# Varieties of groups defined by a single law

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The object of this paper is to show that certain varieties of groups may be defined by a single law in terms of multiplication and inversion. G. HIGMAN and B. H. NEUMANN prove a similar result in [2] for groups defined in terms of right division.

**1.** The variety of all groups is defined in terms of multiplication and inversion by the three laws

a $\quad xx^{-1}=yy^{-1}=e$, say,

b $\quad xe=x=ex$,

c $\quad (xy)z=x(yz)$.

**Lemma 1.** *The variety of all groups is defined by the four laws*

1 $\quad xx^{-1}=yy^{-1}=e$,

2 $\quad xe^{-1}=x=e(ex^{-1})^{-1}$,

3 $\quad [x(ey^{-1})^{-1}](ez^{-1})^{-1}=x\{e[y(ez^{-1})^{-1}]^{-1}\}^{-1}$,

4 $\quad [(xx^{-1})y]y^{-1}=e$.

PROOF. It is not difficult to show that a, b, c imply 1, 2, 3, 4. It remains to prove the converse.

1 and 4 give

$$(ey)y^{-1}=e=yy^{-1}$$

and so by 2

5 $\quad (ey)[e(ey^{-1})^{-1}]^{-1}=y[e(ey^{-1})^{-1}]^{-1}$.

Multiply 5 on the right by $(ey^{-1})^{-1}$ to give

$$\{(ey)[e(ey^{-1})^{-1}]^{-1}\}(ey^{-1})^{-1}=\{y[e(ey^{-1})^{-1}]^{-1}\}(ey^{-1})^{-1}$$

which, by 3, implies

6 $\quad (ey)\{e[(ey^{-1})(ey^{-1})^{-1}]^{-1}\}^{-1}=y\{e[(ey^{-1})(ey^{-1})^{-1}]^{-1}\}^{-1}$.

Apply 1 to 6 to give

$$(ey)e^{-1}=ye^{-1}$$

and so by 2

$$ey=y$$

which is the second part of b.

In

$$y=ey$$

put $y=e^{-1}$ to give

$$e^{-1}=ee^{-1}=e$$

by 1. Thus 2 gives

$$xe=x,$$

the first part of b.

Apply b and $e^{-1}=e$ to 2 to give

$$x=(x^{-1})^{-1}.$$

Apply b, 1 and $x=(x^{-1})^{-1}$ to 3 to give c. Since 1 is a this completes the proof.

The proof of the following lemma is similar to that given by R. A. BULL in [1].

**Lemma 2.** *If $A$ is a word not involving the variables $x, y, z$ and $t$, then the law*

$$\text{(G)} \qquad \{[(yy^{-1})(zx^{-1})^{-1}](tz^{-1})^{-1}\}(At^{-1})^{-1}=x,$$

*implies the laws* 1, 2, 3 *and* $A=e$.

PROOF. In (G) put $y=xx^{-1}$, $t=A$ and $z=x$ to give

$$\text{(1)} \qquad x=[\{[(xx^{-1})(xx^{-1})^{-1}](xx^{-1})^{-1}\}(Ax^{-1})^{-1}](AA^{-1})^{-1}.$$

Putting $t=y=z=x$ in (G) gives

$$x=\{[(xx^{-1})(xx^{-1})^{-1}](xx^{-1})^{-1}\}(Ax^{-1})^{-1}$$

and so by (1)

$$\text{(2)} \qquad x=x(AA^{-1})^{-1}.$$

Put $x=z=t=A$ in (G) to give

$$A=\{[(yy^{-1})(AA^{-1})^{-1}](AA^{-1})^{-1}\}(AA^{-1})^{-1}$$

and then apply (2) three times to give

$$\text{(3)} \qquad A=yy^{-1}.$$

In (3) put $y=x$ to give

$$A=xx^{-1}$$

and hence

$$yy^{-1}=xx^{-1}.$$

Writing $xx^{-1}=e$ we have 1 and $A=e$.

$A=e$ implies, with (2), that

$$x=x(ee^{-1})^{-1}$$

and since, by 1,

$$ee^{-1}=e$$

we have

(4) $$xe^{-1}=x,$$

which is the first part of 2.

In (G) put $y=z=t=x$ and apply 1 to give

$$e(ex^{-1})^{-1}=x$$

which with (4) completes 2.

In (G) put $z=e$, $t=et^{-1}$ and apply 1 and 2 to give

(5) $$[x(et^{-1})^{-1}]t^{-1}=x.$$

Multiply (G) on the right by $z^{-1}$, put $t=e$ and apply 1 and 2 to give

(6) $$\{[e(zx^{-1})^{-1}](ez^{-1})^{-1}\}z^{-1}=xz^{-1}.$$

Apply (5) to (6) to give

(7) $$e(zx^{-1})^{-1}=xz^{-1}.$$

Multiply (G) on the right by $t^{-1}$ and apply 1 and (5) to give

(8) $$[e(zx^{-1})^{-1}](tz^{-1})^{-1}=xt^{-1}.$$

Put $t=t(ez^{-1})^{-1}$ in (8) and apply (5) to give

$$[e(zx^{-1})^{-1}]t^{-1}=x[t(ez^{-1})^{-1}]^{-1}$$

and then apply (7) to give

(9) $$(xz^{-1})t^{-1}=x\{e[(ez^{-1})t^{-1}]^{-1}\}^{-1}.$$

In (9) put $z=ez^{-1}$, $t=et^{-1}$ and apply 2 to give

$$[x(ez^{-1})^{-1}](et^{-1})^{-1}=x\{e[z(et^{-1})^{-1}]^{-1}\}^{-1}$$

which is 3, and so completes the proof.

**2.** Consider the class of groups defined as a subvariety of the variety of all groups by the laws $w_i=w_i'$, $i=1,\dots,r$ where $w_i$ and $w_i'$ are words in variables $x_1,\dots,x_n$, multiplication and inversion. Following [2], we may assume, without loss of generality, that $r=1$ and $w_1'=x_1x_1^{-1}$.

That is, we will consider the subvariety singled out by the law

$$\hat{w}=\hat{w}(x_1,\dots,x_n)=x_1x_1^{-1},$$

(in addition to the group laws).

Define the mapping $\theta$ from the class of all words $w$ in variables $x_1,\dots,x_n$ into itself, inductively as follows,

if $w=x_i$ then $\theta(w)=x_i$,

if $w=w_1w_2$ then $\theta(w)=\theta(w_1)[(x_1x_1^{-1})\theta(w_2)^{-1}]^{-1}$,

if $w=w_1^{-1}$ then $\theta(w)=(x_1x_1^{-1})\theta(w_1)^{-1}$.

It follows, by induction, from the definition of $\theta$ that:

(i) if $G$ is a group the the law $w=\theta(w)$ holds in $G$ for every word $w$, and

(ii) if $G'$ is an algebraic system with a binary operation, multiplication and a unary operation, inversion and $G'$ satisfies the law $x_1x_1^{-1}=x_2x_2^{-1}$ then for any word $w=w(x_1, \dots, x_n)$, $G'$ satisfies the law

$$\theta(w)(x_1x_1^{-1}, x_1x_1^{-1}, \dots, x_1x_1^{-1})=x_1x_1^{-1}.$$

Hence by (i) we may assume that the subvariety under consideration is defined by the law

$$\theta(\hat{w})=\theta(\hat{w}(x_1, \dots, x_n))=x_1x_1^{-1}$$

(in addition to the group laws) and write $\overline{w}=\theta(\hat{w})$.

**Theorem.** *The class of groups defined as a subvariety of the variety of all groups by the law* $\overline{w}=\overline{w}(x_1, \dots, x_n)=x_1x_1^{-1}$ *is the variety of algebraic systems defined by the single law*

$$\text{(H)} \qquad \{[(yy^{-1})(zx^{-1})^{-1}](tz^{-1})^{-1}\}\{[\{[(ss^{-1})r]r^{-1}\}(\overline{w})^{-1}]t^{-1}\}^{-1}=x.$$

Proof. It is not difficult to show that a, b, c and $\overline{w}(x_1, \dots, x_n)=x_1x_1^{-1}$ imply (H). It remains to prove the converse.

By Lemma 2 with $A=\{[(ss^{-1})r]r^{-1}\}(\overline{w})^{-1}$, (H) implies 1, 2, 3 and

$$\text{(A)} \qquad \{[(ss^{-1})r]r^{-1}\}(\overline{w})^{-1}=e.$$

In (A) put $x_i=e$, $i=1, \dots, n$ then by 1, i. e. $xx^{-1}=yy^{-1}=e$ and the fact that $\overline{w}=\theta(\hat{w})$ it follows that $\overline{w}(e, \dots, e)=e$.

Thus (A) reduces to

$$\text{(B)} \qquad \{[(ss^{-1})r]r^{-1}\}e^{-1}=e.$$

Apply 2 to (B) to give

$$[(ss^{-1})r]r^{-1}=e$$

which is law 4 of Lemma 1.

Thus by Lemma 1, (H) implies a, b and c. From (A) and 4 it follows that

$$e(\overline{w})^{-1}=e$$

which by b implies

$$\text{(C)} \qquad (\overline{w})^{-1}=e.$$

But a, b and c imply $e^{-1}=e$ and $(x^{-1})^{-1}=x$ and so from (C) it follows that

$$\overline{w}=e,$$

that is $\overline{w}=\overline{w}(x_1, \dots, x_n)=x_1x_1^{-1}$, which completes the proof.

By putting

$$A=[(ss^{-1})r]r^{-1}$$

in (G) we obtain a single law defining the variety of all groups in terms of multiplication and inversion. The law so obtained has 11 variables on the left hand side. However the law

$$[\{y[z(xv)]\}^{-1}(yz)]^{-1}v^{-1}=x,$$

which contains only 7 variables on the left hand side, can be shown to define the variety of all groups; (the proof is straightforward and is omitted).

Any single law which defines the variety of all groups in terms of multiplication and inversion must have on one side a single variable and on the other side a word with an odd number of variables. By considering such laws when the longer side contains 3 or 5 variables it can be seen that none of these laws define the variety of all groups. Thus the law given above is the shortest single law defining the variety of all groups.

It remains an open question whether the variety of all groups can be defined in terms of multiplication and inversion by a single law involving only 3 different variables.

## References

[1] R. A. BULL, Some Axioms for varieties of groups, *Thesis, University of Manchester*, 1961.
[2] G. HIGMAN and B. H. NEUMANN, Groups as groupoids with one law, *Publ. Math. Debrecen* **2** (1952), 215—221.

*(Received March 31, 1965.)*