Sufficiency of parameter invariance conditions in areal and higher order Kawaguchi spaces

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Introduction

Let V_n be an n-dimensional space coordinatized by x (Latin indices assume the values 1, ..., n throughout this note). The most elementary variational problem s that of minimizing

$$\int L\{x^i, \dot{x}^i\}\, dt$$

relative to differentiable curves $x^i = x^i(t)$ joining given points x_0^i and x_1^i. If the integral is to be independent of the curve parametrization, the then Lagrangian L must satisfy the homogeneity equation

$$L\{x^i, \lambda\dot{x}^i\} = \lambda L\{x^i, \dot{x}^i\} \quad \text{for} \quad \lambda > 0. \tag{1}$$

In this note we consider some equations arising from parameter invariance conditions imposed on more general variational problems. The first part is concerned with problems in which the Lagrangian L depends on higher ordered derivatives of the curve $x^i = x^i(t)$ as in Kawaguchi spaces [1],

$$\int L\{x^i, x_1^i, \ldots, x_m^i\}\, dt \quad \text{where} \quad x_r^i = \frac{d^r x^i}{dt^r} \quad \text{for} \quad r = 1, \ldots, m. \tag{2}$$

If $m = 1$, then parameter invarience implies the homogeneity condition (1), which in turn implies (we use the summation convention on repeated indices)

$$\frac{\partial L}{\partial x_1^i} x_1^i = L.$$

It was shown by ZERMELO [2] and also by LISTER [3] in general that the parameter invariance of (2) implies that L satisfies the m partial differential equations (where δ_1^p is the Kronecker delta)

$$\sum_{r=p}^{m} \frac{r!}{(r-p)!} \frac{\partial L}{\partial x_r^i} x_{r-p+1}^i = \delta_1^p L \quad \text{for} \quad p = 1, 2, \ldots, m. \tag{3}$$

(summation on $i = 1, \ldots, n$). In this note we show that conversely (3) is sufficient to guarantee the parameter invariance of (2).

The second part of this note deals with the multiple integral problems of the variational calculus, as in Areal spaces [4]. In V_n an m-dimensional subspace S_m is defined by the equations $x^i = x^i(t^\alpha)$, where Greek indices assume the values 1 to $m \leqq n$. The functions $x^i(t^\alpha)$ are usually assumed to be of class C^2, and the multiple integrals defined over S_m are of the form

$$(4) \qquad \int_{R_t} \dots \int L\{x^i, \dot{x}^i_\alpha\}\, dt^1 \dots dt^m \quad \text{where} \quad \dot{x}^i_\alpha = \frac{\partial x^i}{\partial t^\alpha};$$

here the Lagrangian L is a function of the $n+nm$ arguments x^i and $\dot{x}^i_\alpha$. Under a transformation $t^\alpha = t^\alpha(\tau^\beta)$ of the Gaussian coordinates of S_m, the partial derivatives $\dot{x}^i_\alpha$ become $\lambda^\beta_\alpha \dot{x}^i_\beta$ (summation on $\beta = 1, \dots, m$) where $\lambda^\alpha_\beta = \partial t^\alpha / \partial \tau^\beta$. If (4) is to be invariant under such transformations then L must satisfy

$$(5) \qquad L\{x^i, \lambda^\beta_\alpha \dot{x}^i_\beta\} = |\lambda^\beta_\alpha| \cdot L\{x^i, \dot{x}^i_\alpha\}$$

where $|\lambda^\beta_\alpha| = \det(\lambda^\alpha_\beta)$ is the Jacobian of the transformation, and should not vanish. It was shown by G. Kobb [5] when $n=3$, $m=2$ and generally by W. Grosz [6] pg. 84, that a necessary condition (assuming differentiability etc.) for (5) to hold is that L satisfies the m^2 partial differential equations

$$(6) \qquad \frac{\partial L}{\partial \dot{x}^i_\alpha} \dot{x}^i_\beta = \delta^\alpha_\beta L$$

where of course summation on $i = 1, \dots, n$ is implied. Rund [7] has also shown these equations to be sufficient, but the sufficiency proof required some assumptions on the path of integration (see Martin [4]) and in this paper an alternate sufficiency proof is gigen which avoids these difficulties.

Finally, the equation (5) is a particular case of a matrix functional equation. Let $[\lambda^\alpha_\beta]$ denote $m \times m$ matrices, while $[x^i_\alpha]$ denotes rectangular m(row)$\times n$(column) matrices. If f is a real valued function on rectangular $m \times n$ matrices, then (5) is equivalent to the functional equation

$$(7) \qquad f\{[\lambda^\alpha_\beta][x^i_\alpha]\} = |\lambda^\alpha_\beta|^k f\{[x^i_\alpha]\}$$

when $k=1$. If $m=1$ this is Euler's homogeneity equation. If $m=n$, so that (7) becomes $f(XY) = |X|^k f(Y)$ for square matrices X and Y, the equation is readily solved. Let I be the identity matrix; it follows that $f(X) = f(XI) = |X|^k f(I) = c|X|^k$ for arbitrary c (see Aczél [8] pg. 242). In this note we discuss the general solution of (7) for arbitrary m and n although our results cannot be considered a complete solution of the functional equation.[1])

Higher ordered Lagrangians

In this section we show that *if L satisfies the m partial differential equations*

$$(3) \qquad \sum_{r=p}^{m} \frac{r!}{(r-p)!} \frac{\partial L}{\partial x^i_r} x^i_{r-p+1} = \delta^p_1 L \quad \text{for} \quad p = 1, \dots, m$$

[1]) As pointed out by Prof. Aczél, equation (7) characterizes the scalar density concomitans of weight k defined on n vectors.

where L is a function of the $n(m+1)$ variables, $L\{x^i, x^i_1, \ldots, x^i_m\}$, then

$$L\left\{x^i, \frac{dx^i}{dt}, \ldots, \frac{d^m x^i}{dt^m}\right\} = \frac{d\tau}{dt} L\left\{x^i, \frac{dx^i}{d\tau}, \ldots, \frac{d^m x^i}{d\tau^m}\right\}$$

for any parameter transformation $\tau = \varphi(t)$ of class C^m *satisfying* $\varphi'(t) > 0$; *we assume that L has continuous partial derivatives for all* x^i_r, *except perhaps for* $(x^1_1)^2 + \ldots + \ldots + (x^n_1)^2 = 0$. (This latter restriction is imposed for simplicity and can be considerably weakened, as indicated below, if so desired).

The following combinatorial formulas will be needed. Let $A^k_\sigma(m)$ denote

$$A^k_\sigma(m) = \begin{cases} 0 & \text{if} \\ \sum\limits_{(q)} \dfrac{1}{q_1! \ldots q_m!} \lambda_1^{q_1} \ldots \lambda_m^{q_m} & \text{if} \quad \sigma \geqq k \end{cases} \tag{8. 1}$$

where the summation is over integral $q_i \geqq 0$ satisfying

$$q_1 + q_2 + \ldots + q_m = k \tag{8. 2}$$

$$q_1 + 2q_2 + \ldots + mq_m = \sigma.$$

The $A^k_\sigma(m)$ are essentially multinomial coefficients [9], satisfying

$$(\lambda_1 z + \lambda_2 z^2 + \ldots + \lambda_m z^m)^k = k! \sum_\sigma A^k_\sigma(m) z^\sigma. \tag{9}$$

From (9) it follows (by differentiating with respect to λ_p) that

$$\frac{\partial A^k_\sigma(m)}{\partial \lambda_p} = \begin{cases} 0 & \text{if} \quad \sigma < p + k - 1, \\ A^{k-1}_{\sigma-p}(m) & \text{if} \quad \sigma \geqq k - p + 1; \end{cases} \tag{10}$$

and also (differentiating with respect to z)

$$\sum_{\alpha=1}^{\sigma-k+2} \alpha \lambda_\alpha A^{k-1}_{\sigma-\alpha+1}(m) = (\sigma+1) A^k_{\sigma+1}(m). \tag{11}$$

Since (3) is to hold for arbitrary variables x^i_r, then also for u^i_r related to the x^i_r by

$$u^i_r = \sum_{s=1}^{r} x^i_s \, r! \, A^s_r(m). \tag{12}$$

Replacing x^i_r throughout (3) by u^i_r, the system of equations becomes

$$\Omega_p \overset{\text{def}}{=} \sum_{r=p}^{m} \frac{\partial L}{\partial u^i_r} r! \sum_{s=1}^{r-p+1} x^i_s (r-p+1) A^s_{r-p+1}(m) = \delta^p_1 L(u^i_r), \tag{13}$$

where the left side defines the m quantities Ω_p.

If $\lambda_1 \neq 0$, we wish to show that by forming linear combinations of the equations (13), one can obtain the system

$$\frac{\partial L(u^i_r)}{\partial \lambda_p} = \frac{1}{\lambda_1} L(u^i_r) \frac{\partial \lambda_1}{\partial \lambda_p} \qquad p = 1, \ldots, m$$

or, expanding and using (10) and (12),

$$\Lambda_p \overset{\text{def}}{=} \sum_{r=p}^{m} \frac{\partial L}{\partial u_r^i} r! \sum_{s=1}^{r} x_s^i A_{r-p}^{s-1}(m) = \frac{1}{\lambda_1} \delta_1^p L. \tag{15}$$

However the required linear combinations of the equations $\Omega_p = \delta_1^p L$ which yield $\Lambda_p = \frac{1}{\lambda_1} \delta_1^p L$ are relatively complicated and it is easier to reverse the process as follows.

Let X_p and Y_p be arbitrary variables, $p=1, \ldots, m$, related by

$$X_p = \sum_{\sigma=1}^{m-p+1} \sigma \lambda_\sigma Y_{p+\sigma-1} \qquad p=1, \ldots, m. \tag{16}$$

Then it is easily verified that, given $\lambda_1 \neq 0$, the Y_p are given by

$$Y_p = \sum_{\sigma=1}^{m-p+1} Q_p^\sigma(\lambda_r) X_{p+\sigma-1} \qquad p=1, \ldots, m \tag{17}$$

where the $Q_p^\sigma(\lambda_r)$ are polynomials in the λ_r except for the presence of λ_1 to negative powers. In particular

$$Y_m = \frac{1}{\lambda_1} X_m; \quad Y_{m-1} = \frac{1}{\lambda_1} X_{m-1} - 2\frac{\lambda_2}{\lambda_1^2} X_m; \quad \text{etc.}$$

Instead of applying (17) to equations (13) and obtaining (15), it is sufficient to apply (16) to equations (15) and verifying that (13) results. Explicitly, we will show that

$$\Omega_p = \sum_{\sigma=1}^{m-p+1} \sigma \lambda_\sigma \Lambda_{p+\sigma-1} \qquad p=1, \ldots, m \tag{18. 1}$$

$$\delta_1^p L = \sum_{\sigma=1}^{m-p+1} \sigma \lambda_\sigma \left(\frac{1}{\lambda_1} \delta_1^{p+\sigma-1} L \right) \qquad p=1, \ldots, m. \tag{18. 2}$$

It will then follow that equations (15) are a consequence of (13) whenever $\lambda_1 \neq 0$. Equation (18. 2) is readily verified since the right hand side is zero unless $p+\sigma=2$; but since $p \geqq 1$ and $\sigma \geqq 1$, this can only happen for $p=\sigma=1$, as required. To verify (18. 1), substituting from the definition of Λ_p given in (15), one obtains

$$\sum_{\sigma=1}^{m-p+1} \sum_{r=p+\sigma-1}^{m} \sum_{s=1}^{r-(p+\sigma-1)+1} \frac{\partial L}{\partial u_r^i} x_s^i r! \, \sigma \lambda_\sigma A_{r-(p+\sigma-1)}^{s-1},$$

and interchanging orders of summation yields

$$\sum_{r=p}^{m} \frac{\partial L}{\partial u_r^i} r! \sum_{s=1}^{r-p+1} x_s^i \sum_{\sigma=1}^{(r-p)-s+2} \sigma \lambda_\sigma A_{(r-p)-\sigma+1}^{s-1}.$$

By (11), this is Ω_p as required.

It follows that applying (17) to (13) will yield (14) whenever $\lambda_1 \neq 0$. But (14) can be integrated since we may write it in the form

$$\left\{\lambda_1 \frac{\partial L}{\partial \lambda_p} - \frac{\partial \lambda_1}{\partial \lambda_p} L\right\} \lambda_1^{-2} = \frac{\partial}{\partial \lambda_p} \{\lambda_1^{-1} L(u_r^i)\} = 0.$$

An integration along a curve[2]) in the λ_r space from $\lambda_r = \delta_r^1$ to an arbitrary λ_r, avoiding $\lambda_1 = 0$, yields

$$L(u_r^i) = \lambda_1 (x_r^i) \quad \text{for} \quad \lambda_1 \neq 0 \tag{19}$$

since, by (8.1) and (12), u_r^i reduces to x_r^i when $\lambda_r = \delta_r^1$.

But (19) is the required formula for parameter invariance since, by setting $\lambda_r = \frac{1}{r!}\frac{d^r \tau}{dt^r}$, equation (19), when fully expanded, becomes

$$L\left\{\sum_{s=1}^{r} \frac{d^s x}{d\tau^s} r! \sum_{(q)} \frac{1}{q_1! \dots q_m!} \left(\frac{\frac{d\tau}{dt}}{1!}\right)^{q_1} \dots \left(\frac{\frac{d^m \tau}{dt^m}}{m!}\right)^{q_m}\right\} = \frac{d\tau}{dt} L\left\{\frac{d^r x^i}{d\tau^r}\right\}.$$

But the arguments on the left are Faa di Bruno's formula [10] for the r^{th} derivative of the composite function $x\{\Phi(t)\}$, verifying the parameter invariance q. e. d.

Multiple integrals

In this case the Lagrangian $L\{x^i, \dot{x}_\alpha^i\}$ is a function of the $n + nm$ variables $\dot{x}_\alpha^i = \partial x^i / \partial t^\alpha$ and x^i where $i = 1, \dots, n$ and $\alpha = 1, \dots, m \leqq n$. As indicated in the introduction we wish to show that if $L\{x^i, \dot{x}_\alpha^i\}$ satisfies the m^2 partial differential equations

$$\frac{\partial L}{\partial \dot{x}_\alpha^i} \dot{x}_\beta^i = \delta_\beta^\alpha L \tag{6}$$

then L satisfies the functional equation

$$L\{x^i, \lambda_\alpha^\beta \dot{x}_\beta^i\} = |\lambda_\beta^\alpha| L\{x^i, \dot{x}_\alpha^i\} \quad \text{for} \quad |\lambda_\beta^\alpha| \neq 0 \tag{5}$$

and hence leads to parameter invarient multiple integrals.

This has already been shown by RUND [7], and we briefly outline his method in order to compare it with the method of this note. Let $[\lambda_\beta^\alpha]$ be an arbitrary $m \times m$ matrix and set $J = |\lambda_\beta^\alpha|$. Assume $J \neq 0$ and let Λ_α^β denote the cofactor of λ_β^α in J. Replace $\dot{x}_\beta^i$ throughout (6) by $u_\beta^i = \lambda_\beta^\gamma \dot{x}_\gamma^i$ and multiply through by Λ_μ^β. The result may be written in the form

$$J \frac{\partial L}{\partial u_\alpha^i} \dot{x}_\nu^i = \Lambda_\nu^\alpha L.$$

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[2]) We are using the fact that $\int_{P_1}^{P_2} \frac{\partial F}{\partial \lambda_r} d\lambda_r = F\big|_{P_1}^{P_2}$, (summation on r). Here P_2 is the point with coordinates $[\lambda_1 \neq 0, \lambda_2, \dots, \lambda_m]$ and P_1 has coordinates $[1, 0, \dots, 0]$. We are assuming that the points P_1 and P_2 can be joined by a curve along which $\lambda_1 \neq 0$, and for which F is continuously differentiable. (This is the condition which should replace the original condition on L concerning differentiability etc.).

But $\partial J/\partial \lambda_\alpha^\nu = \Lambda_\nu^\alpha$ while $\partial L/\partial \lambda_\alpha^\nu = (\partial L/\partial u_\alpha^i) x_\nu^i$ so that this becomes

$$\frac{\partial}{\partial \lambda_\alpha^\nu}\{J^{-1} L\} = 0 \quad \text{provided} \quad J \neq 0.$$

Rund now integrates this equation along a path in the λ_β^α space from the initial point $\lambda_\beta^\alpha = \delta_\beta^\alpha$ to an arbitrary λ_β^α, assuming that along the path $J \neq 0$. Since $J = 1$ at the initial point, the integrated equation becomes (5).

Martin [4], pg. 120, has raised some questions concerning this procedure, mainly concerning the path of integration: is every point in λ_β^α space, even if $J = |\lambda_\beta^\alpha| > 0$, connected to the point δ_β^α by a path along which $J \neq 0$? We will not attempt to answer this questions here.

In this note, the equation (5) is derived from (6) in a way which avoids such problems in the λ_β^α space. Instead we assume simply that (6) holds in a region of the $\dot{x}_\beta^i$ space for which the rank of the matrix $[\dot{x}_\beta^i]$ is m. Since the system (6) is not affected by interchanging the columns of the matrix $[\dot{x}_\beta^i]$ we assume for simplicity that the variables have been re-labeled such that the first m columns $[\dot{x}_\beta^\alpha]$ have non-vanishing determinant in the region in question. We write the matrix in the partitioned form

$$[\dot{x}_\alpha^i] = [\dot{x}_\beta^\alpha : \dot{x}_\beta^p]$$

where $p = m+1, \ldots, n$. (Exception is made if $m = n$. This case is simple and will be treated below). Since $\dot{x}_\beta^\alpha \neq 0$ by assumption, let $[X_\beta^\alpha] = [\dot{x}_\beta^\alpha]^{-1}$, the inverse matrix.

We now perform the following change of variables on the system (6):

$$(20) \qquad \begin{cases} \text{leave } \dot{x}_\beta^\alpha \text{ unchanged} \\ \text{replace } \dot{x}_\beta^p \text{ by } u_\beta^p = X_\beta^\gamma \dot{x}_\gamma^p, \quad \text{for} \quad p = m+1, \ldots, n. \end{cases}$$

Since the matrix $[X_\beta^\gamma]$ has an inverse, this transformation is one-one. Finally, replace $L(x^i, \dot{x}_\alpha^i)$ by

$$(21) \qquad L(x^i, \dot{x}_\alpha^i) = |\dot{x}_\beta^\alpha|\, l\{x^i, \dot{x}_\beta^\alpha, u_\beta^p\}.$$

Since l is arbitrary and $|\dot{x}_\beta^\alpha| \neq 0$, this places no restriction on the system. In the remainder of this section *it will be shown that under the transformation* (20) *and* (21) *the system of equations* (6) *are equivalent to the system*

$$(22) \qquad \frac{\partial l}{\partial \dot{x}_\alpha^\nu} = 0$$

so that the general solution of (6) *is given by*

$$(23) \qquad L\{x^i, \dot{x}_\alpha^i\} = |\dot{x}_\beta^\alpha|\, l\{x^i, X_\beta^\gamma \dot{x}_\gamma^p\} \qquad p = m+1, \ldots, n,$$

for arbitrary l. In the case $m = n$, this reduces to

$$L\{x^i, \dot{x}_j^i\} = |\dot{x}_j^i|\, l\{x^i\}.$$

Further, any function L of the form (23) *satisfies the functional equation* (5), *given* $|\lambda_\beta^\alpha| \neq 0$, *as was required.*

In order to verify the assertion, let $\Delta=|\dot{x}^\alpha_\beta|$. Then, if C^β_α denotes the cofactor of $\dot{x}^\alpha_\beta$ in Δ, it is clear that

$$\frac{\partial\Delta}{\partial\dot{x}^\alpha_\gamma}\dot{x}^\gamma_\beta = C^\alpha_\gamma\dot{x}^\gamma_\beta = \delta^\alpha_\beta\,\Delta. \tag{24}$$

Also, since $X^\sigma_\mu\dot{x}^\mu_\tau=\delta^\sigma_\tau$, it follows that

$$\frac{\partial X^\sigma_\mu}{\partial\dot{x}^\gamma_\alpha}\dot{x}^\mu_\tau = -X^\sigma_\mu\frac{\partial\dot{x}^\mu_\tau}{\partial\dot{x}^\gamma_\alpha} = -X^\sigma_\gamma\,\delta^\alpha_\tau,$$

and multiplying by $X^\tau_\nu\dot{x}^\gamma_\beta$ yields

$$\frac{\partial X^\sigma_\nu}{\partial\dot{x}^\gamma_\alpha}\dot{x}^\gamma_\beta = -X^\sigma_\gamma X^\alpha_\nu\dot{x}^\gamma_\beta = -\delta^\sigma_\beta X^\alpha_\nu. \tag{25}$$

We now write (6) in the form

$$\frac{\partial L}{\partial\dot{x}^\gamma_\alpha}\dot{x}^\gamma_\beta + \frac{\partial L}{\partial\dot{x}^p_\alpha}\dot{x}^p_\beta = \delta^\alpha_\beta L$$

summation on $\gamma=1, \ldots, m$ and $p=m+1, \ldots, n$. Replacing L by $L=\Delta l$ as in (21), this equation becomes

$$\frac{\partial\Delta}{\partial\dot{x}^\gamma_\alpha}\dot{x}^\gamma_\beta\, l+\Delta\frac{\partial l}{\partial\dot{x}^\gamma_\alpha}\dot{x}^\gamma_\beta+\Delta\frac{\partial l}{\partial u^p_\nu}\frac{\partial X^\sigma_\nu}{\partial\dot{x}^\gamma_\alpha}\dot{x}^\gamma_\beta\,\dot{x}^p_\sigma+\Delta\frac{\partial l}{\partial u^p_\nu}X^\alpha_\nu\,\dot{x}^p_\beta = \delta^\alpha_\beta\,\Delta l. \tag{26}$$

Using (24), the first term cancels with the right hand side, while by (25) the third and fourth terms cancell. Since $\Delta\neq0$, this equation reduces to

$$\frac{\partial l}{\partial\dot{x}^\gamma_\alpha}\dot{x}^\gamma_\beta = 0$$

and multiplying by X^β_ν yields (22). If $m=n$, then the variables u^p_β are missing in (26), but the result is the same. It remains to show that if L is of the form specified in (23) then the functional equation (5) is satisfied. This will be shown with more generality in the next section.

The matrix functional equation

In this section we discuss a generalization of the functional equation treated (with differentiability assumptions) in the previous section. As before $[\lambda^\alpha_\beta]$ denotes an $m\times m$ matrix, $[x^i_\alpha]$ an m(row)xn(column) matrix where now n may be less than m, and f is a real valued function on rectangular $m\times n$ matrices, satisfying

$$f\{[\lambda^\alpha_\beta][x^i_\alpha]\} = |\lambda^\alpha_\beta|^k f\{[x^i_\alpha]\} \tag{7}$$

We *do not* give a complete solution of (7) since this seems to require many distinctions in the possible ways in which the rank of the matrix $[x^i_\alpha]$ can be m. This is analogous to the case $F\{\lambda x, \lambda y\} = \lambda^k F(x, y)$ which requires the four distinct cases (ACZÉL [8] p. 160): $y \neq 0$, $y = 0$ and $x \neq 0$, $y = x = 0$ and $k \neq 0$, $y = x = 0$ and $k = 0$. Here, the matrix $[x_\alpha^i]$ can have rank m in $\dfrac{n!}{m!(n-m)!}$ ways (the number of ways of choosing m distinct columns from the total of n columns).

In this section we merely show that

$$(27) \qquad f\{[x_\alpha^i]\} = |x_\beta^\alpha|^k F\{X_\beta^\gamma x_\gamma^p\}, \qquad |x_\beta^\alpha| \neq 0$$

for arbitrary F if the rank of the matrix $[x_\beta^\alpha : x_\beta^p]$ is m and $|x_\beta^\alpha|$ is the non-vanishing determinant, while

$$f\{[x_\alpha^i]\} = 0$$

if the rank of the matrix $[x_\alpha^i]$ is less than m, while $k \neq 0$. Here X_β^γ and x_γ^p have the same interpretation[3]) as in the previous section, and F is an arbitrary function of the $(n-m)m$ variables $X_\beta^\gamma x_\gamma^p$. In the case $m = 1$ equation (27) reads.

$$f\{x^1, x^2, \dots, x^n\} = (x^1)^k F\left\{\frac{x^2}{x^1}, \dots, \frac{x^n}{x^1}\right\}, \qquad x^1 \neq 0.$$

The above assertion may be verified as follows. If the rank of $[x_\alpha^i]$ is less than m, then $[x_\alpha^i]$ is row-equivalent to a matrix with zero first row. Alternatively there exists a non singular matrix $[\lambda_\beta^\alpha]$ such that the first row in $[\lambda_\beta^\alpha][x_\alpha^i]$ is zero. But then if $[\sigma_\beta^\alpha]$ is the matrix

$$[\sigma_\beta^\alpha] = \begin{bmatrix} \sigma & 0 \dots 0 \\ 0 & 1 \dots 0 \\ \vdots & \vdots \quad \vdots \\ 0 & 0 \dots 1 \end{bmatrix},$$

the matrix $[\sigma_\beta^\alpha][\lambda_\beta^\alpha][x_\alpha^i]$ is independent of σ. It follows by substituting in (7) that the right hand side of the equation

$$f\{[\sigma_\beta^\alpha][\lambda_\beta^\alpha][x_\alpha^i]\} = \sigma^k |\lambda_\beta^\alpha|^k f\{[x_\alpha^i]\}, \qquad k \neq 0$$

is independent of σ and since $|\lambda_\beta^\alpha| \neq 0$, this implies $f\{[x_\alpha^i]\} = 0$.

If the matrix $[x_\beta^i] = [x_\beta^\alpha : x_\beta^p]$ has rank m (which implies that $n \geqq m$), and if further $|x_\beta^\alpha| \neq 0$, the $[\lambda_\beta^\alpha]$ in (7) can be chosen as the inverse matrix $[X_\beta^\alpha] = [x_\beta^\alpha]^{-1}$, resulting in the equation

$$f\{[X_\beta^\alpha][x_\beta^\alpha : x_\beta^p]\} = f\{[\delta_\beta^\alpha : X_\beta^\gamma x_\gamma^p]\} = |X_\beta^\alpha|^k f\{[x_\alpha^i]\}.$$

Since $|X_\beta^\alpha|^k = |x_\beta^\alpha|^{-k}$ it follows that f has the form specified in (27).

Conversely, if f is given by (27) then

$$(28) \qquad f\{[\lambda_\beta^\alpha][x_\alpha^i]\} = f\{[\lambda_\beta^\gamma x_\gamma^\alpha : \lambda_\beta^\gamma x_\gamma^p]\} = |\lambda_\beta^\gamma x_\gamma^\alpha|^k F\{Y_\beta^\gamma y_\gamma^p\}$$

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[3]) This is but a slight generalization of the results of W. GROSZ [6], pg. 82—83, for $k = 1$. In his notation $p_0 = |x_\beta^\alpha|$ and $p_{h,k} = -p_0 X_h^\gamma x_\gamma^{m+k}$.

where $y^p_\gamma = \lambda^\nu_\gamma x^p_\nu$ and $[Y^\gamma_\beta] = [\lambda^\nu_\beta x^\gamma_\nu]^{-1} = [x^\alpha_\beta]^{-1}[\lambda^\alpha_\beta]^{-1}$. Therefore

$$[Y^\gamma_\beta y^p_\gamma] = [X^\alpha_\beta][\lambda^\alpha_\beta]^{-1}[\lambda^\alpha_\beta][x^p_\beta] = [X^\gamma_\beta x^p_\gamma].$$

Hence (28) becomes

$$f\{[\lambda^\alpha_\beta][x^i_\alpha]\} = |\lambda^\alpha_\beta|^k\, |x^\alpha_\beta|^k\, F\{X^\gamma_\beta x^p_\gamma\} = |\lambda^\alpha_\beta|^k f\{[x^i_\alpha]\}$$

which verifies the assertion that the function f in (27) satisfies (7), and also completes the proof of the previous section.

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[9] For example E. NETTO Combinatorik, *N. Y.,* (1927) p. 60.

[10] For example C. JORDAN, Calculus of Finite Differences, *Sopron*, 1939, p. 33; or M. A. MCKIERNAN, On the n^{th} derivative of composite functions, *Math. Monthly,* **63** (1956), 331—333.

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