On Hausdorff and related moment problems

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The Hausdorff moment problem relates to obtaining necessary and sufficient conditions involving a real sequence $\{\lambda_n\}$ so that there exists a function $g(t)$ of bounded variation over [0, 1] and $\lambda_n = \int_0^1 t^n dg(t)$. Different solutions of this problem have been given by HAUSDORFF [6], RAMANUJAN [10] and JAKIMOVSKI [7]. A related question is to obtain necessary and sufficient conditions for the representation $\lambda_n = \int_0^1 t^n g(t)\,dt$, with $g(t)$ belonging to a specified function space. This has been discussed in HAUSDORFF [6], RAMANUJAN [12], BERMAN [1, 2], LORENTZ [9] and GEHRING [4]. The above problems with the modifications of the type that $\lambda_n = \int_0^1 t^{n+\alpha} dg(t)$ or $\lambda_n = \int_0^1 t^{n+\alpha} g(t)\,dt$ form the content of [3] and [8]. The main tool in most of these investigations is the uniform approximation of continuous functions in [0, 1] by polynomials or power series suggested by them. In this paper we obtain certain sufficient conditions for the above representations under a fairly general set-up and indicate how some of the earlier results are included as special cases of these results.

Notation. Throughout the paper the following assumptions are made, unless a statement to the contrary is made.

1. $R > 1$ is a fixed real number;
2. $\varphi(n)$ denotes a sequence of non-negative integers or $+\infty$;
3. For each fixed n, $(n = 0, 1, 2, 3, \ldots)$, $\{\alpha_{nm}\}$ is a sequence of real numbers such that $\alpha_{nm} = 0$, for $m > \varphi(n)$ and $0 \leqq \alpha_{nm} \leqq 1$ for all n and m;
4. $c_{mn}(z)$ are power-series satisfying

(4. 1) $$c_{nm}(z) = 0 \quad \text{for} \quad m > \varphi(n), \qquad n = 0, 1, 2, \ldots;$$

(4. 2) $$\text{for } c_{nm}(z) = \sum_{k=0}^{\infty} a_{nmk} z^k, \quad (0 \leqq m \leqq \varphi(n)), \quad \text{we have } \sum_{k=0}^{\infty} |a_{nmk}| R^k < \infty$$

(4. 3) $$\text{for each } p = 0, 1, 2, 3, \ldots, A_{np}(z) = \sum_{m=0}^{\infty} c_{nm}(z)\, \alpha_{nm}^p \to z^p$$

uniformily in $|z| \leqq R$, as $n \to \infty$;

* The third author acknowledges support by the National Science Foundation. NSF 05261.

5. In the space P of power series $g(z)=\sum_0^\infty a_n z^n$, with radius of convergence greater than 1, set $L(g)=\sum_0^\infty a_n\lambda_n$, where $\{\lambda_n\}$ is a fixed bounded sequence. $L(g)$ is well defined and is a linear functional on P.

We list below some special examples of functions described on (4).

Example 1. $c_{nm}(z)=\binom{n}{m}(1-z)^{n-m}z^m$, $0\leqq m\leqq n$ and $=0$ for $m>\varphi(n)=$ $=n$ and $\alpha_{nm}=m/n$.

The function described above is the kernel of the well-known Bernstein approximation for analytic functions; see, for instance, Lorentz ([9], Theorem 4. 1. 1).

Example 2. $c_{nm}(z)=\binom{n}{m}[z+Q_n(z)]^m[1-z-Q_n(z)]^{n-m}$ for $0\leqq m\leqq n$ and $=0$, $m>\varphi(n)=n$ where for each n, $|z+Q_n(z)|\leqq R$ for $|z|\leqq R$, $R>1$ being fixed, α_{nm} are chosen as in Example 1, and $Q_n(z)=\sum_r q_{nr}z^r$ with $\varepsilon_n=\sum_r|q_{nr}|R^r\to 0$ as $n\to\infty$.

Example 3. The Lagrange interpolatory functions.

Example 4. In Example 2, choose $Q_n(z)=\dfrac{a_n(1-z)}{1+a_n}$, with $a_n>0$ and $a_n\to 0$.

Theorems

In this section we prove certain sufficient conditions for a given sequence $\{\lambda_n\}$ to be a Hausdorff moment sequence. We start with the following lemma.

Lemma 1. *Suppose* $R>1$; *if* $f_n(z)=\sum_{k=0}^\infty a_{nk}z^k\to z^p$ *uniformly in* $|z|\leqq R$ *and if* $\{\lambda_n\}$ *is a bounded sequence, then* $\lim_{n\to\infty}L(f_n)=\lambda_p$.

Proof. We first prove that for each $\varepsilon>0$ there corresponds a $N(\varepsilon)$ such that for $n>N(\varepsilon)$, we have $|a_{nk}-\delta_{kp}|<\varepsilon R^{-k}$, where δ_{kp} are the Kronecker symbols.

For any $k\geqq 0$,

$$a_{nk}=\frac{1}{2\pi i}\oint_{|z|=R}\frac{f_n(z)}{z^{k+1}}\,dz$$

and

$$\delta_{kp}=\frac{1}{2\pi i}\oint_{|z|=R}\frac{z^p}{z^{k+1}}\,dz.$$

Consequently

$$|a_{nk}-\delta_{kp}|=\left|\frac{1}{2\pi i}\oint_{|z|=R}\frac{f_n(z)-z^p}{z^{k+1}}\,dz\right|\leqq\max_{|z|=R}\frac{|f_n(z)-z^p|}{R^k}$$

and the conclusion above easily follows.

Now to complete the proof of the lemma, consider any $\varepsilon > 0$ and the corresponding $N(\varepsilon)$ indicated above. For $n > N(\varepsilon)$,

$$\begin{aligned} |L(f_n) - \lambda_p| &= |\sum_k a_{nk} \lambda_k - \lambda_p| \\ &= |\sum_k (a_{nk} - \delta_{kp}) \lambda_k| \\ &\leqq \sup_k |\lambda_k| \sum_k |a_{nk} - \delta_{kp}| \leqq \sup_k |\lambda_k| \varepsilon \sum_k R^{-k} \end{aligned}$$

and the proof of the lemma is complete.

We state a theorem below and omit its proof remarking that it runs on lines parallel to that of a known theorem (JAKIMOVSKI [7], Theorem 1) and that Lemma 1 plays now the rôle of Theorem 3 of [7].

Theorem 1. *Let $\varphi(n)$ be finite for $n \geqq 0$. If for the bounded sequence $\{\lambda_n\}$ we have*

$$\sup_n \sum_{k=0}^{\infty} |b_{nk}| \sum_{m=0}^{\varphi(k)} |L(c_{km})| < \infty \tag{1. 1}$$

for some regular matrix (i. e., Toeplitz matrix) $\mathbf{B} = (b_{nk})$, then $\{\lambda_n\}$ is a Hausdorff moment-sequence.

The following remarks on Theorem 1 are relevant.

1. The following converse of Theorem 1 is easily proved: If $\sup_n \max_{0 \leqq z \leqq 1} \sum_{m=0}^{\varphi(n)} |c_{nm}(z)| < \infty$ and $\{\lambda_n\}$ is a Hausdorff moment sequence then for *each* regular matrix $B = (b_{nk})$, we have that (1. 1) holds.

2. Theorem 1 of [7] referred to earlier is itself a special case of our Theorem 1 by choosing the $c_{nm}(z)$ and α_{nm} as in Example 1.

3. In the case of Example 2, the condition (1. 1) becomes

$$\sup_n \sum_{k=0}^{\infty} |b_{nk}| \sum_{m=0}^{n} \binom{n}{m} |L([z + Q_n(z)]^m) [1 - z - Q_n(z)]^{n-m}| < \infty$$

and considering Example 4, it follows that a bounded sequence $\{\lambda_n\}$ will be a Hausdorff moment sequence if, in particular, $a_n > 0$, $a_n \to 0$ and

$$\sup_n (1 + a_n)^{-n} \sum_{m=0}^{n} \binom{n}{m} \left| \sum_{r=0}^{m} \binom{m}{r} a_n^{m-r} \Delta^{n-m} \lambda_r \right| < \infty.$$

Indeed the above condition is also seen to be necessary.

4. It is no serious limitation to assume that the sequence $\{\lambda_n\}$ is bounded since a moment sequence is necessarily bounded.

Theorem 2. *Let $\varphi(n) = O(n)$ and let for each fixed $n, \{c_{nM}(z)\}$, $M = 0, 1, \ldots, \varphi(n)$, denote some permutation of $\{c_{nm}(z)\}$, $m = 0, 1, 2, \ldots \varphi(n)$. If for a bounded sequence $\{\lambda_n\}$,*

$$\sup_M \sum_{n=0}^{\infty} |L(c_{nM})| < \infty \tag{2. 1}$$

then $\{\lambda_n\}$ is a Hausdorff moment sequence.

PROOF. Let

$$\sup_M \sum_{n=0}^{\infty} |L(c_{nM})| = K.$$

For each fixed n, let $N_n = \max_{j=0,1,\dots,n} \varphi(j)$. Then

$$(N_n+1)K \geqq \sum_{M=0}^{N_n} \sum_{k=0}^{\infty} |L(c_{kM})| = \sum_{k=0}^{\infty} \sum_{M=0}^{N_n} |L(c_{kM})| \geqq \sum_{k=0}^{n} \sum_{M=0}^{\varphi(k)} |L(c_{kM})|.$$

Thus, for each fixed n,

$$\infty > K^* = \frac{(N_n+1)K}{(n+1)} \geqq \frac{1}{n+1} \sum_{k=0}^{n} \sum_{M=0}^{\varphi(k)} |L(c_{kM})| =$$

$$= \sum_{k=0}^{n} \frac{1}{n+1} \sum_{M=0}^{\varphi(k)} |L(c_{kM})| = \sum_{k=0}^{n} \frac{1}{n+1} \sum_{m=0}^{\varphi(k)} |L(c_{km})|$$

and the result follows by taking the matrix **B** in Theorem 1 to be that of the $(C, 1)$ method.

Corollary and **Remark.** Choosing the $c_{nm}(z)$ as in Example 1 it follows that

$$\sup \sum_{n=0}^{\infty} \left| \binom{n}{m} \Delta^{n-m} \lambda_m \right| < \infty \tag{2.2}$$

implies that $\{\lambda_n\}$ is necessarily a moment sequence, thus enabling us to make the following remark on Theorem 219 of HARDY [5]. If the quasi-Hausdorff method (H^*, μ_n) is conservative (i. e.) if it is a Kojima matrix, then it is necessary that μ_n is a moment sequence. (see also RAMANUJAN [11], p. 201).

We consider next the case the $\varphi(n)$ is not necessarily finite for any n and prove, with the notations set out earlier,

Theorem 3. *Let $\varphi(n)$ be not necessarily finite for any n. Suppose also the following conditions hold:*

(i) *for each* $p \geqq 0, \quad \sum_{m=0}^{\varphi(n)} |\alpha_{nm}|^p \sum_{k=0}^{\infty} |a_{nmk}| < \infty$

and

(ii) *for $\{\lambda_n\}$ bounded,* $\sup_n \sum_{m=0}^{\varphi(n)} |L(c_{nm})| < \infty.$

Then $\{\lambda_n\}$ is a Hausdorff moment sequence.

PROOF. We first prove that under the assumptions of the theorem

$$\lim_{n\to\infty} \sum_{m=0}^{\varphi(n)} L(c_{nm}) \alpha_{nm}^p = \lambda_p, \quad \text{for each } p \geqq 0. \tag{3.1}$$

Let p be fixed. The uniform convergence, in $|z| \leqq R$, of $\sum_{m=0}^{\varphi(n)} c_{nm}(z)\alpha_{nm}^p$ to z^p implies, by the Wierstrass's theorem for double series, that

$$A_{np}(z) = \sum_{m=0}^{\varphi(n)} c_{nm}(z)\alpha_{nm}^p = \sum_{j=0}^{\infty} b_{npj}\, z^j,$$

where

$$b_{npk} = \sum_{m=0}^{\varphi(n)} a_{nmk}(\alpha_{nm})^p.$$

We now have $A_{np}(z) \to z^p$ uniformly in $|z| \leqq R$, as $n \to \infty$. Therefore, by Lemma 1,

$$\lim_{n\to\infty} L(A_{np}) = \lim_{n\to\infty} \sum_{j=0}^{\infty} b_{npj}\, \lambda_j = \lambda_p. \tag{3. 2}$$

This is true for $p = 0, 1, 2, 3, \ldots$.
Now

$$\sum_{m=0}^{\varphi(n)} L(c_{nm})\alpha_{nm}^p = \sum_{m=0}^{\varphi(n)} \alpha_{nm}^p \sum_{k=0}^{\infty} a_{nmk}\, \lambda_k$$

and

$$\sum_{m=0}^{\varphi(n)} |\alpha_{nm}|^p \sum_{k=0}^{\infty} |a_{nmk}|\, |\lambda_k| \leqq \sup_k |\lambda_k| \sum_{m=0}^{\varphi(n)} |\alpha_{nm}|^p \sum_{k=0}^{\infty} |a_{nmk}| < \infty$$

and therefore interchanging the order of summation we have,

$$\sum_{m=0}^{\varphi(n)} L(c_{nm})\alpha_{nm}^p = \sum_{k=0}^{\infty} \lambda_k \sum_{m=0}^{\varphi(n)} \alpha_{nm}^p\, a_{nmk} = \sum_{k=0}^{\infty} b_{npk}\, \lambda_k$$

and therefore (3. 1) follows from (3. 2).

It follows from (3. 1) that for each polynomial $P(z)$ we have

$$\lim_{n\to\infty} \sum_{m=0}^{\varphi(n)} L(c_{nm})\, P(\alpha_{nm}) = L(P).$$

Also,

$$\left| \sum_{m=0}^{\varphi(n)} L(c_{nm})\, P(\alpha_{nm}) \right| \leqq \|P\| \sum_{m=0}^{\varphi(n)} |L(c_{nm})| = K\|P\|,$$

for each n, the norm above being the supremum norm over $[0, 1]$. Letting $n \to \infty$, it follows that for each polynomial P over $[0, 1]$, $|L(P)| \leqq K||P||$; i. e. L is a linear continuous functional over $P[0, 1]$, the space of polynomials over $[0, 1]$. Since $P[0, 1]$ is dense in $C[0, 1]$, L can be extended to $C[0, 1]$ and the result in the theorem is now an easy consequence of the Riesz representation theorem.

Our final result in Theorem 4 is a sufficient condition for a bounded sequence $\{\lambda_n\}$ to be a moment sequence of the form $\lambda_n = \int_0^1 t^n f(t)\, dt$, where f is in a specified function-space $X(C)$ of the Koethe type. For definition and relevant details regarding these see LORENTZ ([9], pp. 65—69).

Theorem 4. *Let the function-space $X(C)$ have the property of rearrangement-invariant norm and uniform absolute continuity for functions with norm less than* 1. *Let $\varphi(n)<\infty$, for each n. Let $0=\alpha_{n0}<\alpha_{n1}<\alpha_{n2}<\dots<\alpha_{n,\varphi(n)}=1$ and $\sup_m |\alpha_{nm}-\alpha_{n,m-1}|\to 0$ as $n\to\infty$. Given a bounded sequence $\{\lambda_n\}$, define the functions $f_n(x)$ as follows:*

$$f_n(x)=(\alpha_{nm}-\alpha_{n,m-1})^{-1}L(c_{nm}),\ \alpha_{n,m-1}\leqq x<\alpha_{nm},\ m=0,1,2,\dots\varphi(n).$$

Then, if $\|f_n\|_{X(C)}\leqq M$, for each fixed n, then $\lambda_n=\int_0^1 t^n f(t)\,dt$, for a suitable function f in the space $X(C)$, with $\|f\|_{X(C)}\leqq M$.

Partial proof of Theorem 4. We have, in the notation set out earlier, $\sum_{m=0}^{\varphi(n)} c_{nm}(z)\alpha_{nm}^p\to z^p$, uniformly in $|z|\leqq R$, $R>1$,

i. e.
$$\sum_{m=0}^{\varphi(n)}\alpha_{nm}^p\sum_{k=0}^{\infty}a_{nmk}z^k=\sum_{k=0}^{\infty}z^k\sum_{m=0}^{\varphi(n)}a_{nmk}\alpha_{nm}^p=\sum_{k=0}^{\infty}b_{nk}^{(p)}z^k,$$

say, and the above expression converges uniformly in $|z|\leqq R$ to z^p, for each p. Therefore, by Lemma 1,

$$\sum_{k=0}^{\infty}b_{nk}^{(p)}\lambda_k\to\lambda_p\quad\text{for each } p,\quad\text{as}\quad n\to\infty;$$

i. e.

$$\sum_{k=0}^{\infty}\lambda_k\sum_{m=0}^{\varphi(n)}a_{nmk}\alpha_{nm}^p=\sum_{m=0}^{\varphi(n)}\alpha_{nm}^p\sum_{k=0}^{\infty}a_{nmk}\lambda_k\to\lambda_p\quad\text{as}\quad n\to\infty;$$

i. e.

$$\lim_n\sum_{m=0}^{\varphi(n)}L(c_{nm})\alpha_{nm}^p=\lambda_p,\quad\text{for}\quad p=0,1,2,3,\dots.$$

But the left-side above is $\int_0^1 f_n(x)g_n(x)\,dx$, where the $f_n(x)$ are the functions defined in the statement of the theorem and

$$g_n(x)=\alpha_{nm}^p,\quad\text{for}\quad\alpha_{n,m-1}\leqq x<\alpha_{nm},\quad m=0,1,\dots\varphi(n).$$

Hereafter the proof is routine and we refer the reader to the proof of Theorem 3. 8. 4 in Lorentz [9].

Corollary. For the space $L_p(p>1)$, the condition $\|f_n\|\leqq M$, reduces to

$$\sum_{m=0}^{\varphi(n)}\frac{|L(C_{nm})|^p}{(\alpha_{nm}-\alpha_{n,m-1})^{p-1}}\leqq M,\qquad n=0,1,\dots.$$

The choice of $c_{nm}(z)$ as in Example 1 yields the well-known solution of the moment problem for the space $L_p(p>1)$.

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(Received July 28, 1964.)