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# Some structural considerations on the Finslerian gravitational field

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**Abstract.** Some generalized connection structures of the Finslerian gravitational field are constructively considered from the vector bundle-like standpoint, where the independent internal variables are generalized in various ways: The Finslerian independent variables $(x^\kappa; y^i)$ $(\kappa, i = 1,2,3,4)$ are generalized to $(x^\kappa;\ (y^i, y^0))$ ($y^0$ is a scalar), or $((x^\kappa, x^0); y^i)$ ($x^0$ is a scalar), or $(x^\kappa; y^i;\ z^a)$ ($z^a$ is a vector), or $(x^\kappa;\ y^{(\alpha)i})$ $(\alpha = 1,2,\dots,n;\ y^{(\alpha)i}$ is a vector).

In this paper, some generalized connection structures of the Finslerian gravitational field are considered by generalizing the independent internal variables in various ways: The Finslerian independent variables $(x^\kappa;\ y^i)$ $(\kappa, i = 1,2,3,4;\ y^i$ is an internal vector associated with each point $x^\kappa)$ are generalized to $(x^\kappa;\ (y^i, y^0))$ ($y^0$ is a scalar), or $((x^\kappa, x^0);\ y^i)$ ($x^0$ is a scalar), or $(x^\kappa;\ y^i;\ z^a)$ ($z^a$ is a vector), or $(x^\kappa;\ y^{(\alpha)i})$ $(\alpha = 1,2,\dots,n;\ y^{(\alpha)i}$ is a vector).

## §1. Introduction

In the Finslerian gravitational field [3], the independent variables are chosen as $(x^\kappa; y^i)$ $(\kappa, i = 1,2,3,4)$, where $y$ means the independent internal vector associated with each point $x$. That is to say, the Finslerian gravitational field is regarded as a nonlocal field [7] nonlocalized by the internal vector $y$.

In other words, the Finslerian gravitational field is considered a unified field between the external $(x)$-field spanned by points $\{x\}$ and the internal $(y)$-field spanned by vectors $\{y\}$. Therefore, from the vector bundle-like standpoint [6], the Finslerian gravitational field is regarded as the unified field over the total space of the vector bundle, whose base manifold is the $(x)$-field and fibre at each point $x$ is the $(y)$-field. The base $(x)$-field is nothing but the Einstein's gravitational field.

In the total space, the so-called adapted frame is set properly in the form [6]

$$
\begin{aligned}
& d\zeta^A = (dx^\kappa,\ \delta y^i = dy^i + N^i_\lambda dx^\lambda), \\
& \frac{\partial}{\partial \zeta^A} = \left( \frac{\delta}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} - N^i_\lambda \frac{\partial}{\partial y^i},\ \frac{\partial}{\partial y^i} \right),
\end{aligned} \tag{1.1}
$$

where $N^i_\lambda$ denotes the nonlinear connection, which represents physically the interaction between the $(x)$- and $(y)$-fields. On the basis of (1.1), the connection structure $\Gamma$ is prescribed by

$$
\nabla_{\frac{\partial}{\partial \zeta^C}} \frac{\partial}{\partial \zeta^B} = \Gamma_B{}^A{}_C \frac{\partial}{\partial \zeta^A}; \quad \Gamma_B{}^A{}_C \equiv \left( F_\lambda{}^\kappa{}_\mu, F_j{}^i{}_\mu, C_\lambda{}^\kappa{}_k, C_j{}^i{}_k \right). \tag{1.2}
$$

Then, the covariant derivatives, for an arbitrary vector $V^A = (V^\kappa, V^i)$, can be defined as follows:

$$
\begin{aligned}
& DV^\kappa = (V^\kappa{}_{|\mu}) dx^\mu + (V^\kappa|_k)\delta y^k, \qquad DV^i = (V^i{}_{|\mu}) dx^\mu + (V^i|_k)\delta y^k; \\
& V^\kappa{}_{|\mu} = \frac{\delta V^\kappa}{\delta x^\mu} + F_\lambda{}^\kappa{}_\mu V^\lambda, \qquad V^\kappa|_k = \frac{\partial V^\kappa}{\partial y^k} + C_\lambda{}^\kappa{}_k V^\lambda; \\
& V^i{}_{|\mu} = \frac{\delta V^i}{\delta x^\mu} + F_j{}^i{}_\mu V^j, \qquad V^i|_k = \frac{\partial V^i}{\partial y^k} + C_j{}^i{}_k V^j.
\end{aligned} \tag{1.3}
$$

By the way, the metrical structure $G$ of the total space is expressed as

$$
G = G_{AB}\, d\zeta^A \otimes d\zeta^B = g_{\lambda\kappa}(x,y) dx^\kappa \otimes dx^\lambda + g_{ij}(x,y)\delta y^i \otimes \delta y^j. \tag{1.4}
$$

And the metrical connection can be determined by the metrical conditions $g_{\lambda\kappa|\mu} = 0$, $g_{\lambda\kappa}|_k = 0$, $g_{ij|\mu} = 0$ and $g_{ij}|_k = 0$. The orthogonal formula (1.4) with respect to the adapted frame (1.1) is obtained from the (general) unorthogonal formula such as

$$
G = \widetilde{g}_{\lambda\kappa}\, dx^\kappa \otimes dx^\lambda + 2\widetilde{g}_{\lambda i}\, dx^\lambda \otimes dy^i + \widetilde{g}_{ij}\, dy^i \otimes dy^j, \tag{1.5}
$$

where the following relations are induced:

$$
\begin{aligned}
& g_{\lambda\kappa} = \widetilde{g}_{\lambda\kappa} - \widetilde{g}_{ij} N^i_\lambda N^j_\kappa \quad \left( = \widetilde{g}_{\lambda\kappa} - \widetilde{g}_{\lambda i}\, \widetilde{g}_{\kappa j}\, \widetilde{g}^{ij} \right), \\
& \widetilde{g}_{\lambda i} = N^j_\lambda\, \widetilde{g}_{ij} \quad \left( \text{or } N^j_\lambda = \widetilde{g}_{\lambda i}\, \widetilde{g}^{ij} \right), \\
& g_{ij} = \widetilde{g}_{ij}.
\end{aligned} \tag{1.6}
$$

As is understood from (1.2) or (1.3), the connection structure depends essentially on the independent variables or the adapted frame, so that the connection structure can be generalized by generalizing the independent variables in various ways. From this viewpoint, in this paper, we shall consider some generalized connection structures of the Finslerian gravitational field.

## §2. On the generalized connection structures I

In this Section, we shall generalize the independent variables to the form $(x^\kappa;\ (y^i, y^0))$ or $((x^\kappa, x^0);\ y^i)$, where $y^0$ or $x^0$ is an independent scalar.

First, in the case of $(x^\kappa;\ y^a = (y^i, y^0))$, the fibre becomes the 5-dimensional vector space. Then, the adapted frame is generalized as follows:

$$\begin{gathered}\big(dx^\kappa,\ \delta y^a = (\delta y^i = dy^i + N^i_\lambda dx^\lambda,\ \delta y^0 = dy^0 + N^0_\lambda dx^\lambda)\big)\,, \\ \left(\frac{\delta}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} - N^a_\lambda \frac{\partial}{\partial y^a} = \frac{\partial}{\partial x^\lambda} - N^i_\lambda \frac{\partial}{\partial y^i} - N^0_\lambda \frac{\partial}{\partial y^0},\right. \\ \left.\frac{\partial}{\partial y^a} = \left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^0}\right)\right).\end{gathered} \tag{2.1}$$

One new nonlinear connection $N^0_\lambda$ must be introduced additionally. Therefore, the generalized connection structure can be written in the form

$$\begin{aligned}&\Gamma_B{}^A{}_C \equiv (F_\lambda{}^\kappa{}_\mu,\ F_b{}^a{}_\mu,\ C_\lambda{}^\kappa{}_c,\ C_b{}^a{}_c)\,; \quad F_b{}^a{}_\mu \equiv \left(F_j{}^i{}_\mu\, F_0{}^0{}_\mu\right), \\ &C_\lambda{}^\kappa{}_c \equiv (C_\lambda{}^\kappa{}_k,\ C_\lambda{}^\kappa{}_0)\,, \quad C_b{}^a{}_c \equiv \left(C_j{}^i{}_k,\ C_0{}^0{}_k,\ C_j{}^i{}_0,\ C_0{}^0{}_0\right).\end{aligned} \tag{2.2}$$

The metrical structure is expressed as

$$\begin{aligned}G &= g_{\lambda\kappa}\, dx^\kappa \otimes dx^\lambda + g_{ab}\, \delta y^a \otimes \delta y^b \\ &= g_{\lambda\kappa}\, dx^\kappa \otimes dx^\lambda + g_{ij}\, \delta y^i \otimes \delta y^j + g_{00}\, \delta y^0 \otimes \delta y^0.\end{aligned} \tag{2.3}$$

Second, in the case of $(x^a = (x^\kappa, x^0);\ y^i)$, the base manifold becomes 5-dimensional. In this case, the adapted frame is given by

$$\begin{gathered}(dx^a = (dx^\kappa, dx^0), \quad \delta y^i = dy^i + N^i_a dx^a = dy^i + N^i_\lambda dx^\lambda + N^i_0\, dx^0), \\ \left(\frac{\delta}{\delta x^a} = \left(\frac{\delta}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} - N^i_\lambda \frac{\partial}{\partial y^i},\ \frac{\delta}{\delta x^0} = \frac{\partial}{\partial x^0} - N^i_0 \frac{\partial}{\partial y^i}\right), \frac{\partial}{\partial y^i}\right).\end{gathered} \tag{2.4}$$

Another new nonlinear connection $N_0^i$ must be introduced. Then, the connection structure is introduced in the form

$$
\begin{aligned}
\Gamma_B{}^A{}_C &= \left(F_b{}^a{}_c,\, F_j{}^i{}_c,\, C_b{}^a{}_k,\, C_j{}^i{}_k\right);\\
F_b{}^a{}_c &= \left(F_\lambda{}^\kappa{}_\mu,\, F_0{}^0{}_\mu,\, F_\lambda{}^\kappa{}_0,\, F_0{}^0{}_0\right),\\
F_j{}^i{}_c &= \left(F_j{}^i{}_\mu,\, F_j{}^i{}_0\right),\\
C_b{}^a{}_k &= \left(C_\lambda{}^\kappa{}_k,\, C_0{}^0{}_k\right).
\end{aligned}
\tag{2.5}
$$

The generalized metrical structure is given by

$$
\begin{aligned}
G &= g_{ab}\, dx^a \otimes dx^b + g_{ij}\, \delta y^i \otimes \delta y^j\\
&= g_{\lambda\kappa} dx^\kappa \otimes dx^\lambda + g_{00} dx^0 \otimes dx^0 + g_{ij} \delta y^i \otimes \delta y^j.
\end{aligned}
\tag{2.6}
$$

Here, different from the Finslerian gravitational field, if we take the Riemannian (Einstein's) gravitational field and decompose the coordinate $x^\kappa$ into the 3-dimensional space part $x^\phi$ and the 1-dimensional time part $x^0 = t$, then we can adopt the following adapted frame:

$$
(dx^\phi,\ \delta t = dt + N_\phi^0 dx^\phi),\ \left(\frac{\delta}{\delta x^\phi} = \frac{\partial}{\partial x^\phi} - N_\phi^0 \frac{\partial}{\partial t},\ \frac{\partial}{\partial t}\right),
\tag{2.7}
$$

where the fibre $x^0 = t$ is associated with each point $x^\phi$ of the 3-dimensional base manifold (with the Riemann metric $\gamma_{\phi\psi}$). On the other hand, if the 3-dimensional vector space spanned by $\{x^\phi\}$ (with the Riemann metric $h_{\phi\psi}$) is attached as the fibre to the 1-dimensional base $x^0 = t$, then the adapted frame is given by

$$
(dt,\ \delta x^\phi = dx^\phi + N_0^\phi dt),\ \left(\frac{\delta}{\delta t} = \frac{\partial}{\partial t} - N_0^\phi \frac{\partial}{\partial x^\phi},\quad \frac{\partial}{\partial x^\phi}\right).
\tag{2.8}
$$

Comparing (2.7) and (2.8), we can say that the nonlinear connections $N_\phi^0$ and $N_0^\phi$ play the different role in each case. These situations have been considered [2] with respect to the difference of the Threading metric ($g_{\phi\psi} = \gamma_{\phi\psi} + N_\phi^0 N_\psi^0 h_{00}$) and the Slicing metric ($g_{00} = \gamma_{00} + N_0^\phi N_0^\psi h_{\phi\psi}$), where $\gamma_{\lambda\kappa} \equiv (\gamma_{\phi\psi}, \gamma_{00})$ and $h_{\lambda\kappa} \equiv (h_{\phi\psi}, h_{00})$ mean the 4-dimensional Riemann metrics of the total spaces of the corresponding 4-dimensional Riemannian vector bundles.

## §3. On the generalized connection structures II

Now, the generalized connection structures mentioned in Section 2 are considered within the framework of the first-order vector bundle-like standpoint. If another vector $z^a$ $(a = 1, 2, 3, 4)$ is chosen, besides $y^i$, as the independent variable at one more microscopic level than the Finslerian one, then the connection structure of this case must be considered from the second-order vector bundle-like standpoint. Then, the adapted frame is generalized as follows [4]:

$$(3.1)\qquad \begin{gathered}(dx^\kappa,\ \delta y^i = dy^i + N^i_\lambda dx^\lambda,\ \delta z^a = dz^a + M^a_\lambda\, dx^\lambda + L^a_i dy^i),\\ \Big(\frac{\delta}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} - N^i_\lambda \frac{\partial}{\partial y^i} - M^a_\lambda \frac{\partial}{\partial z^a}, \frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - L^a_i \frac{\partial}{\partial z^a},\ \frac{\partial}{\partial z^a}\Big).\end{gathered}$$

Three kinds of nonlinear connections $N^i_\lambda$, $M^a_\lambda$ and $L^a_i$ must be introduced. In this case, the connection structure is generalized as follows:

$$(3.2)\qquad \begin{gathered}\Gamma_B{}^A{}_C = \Big(F_\lambda{}^\kappa{}_\mu,\ F_j{}^i{}_\mu,\ F_b{}^a{}_\mu;\quad C_\lambda{}^\kappa{}_k,\ C_j{}^i{}_k,\ C_b{}^a{}_k;\\ H_\lambda{}^\kappa{}_c,\ H_j{}^i{}_c,\ H_b{}^a{}_c\Big).\end{gathered}$$

Nine components of connection coefficients must be introduced. For example, the covariant derivatives are defined in the form

$$(3.3)\qquad \begin{gathered}V^\kappa{}_{|\mu} = \frac{\delta V^\kappa}{\delta x^\mu} + F_\lambda{}^\kappa{}_\mu V^\lambda, \qquad V^i|_k = \frac{\delta V^i}{\delta y^k} + C_j{}^i{}_k V^j,\\ V^a\|_c = \frac{\partial V^a}{\partial z^c} + H_b{}^a{}_c V^b,\end{gathered}$$

etc. And the metrical structure is given by

$$(3.4)\qquad G = g_{\lambda\kappa} dx^\kappa \otimes dx^\lambda + g_{ij}\delta y^i \otimes \delta y^i + g_{ab}\delta z^a \otimes \delta z^b.$$

Further generalizations such as $((x^\kappa, x^0);\ y^i;\ z^a)$, or $(x^\kappa;\ (y^i, y^0);\ z^a)$, etc. can be considered in various ways. Of course, $z^a$ can be specialized to a scalar $z^0$, as in the case of $(x^\kappa;\ y^i;\ z^0)$.

According to several authors [1] [6], if the intrinsic transformations of the independent variables are regarded as the gauge transformations, then the second-order gauge transformations, which are defined as the gauge transformations over the second-order vector bundle, should be expressed in the form

$$(3.5)\qquad \tilde{x}^\kappa = X^\kappa(x^\lambda), \quad \tilde{y}^i = Y^i(x^\kappa, y^j), \quad \tilde{z}^a = Z^a(x^\kappa, y^i, z^b),$$

which respond suitably to the adapted frame (3.1). Such specializations as $\widetilde{y}^i = Y^i_j y^j$ ($Y^i_j = \frac{\partial Y^i}{\partial y^j}$), $\widetilde{z}^a = Z^a_b z^b$ ($Z^a_b = \frac{\partial Z^a}{\partial z^b}$), etc. can be considered properly. If the gauge invariance of the adapted frame is required in the sense that $\frac{\delta}{\delta x^\lambda} = X^\kappa_\lambda \frac{\delta}{\delta \widetilde{x}^\kappa}$ ($X^\kappa_\lambda = \frac{\partial X^\kappa}{\partial x^\lambda}$), $\frac{\delta}{\delta y^i} = Y^j_i \frac{\delta}{\delta \widetilde{y}^j}$ and $\frac{\partial}{\partial z^a} = Z^b_a \frac{\partial}{\partial \widetilde{z}^b}$, then the gauge transformations of the nonlinear connections can be obtained as follows: Because of $\frac{\delta}{\delta \widetilde{x}^\kappa} = \frac{\partial}{\partial \widetilde{x}^\kappa} - \widetilde{N}^i_\kappa \frac{\partial}{\partial \widetilde{y}^i} - \widetilde{M}^a_\kappa \frac{\partial}{\partial \widetilde{z}^a}$, $\frac{\delta}{\delta \widetilde{y}^j} = \frac{\partial}{\partial \widetilde{y}^j} - \widetilde{L}^a_j \frac{\partial}{\partial \widetilde{z}^a}$, etc.,

$$
\begin{aligned}
\widetilde{N}^i_\kappa &= \overline{X}^\lambda_\kappa Y^i_j N^j_\lambda - \overline{X}^\lambda_\kappa \frac{\partial Y^i}{\partial x^\lambda},\\
\widetilde{M}^a_\kappa &= \overline{X}^\lambda_\kappa Z^a_b M^b_\lambda + \overline{X}^\lambda_\kappa \frac{\partial Z^a}{\partial y^i} N^i_\lambda - \overline{X}^\lambda_\kappa \frac{\partial Z^a}{\partial x^\lambda},\\
\widetilde{L}^a_i &= \overline{Y}^j_i Z^a_b L^b_j - \overline{Y}^j_i \frac{\partial Z^a}{\partial y^j},
\end{aligned}
\tag{3.6.}
$$

where $\overline{X}^\lambda_\kappa$, $\overline{Y}^j_i$ (and $\overline{Z}^b_a$) are the inverse matrices of $X^\kappa_\lambda$, $Y^i_j$ (and $Z^a_b$), respectively. In the same manner, if the gauge invariant conditions are imposed on the covariant derivatives (3.3) in the sense that for example, $V^\kappa{}_{|\lambda} = X^\kappa_\alpha \overline{X}^\beta_\lambda \widetilde{V}^\alpha{}_{|\widetilde{\beta}}$ with $\widetilde{V}^\alpha{}_{|\widetilde{\beta}} = \frac{\delta \widetilde{V}^\alpha}{\delta \widetilde{x}^\beta} + \widetilde{F}_\gamma{}^\alpha{}_\beta \widetilde{V}^\gamma \left(\widetilde{V}^\gamma = X^\gamma_\alpha V^\alpha\right)$, etc., then the gauge transformations of the connection coefficients (3.2) or (3.3) can be obtained as follows: For example,

$$
\begin{aligned}
\widetilde{F}_\lambda{}^\kappa{}_\mu &= X^\kappa_\alpha \overline{X}^\beta_\lambda \overline{X}^\gamma_\mu F_\beta{}^\alpha{}_\gamma - \overline{X}^\beta_\lambda \overline{X}^\gamma_\mu \frac{\delta X^\kappa_\beta}{\delta x^\gamma},\\
\widetilde{C}_j{}^i{}_k &= Y^i_m \overline{Y}^n_j \overline{Y}^\ell_k C_n{}^m{}_\ell - \overline{Y}^m_j \overline{Y}^n_k \frac{\delta Y^i_m}{\delta y^n},\\
\widetilde{H}_b{}^a{}_c &= Z^a_d \overline{Z}^e_b \overline{Z}^f_c H_e{}^d{}_f - \overline{Z}^e_b \overline{Z}^f_c \frac{\partial Z^a_e}{\partial z^f},
\end{aligned}
\tag{3.7}
$$

etc. The gauge invariant metric tensors are also considered in the sense that $\widetilde{g}_{\lambda\kappa} = \overline{X}^\alpha_\lambda \overline{X}^\beta_\kappa g_{\alpha\beta}$, $\widetilde{g}_{ij} = \overline{Y}^m_i \overline{Y}^n_j g_{mn}$ and $\widetilde{g}_{ab} = \overline{Z}^c_a \overline{Z}^d_b g_{cd}$.

## §4. On the generalized connection structures III

If we further generalize the second-order vector bundle-like standpoint by adopting many vectors $y^{(\alpha)}$ ($\alpha = 1, 2, 3, \ldots, n$) at more microscopic level, then we must take the higher-order space-like standpoint [5], because we must take account of the interactions of internal vectors $y^{(\alpha)}$ and $y^{(\beta)}$.

Then, the form $\delta y^{(\alpha)}$ is prescribed as, by adopting the base connection-like form of $y^{(\alpha)}$ [5],

$$\delta y^{(\alpha)i} = M^{(\alpha)i}_{\ \ j}\left(dy^{(\alpha)j} + \sum_{\beta=1}^{\alpha-1}\Lambda^{(\alpha)j}_{(\beta)k}\, dy^{(\beta)k} + \Lambda^{(\alpha)j}_{(0)\lambda}\, dx^{\lambda}\right), \tag{4.1}$$

from which we can obtain

$$M^{(\alpha)i}_{\ \ j} dy^{(\alpha)j} = \delta y^{(\alpha)i} - \sum_{\beta=1}^{\alpha-1}\Pi^{(\alpha)i}_{(\beta)k}\, dy^{(\beta)k} - \Pi^{(\alpha)i}_{(0)\lambda}\, dx^{\lambda}, \tag{4.2}$$

where $\Pi^{(\alpha)i}_{(\beta)k} = M^{(\alpha)i}_{\ \ j}\Lambda^{(\alpha)j}_{(\beta)k}$ and $\Pi^{(\alpha)i}_{(0)\lambda} = M^{(\alpha)i}_{\ \ j}\Lambda^{(\alpha)j}_{(0)\lambda}$. In general, the following conditions hold good ($\alpha, \beta, \gamma = 0, 1, 2, \ldots, n$):

$$M^{(\alpha)i}_{\ \ j}\Lambda^{(\alpha)j}_{(\gamma)k} + \Pi^{(\alpha)i}_{(\gamma)j}M^{(\gamma)j}_{\ \ k} + \sum_{\beta=\gamma+1}^{\alpha-1}\Pi^{(\alpha)i}_{(\beta)j}M^{(\beta)j}_{\ \ \ell}\Lambda^{(\beta)\ell}_{(\gamma)k} = 0. \tag{4.3}$$

Taking account of the above formulations, we can write the adapted frame in the form

$$\left(dx^{\kappa},\ \delta y^{(\alpha)i} = N^{(\alpha)i}_{\ \ \lambda}\, dx^{\lambda} + \sum_{\beta=1}^{\alpha-1}\Psi^{(\alpha)i}_{(\beta)k}\, dy^{(\beta)k}\right), \tag{4.4}$$

$$\left(\frac{\delta}{\delta x^{\lambda}} = \frac{\partial}{\partial x^{\lambda}} - N^{(\alpha)i}_{\ \ \lambda}\frac{\delta}{\delta y^{(\alpha)i}},\quad \frac{\delta}{\delta y^{(\alpha)k}} = \sum_{\beta=\alpha+1}^{n}\left(\Psi^{-1}\right)^{(\beta)\ell}_{(\alpha)k}\frac{\partial}{\partial y^{(\beta)\ell}}\right),$$

where $N^{(\alpha)i}_{\ \ \lambda} = M^{(\alpha)i}_{\ \ j}\Lambda^{(\alpha)j}_{(0)\lambda}$, $\Psi^{(\alpha)i}_{(\beta)k} = M^{(\alpha)i}_{j}\widetilde{\Lambda}^{(\alpha)j}_{(\beta)k}$ and $\widetilde{\Lambda}^{(\alpha)j}_{(\beta)k} = \delta^{(\alpha)}_{(\beta)}\delta^{j}_{k} + \Lambda^{(\alpha)j}_{(\beta)k}$. Those quantities such as $\Lambda^{(\alpha)}_{(\beta)}$, $\widetilde{\Lambda}^{(\alpha)}_{(\beta)}$, $\Psi^{(\alpha)}_{(\beta)}$, etc. represent the interactions of $y^{(\alpha)}$ and $y^{(\beta)}$.

On the basis of the adapted frame (4.4), the connection structure is given by

$$\Gamma_{B}{}^{A}{}_{C} \equiv \left(F_{\lambda}{}^{\kappa}{}_{\mu},\ F_{(\beta)j}{}^{(\alpha)i}{}_{\mu},\ C_{\lambda}{}^{\kappa}{}_{(\gamma)k},\ C_{(\beta)j}{}^{(\alpha)i}{}_{(\gamma)k}\right), \tag{4.5}$$

and the metrical structure is given by

$$G = g_{\lambda\kappa}\, dx^{\kappa}\otimes dx^{\lambda} + g_{(\alpha)i(\beta)j}\delta y^{(\alpha)i}\otimes\delta y^{(\beta)j}. \tag{4.6}$$

The most generalized connection structure (4.5) can be specialized in various ways: For example, we can reduce ${F_{(\beta)j}}^{(\alpha)i}{}_{\mu}$ and ${C_{(\beta)j}}^{(\alpha)i}{}_{(\gamma)k}$ to ${F_{(\alpha)j}}^{(\alpha)i}{}_{\mu}$ and ${C_{(\alpha)i}}^{(\alpha)i}{}_{(\gamma)k}$, and further ${C_{(\alpha)j}}^{(\alpha)i}{}_{(\gamma)k}$ to ${C_{(\alpha)j}}^{(\alpha)i}{}_{(\alpha)k}$, etc.

Thus, as is understood from the above, the generalized connection structures of the Finslerian gravitational field can be considered by generalizing the independent variables or the adapted frame in various ways. Of course, these theories can be applied to any kind of internal variable such as scalar, vector, spinor, etc.

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