On the number of independent complete subgraphs

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At the Symposium on theory of graphs in Smolenice P. ERDŐS has asked the following problem (see [1]):

Let $l > 1$, assume that G has ln vertices and every vertex has valency $\geqq (l-1)n$. Is it then true that G contains n independent complete l-gons (i. e. no two of which have a common vertex).

This conjecture was not still proved in general, but it was proved for some values of l at arbitrary n. For $l=2$ it is a consequence of a well known theorem of Dirac and for $l=3$ it was proved by CORRÁDI and HAJNAL in [2]. For $l=4$ CORRÁDI proved that G contains $n-1$ independent complete quadrilaterals and an n-th complete quadrilateral independent of the previous ones with perhaps one edge missing.

In this paper one goes to the problem from another side. The problem is solved for arbitrary l, but only for the values $n=2$ and $n=3$. (For $n=1$ the conjecture holds trivially.) Let $n=2$. The graph G has $2l$ vertices and every vertex has valency $\geqq 2l-2$. So in the complement $\bar{G}$ of G every vertex has valency $\leqq 1$, i. e. every component of $\bar{G}$ is either a single edge with its end vertices, or an isolated vertex. Therefore we can number the vertices of $\bar{G}$ as follows. If two vertices are joined by an edge in $\bar{G}$, we assign the number 1 to one of them and the number 2 to the other. Then we decompose the set of isolated vertices of $\bar{G}$ into two disjoint sets of equal cardinality (this is possible, as the number of isolated vertices is even), to all vertices

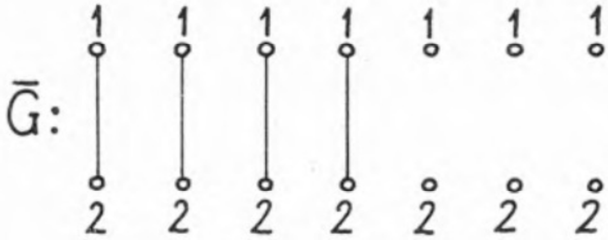

Fig. 1

of one of the sets we assign the number 1, to all vertices of the other set we assign the number 2 (see fig. 1). The set of all vertices numbered by 1 (resp. by 2) consists of l vertices and is evidently independent in $\bar{G}$, therefore it generates a complete l-gon in G.

Let $n=3$. The graph G has $3l$ vertices and every vertex has valency $\geqq 3l-3$. So in the complement $\bar{G}$ of G every vertex has valency $\leqq 2$; i. e. every component of $\bar{G}$ is either a circuit, or a path, or an isolated vertex. Now we shall construct the set K as follows. If a circuit in $\bar{G}$ has the number of vertices congruent with 2 modulo 3, we choose two neighbouring vertices of that circuit and these will be contained in K. If a circuit in $\bar{G}$ has the number of vertices congruent with 1 modulo 3, we choose one vertex of that circuit and it will be contained in K. All vertices of $\bar{G}$ which are not contained in any circuit will be also contained in K. Other vertices are not contained in K. All components of the subgraph of $\bar{G}$ generated by the set K are paths or isolated vertices and the cardinality of K is divisible by three (as the number of vertices not contained in K is evidently divisible by three). So it is possible to number all vertices of K by the numbers 1, 2 and 3, so that neighbouring vertices in $\bar{G}$ might have different numbers and the sets of vertices equally numbered have equal cardinality. When a circuit has the number of vertices congruent with 0 modulo 3, we can go around it, starting with an arbitrary vertex, and number its vertices mutually by 1, 2 and 3. Now let a_1, a_2, a_3 be some permutation of the numbers 1, 2, 3. When a circuit has the number of vertices congruent with 1 modulo 3 and its vertex belonging to K has number a_1, we start in one vertex neighbouring with it and go along the circuit (with exception of the vertex belonging to K) numbering its vertices mutually by a_2, a_1, a_3. When a circuit has the number of vertices congruent with 2 modulo 3 and its vertices belonging to K have numbers a_1 and a_2, we start in one vertex neighbouring with the vertex numbered by a_1 and go along the circuit (with exception of the vertices belonging to K) numbering its vertices

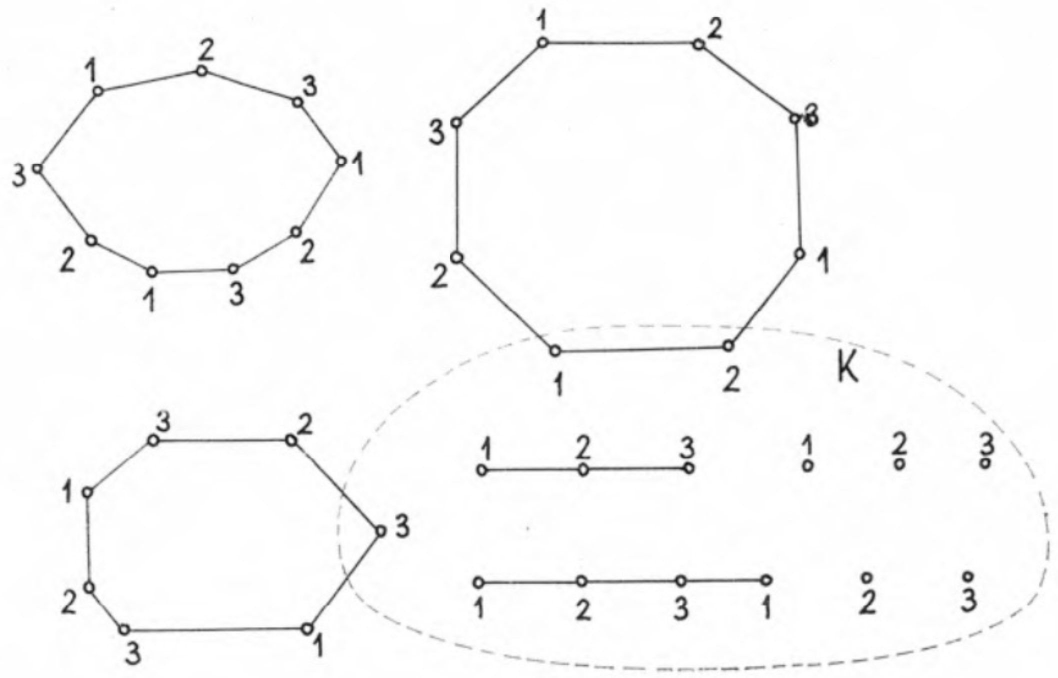

Fig. 2

mutually by a_2, a_3, a_1. This numbering is seen on fig. 2. In such a numbering the set of vertices numbered by the same number has again the cardinality l and is independent in $\overline{G}$, therefore it generates a complete l-gon in G.

References

[1] Theory of Graphs and Its Applications. Proceedings of the Symposium held in Smolenice, June 1963. *Praha* (1964).
[2] K. Corrádi—A. Hajnal, On the number of independent circuits in a graph, *Acta Math. Acad. Sci. Hungar.* **14** (1963), 423—439.

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