm.

We have the following characterization of the dual spaces.

Theorem 9. *The dual space of a topological vector antilattice is an abstract Lebesgue space.*

From Riesz' theorem cited in § 2 we infer that the dual space A^* of a topological vector antilattice A is a vector lattice. By classical results on normed spaces we also know that A^* is a Banach space in the norm mentioned above. In order to verify (ii′), we show:

$$\|(f \vee 0)+(-f \vee 0)\| = \|f\|$$

which is equivalent to (ii′) because of $f \vee -f = f \vee 0 + (-f \vee 0)$. We have clearly:

$$\|(f \vee 0)+(-f \vee 0)\| = \sup\{(f(x) \vee 0)+(-f(x) \vee 0) \text{ for } \|x\| \leqq 1\} =$$
$$= \sup\{\sup[f(y) \text{ for } 0 \leqq y \leqq x] + \sup[-f(z) \text{ for } 0 \leqq z \leqq x] \text{ for } \|x\| \leqq 1\} =$$
$$= \sup\{\sup[f(y)-f(z) \text{ for } y, z \in [0, x]] \text{ for } \|x\| \leqq 1\}.$$

By the Riesz interpolation property, the closed intervals are additive, that is, the set of all $y-z$ with $y, z \in [0, x]$ coincides with the interval $[-x, x]$. Hence

$$\|(f \vee 0)+(-f \vee 0)\| = \sup\{\sup[f(v) \text{ for } v \in [-x, x]] \text{ for } \|x\| \leqq 1\} =$$
$$= \sup\{f(v) \text{ for } \|v\| \leqq 1\} = \|f\|,$$

as claimed. To establish (iii), let $f, g \geqq 0$. Then f and g are isotone and we have

$$f(x)+g(y) \leqq f(w)+g(w) \qquad \text{if } x, y \leqq w.$$

Therefore, if to x and y in the unit ball w is chosen again in the unit ball, then

$$\|f\|+\|g\| = \sup\{f(x) \text{ for } \|x\| < 1\} + \sup\{f(y) \text{ for } \|y\| < 1\} \leqq$$
$$\leqq \sup\{f(w)+g(w) \text{ for } \|w\| < 1\} = \|f+g\|.$$

The converse inequality being always true, the proof of Theorem 9 is finished.

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