On the geometric means of integral functions*)

By SATYA NARAIN SRIVASTAVA (Lucknow)

1. Let $f(z)$ be an integral function of order ϱ. Let $n(r)$ denote the number of zeros of $f(z)$ for $|z|\leqq r$ and let $N(r)=\int\limits_0^r \frac{n(x)}{x}\,dx$, $f(0)\neq 0$. Let $G(r)$ and $g(r)$ denote the geometric means of $|f(z)|$ on the circumference $|z|=r$ and $|z|\leqq r$ respectively. Then ([1], pp. 144, 108)

$$G(r)=\exp\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}\log|f(re^{i\theta})|\,d\theta\right\} \tag{1. 1}$$

$$g(r)=\exp\left\{\frac{1}{\pi r^2}\int\limits_0^r\int\limits_0^{2\pi}\log|f(xe^{i\theta})|\,x\,dx\,d\theta\right\}. \tag{1. 2}$$

Consider a function $\varphi(r)$ with the following properties: (i) $\varphi(r)$ is continuous for $r>r_0$; (ii) $\varphi(r)>0$ for $r>r_0$ and (iii) $\varphi(lr)\sim\varphi(r)$ as $r\to\infty$ for every constant $l>0$. Further, for $0<\varrho<\infty$, let

(1. 3)
$$\lim_{r\to\infty}\,\begin{matrix}\sup\\ \inf\end{matrix}\,\frac{\log g(r)}{r^\varrho\,\varphi(r)}=\begin{cases}p\\ q\end{cases},\quad(0<q\leqq p<\infty);\quad \lim_{r\to\infty}\,\begin{matrix}\sup\\ \inf\end{matrix}\,\frac{N(r)}{r^\varrho\,\varphi(r)}=\begin{cases}c\\ d\end{cases},\quad(0<d\leqq c<\infty).$$

We shall obtain some of the properties of $G(r)$ and $g(r)$.

2. Theorem 1. *If $f(z)$ be an integral function of finite order ϱ*

$$\frac{2}{(\varrho+2)}\,\frac{d}{c}\leqq\lim_{r\to\infty}\,\begin{matrix}\sup\\ \inf\end{matrix}\,\frac{\log g(r)}{N(r)}\leqq\frac{2}{(\varrho+2)}\,\frac{c}{d}, \tag{2. 1}$$

where c and d are given by (1. 3).

PROOF. From (1. 2), we have

$$\log g(r)=\frac{1}{\pi r^2}\int\limits_0^r\int\limits_0^{2\pi}\log|f(xe^{i\theta})|\,x\,dx\,d\theta=\frac{2}{r^2}\int\limits_0^r\left\{\int\limits_0^x\frac{n(t)}{t}\,dt+\log|f(0)|\right\}x\,dx,$$

*) This work has been supported by Senior Research Fellowship award of C. S. I. R., New Delhi (INDIA).

on using Jensen's theorem,

$$\log g(r) = \frac{2}{r^2}\int_0^r N(x)x\,dx + \log|f(0)|.$$

Therefore,

$$\frac{1}{2}r^2\log\left\{\frac{g(r)}{|f(0)|}\right\} = \int_0^r N(x)x\,dx. \tag{2. 2}$$

Writing (2. 2) as

$$\frac{1}{2}r^2(1+\delta)^2\log\left[\frac{g\{r(1+\delta)\}}{|f(0)|}\right] = \int_0^{r_0} N(x)x\,dx + \int_{r_0}^{r} N(x)x\,dx + \int_r^{(1+\delta)r} N(x)x\,dx$$

$$< O(1) + (c+\varepsilon)\int_{r_0}^{r} x^{\varrho+1}\varphi(x)\,dx + N\{r(1+\delta)\}r^2\delta\left(1+\frac{\delta}{2}\right)\sim$$

$$\sim (c+\varepsilon)\varphi(r)\frac{r^{\varrho+2}}{(\varrho+2)} + N\{r(1+\delta)\}r^2\delta\left(1+\frac{\delta}{2}\right), \quad \text{by [2], Lemma 5} \quad \delta\geqq 0$$

Proceeding to limits and using (1. 3), we get

$$\frac{1}{2}(1+\delta)^{\varrho+2}p \leqq \frac{c}{(\varrho+2)} + c\delta(1+\delta)^{\varrho}\left(1+\frac{\delta}{2}\right). \tag{2. 3}$$

Similarly, we obtain

$$\frac{1}{2}(1+\delta)^{\varrho+2}q \geqq \frac{d}{(\varrho+2)} + d\delta\left(1+\frac{\delta}{2}\right). \tag{2. 4}$$

Putting $\delta=0$ in (2. 3) and (2. 4), we have

$$\frac{p}{2} \leqq \frac{c}{(\varrho+2)} \quad \text{and}\ \frac{q}{2} \geqq \frac{d}{(\varrho+2)} \tag{2. 5}$$

From (1. 3), we obtain

$$\frac{q-\varepsilon}{c+\varepsilon} < \frac{\log g(r)}{N(r)} < \frac{p+\varepsilon}{d-\varepsilon}.$$

Taking limits and using (2. 5) we get

$$\frac{2}{(\varrho+2)}\frac{d}{c} \leqq \lim_{r\to\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\log g(r)}{N(r)} \leqq \frac{2}{(\varrho+2)}\frac{c}{d}.$$

Corollary 1. If $c=d$,

$$(\varrho+2)\log g(r)\sim 2N(r).$$

3. Theorem 2. *If* $\log g(r) \sim pr^{\varrho}\Phi(r)$ *then* $N(r) \sim \frac{(\varrho+2)}{2} pr^{\varrho}\varphi(r)$ *and conversely.*

PROOF. From (2. 5), if $c=d$, $p=q=\frac{2c}{\varrho+2}$.
Suppose now $p=q$. We shall show that $c=d$. If $0<\eta<1$, we have from (2. 2)

$$N(r)\eta < \frac{1}{r^2}\int_r^{r+\eta r} N(x)x\,dx = \frac{1}{2}(1+\eta)^2\log\left\{\frac{g(r+\eta r)}{|f(0)|}\right\} - \frac{1}{2}\log\left\{\frac{g(r)}{|f(0)|}\right\} =$$

$$= \frac{1}{2}p(1+\eta)^{\varrho+2}r^{\varrho}\varphi(r+\eta r) - \frac{1}{2}pr^{\varrho}\varphi(r) + o\big(r^{\varrho}\varphi(r)\big).$$

Hence,

$$\limsup_{r\to\infty}\frac{N(r)}{r^{\varrho}\varphi(r)} \leqq \frac{p}{2}(\varrho+2)+\beta\eta,$$

where β is a constant. Since η is arbitrary we get

$$\limsup_{r\to\infty}\frac{N(r)}{r^{\varrho}\varphi(r)} \leqq \frac{p}{2}(\varrho+2).$$

By considering the integral $\frac{1}{2}\log\left\{\frac{g(r)}{|f(0)|}\right\} - \frac{(1-\eta)^2}{2}\log\left\{\frac{g(r-\eta r)}{|f(0)|}\right\}$, we get

$$\liminf_{r\to\infty}\frac{N(r)}{r^{\varrho}\varphi(r)} \geqq \frac{p}{2}(\varrho+2)$$

and hence

$$N(r) \sim \frac{p}{2}(\varrho+2)r^{\varrho}\varphi(r).$$

4. Theorem 3. *If* $f(z)$ $\big(f(0)\neq 0\big)$ *is an integral function of order* ϱ,

$$\limsup_{r\to\infty}\frac{\log\left\{r\frac{G'(r)}{G(r)}\right\}}{\log r} \leqq \varrho,$$

where $G'(r)$ *is the derivative of* $G(r)$ *and* $r\to\infty$ *through almost all values of* r.

PROOF. From (1. 1), we have

$$(4.\ 1) \qquad \log G(r) = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta = \int_0^r \frac{n(x)}{x}\,dx + \log|f(0)|,$$

on using Jensen's theorem.

Differentiating (4. 1) with respect to r, we get

$$\frac{G'(r)}{G(r)} = \frac{n(r)}{r} \tag{4. 2}$$

for almost all values of r.

Taking limits on both the sides of (4. 2), we get

$$\limsup_{r\to\infty} \frac{\log\left\{r\frac{G'(r)}{G(r)}\right\}}{\log r} = \limsup_{r\to\infty} \frac{\log n(r)}{\log r} \leqq \varrho,$$

where $r\to\infty$ through almost all values of r.

Further, we note that if $f(z)$ $(f(0)\neq 0)$ is an integral function of finite non-integral order ϱ,

$$\limsup_{r\to\infty} \frac{\log\left\{r\frac{G'(r)}{G(r)}\right\}}{\log r} = \varrho,$$

where $r\to\infty$ through almost all values of r.

5. Theorem 4. *If $f(z)$ $(f(0)\neq 0)$ is an integral function of order ϱ and finite-type T,*

(i) $$\limsup_{r\to\infty} \frac{\log G(r)}{r^{\varrho}\log r} \leqq \limsup_{r\to\infty} \frac{n(r)}{r^{\varrho}} \leqq e\varrho T,$$

(ii) $$\liminf_{r\to\infty} \frac{\log G(r)}{r^{\varrho}\log r} \leqq \liminf_{r\to\infty} \frac{n(r)}{r^{\varrho}} \leqq \varrho T.$$

PROOF. From (4. 1), we have

$$\log G(r) = \int_{r_0}^{r} \frac{n(x)}{x}\,dx + O(1) \quad \leqq n(r)(\log r - \log r_0) + O(1).$$

Taking limits and using (2. 5. 14) and (2. 5. 15) [3], p. 16, we get (i) and (ii).

References

[1] G. PÓLYA and G. SZEGŐ, Aufgaben und Lehrsätze aus der Analysis I, *Berlin*, 1925.
[2] G. H. HARDY and W. W. ROGOSINSKI, Notes on Fourier Series (III) *Quart. J. Math. Oxford* **16** (1945), 49—58.
[3] R. P. BOAS, Entire Functions, *New York*, 1954.

(Received February 2, 1965.)