The block-cutpoint-tree of a graph

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It has often been observed that a connected graph with quite a few cutpoints bears a resemblance to a tree (see Figure 1A). In this note this idea will be made definite by associating with every connected graph G a tree $T(G)$ which displays this resemblance. The definition of $T(G)$ is a natural extension of the definitions of the block-graph $B(G)$ and the cutpoint-graph $C(G)$, defined in [1]. We recall the following definitions from that paper. A *cutpoint* c of a connected graph G is a point whose removal results in a disconnected graph. A *block* B of G is a maximal connected subgraph of G which has no cutpoints. The *block-graph* $B(G)$ is the graph whose points are the blocks of G and in which two points are adjacent whenever the corresponding blocks have a point (which must be a cutpoint of G) in common (see Figure 1B). The*cutpoint-graph* $C(G)$ is the graph whose points are the cutpoints of G, and in which two points are adjacent provided they lie on a common block of G (see Figure 1C).

We now define the *block-cutpoint-tree* $T(G)$ as the graph whose set of points is the union of the set of blocks and the set of cutpoints of G, and in which two points are adjacent only if one corresponds to a block B of G, and the other to a cutpoint c of G and $c \in B$ in G (see Figure 1D).

Theorem 1. *If G is connected, $T(G)$ is a tree.*

PROOF. If $T(G)$ has a cycle, this cycle must contain at least two blocks of G, say B_1 and B_2, as points. There are two paths from B_1 to B_2 in $T(G)$, and therefore also in G. But then B_1 and B_2 are contained in the same block of G, which is nonsense. Hence $T(G)$ is acyclic. As the connectedness of G implies that of $T(G)$, we conclude that $T(G)$ is a tree.

A graph is *bicolorable* if its points *can be* divided into two classes (colors), such that points of the same class are never adjacent. A graph is *bicolored* if its points *are* divided into two classes, such that points of the same class are never adjacent. A bicolorable connected graph can be bicolored in only one way, except for the naming of the colors. Hence we do not need to distinguish between bicolorable and bicolored connected graphs. The block-cutpoint tree $T(G)$ of a connected graph G is bicolorable by assigning one color (say blue) to the points corresponding to the blocks of G, and another color (say coral) to the points

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corresponding to the cutpoints of G. As stated above, this is essentially the only way this tree can be bicolored. We also note that the endpoints of $T(G)$ are all colored blue.

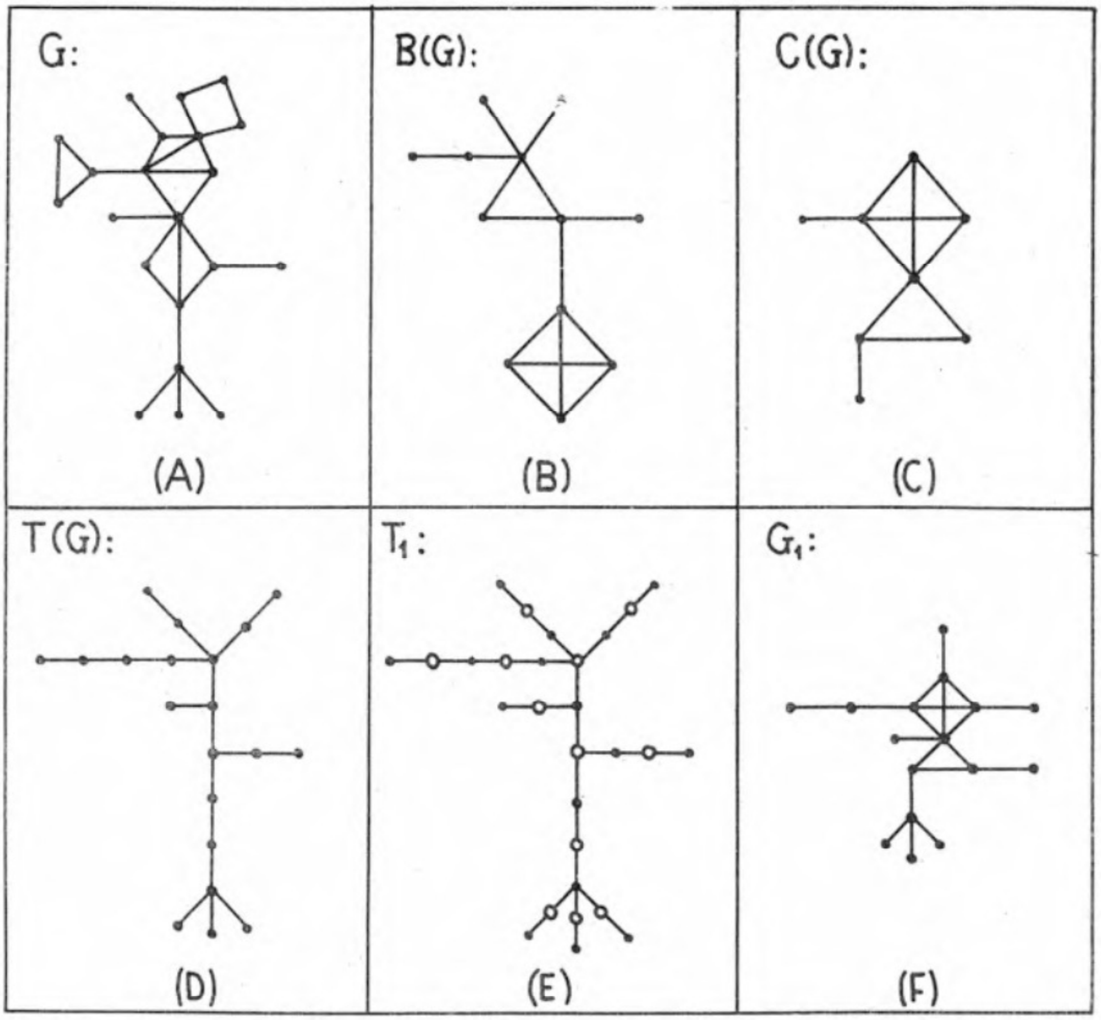

Fig. 1

We now define a *bc-tree* as a bicolored tree in which every endpoint has the same color. Alternately, we may define a bc-tree to be a tree T which has a point u, such that for every two endpoints e_1, e_2 of T, $d(u, e_1) \equiv d(u, e_2) \bmod 2$, where $d(u, v)$ is the distance between u and v.

Theorem 2. *Every bc-tree is the block-cutpoint-tree of a connected graph and conversely.*

PROOF. We only need to show the direct part of the theorem. We shall construct a graph G, corresponding to an arbitrary bc-tree T such that $T \cong T(G)$ (is isomorphic).

Let T be colored blue and coral with blue endpoints. We first construct a bicolored tree T_1 which has all the points and lines of T and in addition for every endpoint b_i of T a new coral point c_i and a new line b_ic_i, as shown in Figure 1E. We now construct a graph G_1, whose points are the coral points of T_1, and in which two points are adjacent whenever the distance between them in T_1 is 2. It is easy to verify that $T(G_1) \cong T$; see Figure 1F.

We briefly note some properties of the graph G_1 constructed in the preceding proof. If G is another graph with $T(G) \cong T$, then the number of points of G_1 is less than or equal to the number of points of G. Also, G_1 is the only graph with complete blocks with this property. The set of graphs G_1 contains the one-point graph and further all connected graphs with complete blocks in which every point is a cutpoint or an endpoint, except for the 2-point complete graph. If every (complete) block of G_1 is replaced by one of its Hamilton cycles, we obtain a graph H which is also minimal as to number of lines among all graphs G for which $T(G) \cong T$. However, H is not unique, as different choices of Hamilton cycles may lead to non-isomorphic graphs. Finally, the graph obtained from G_1 by deleting its endpoints is its cutpoint-graph, and that of all graphs G with $T(G) \cong T$.

Graphs with the same block-cutpoint-tree also have the same block-graph and the same cutpoint-graph. The converse also holds

Theorem 3. *Two connected graphs have the same block-graph if and only if they have the same block-cutpoint-tree.*

PROOF. The blocks of $B(G)$ correspond to the cutpoints of G. We construct a new graph G^* whose points are the points of $B(G)$ and the blocks of $B(G)$, with two points adjacent in G^* whenever one is a block in $B(G)$ and the other a point in that block. Clearly, $G^* \cong T(G)$.

Conversely, construct a graph G' which has as points the blue points of $T(G)$, and let two points be adjacent in G' if and only if they are adjacent to the same coral point in $T(G)$. Then $G' \cong B(G)$. As both constructions are unique, the correspondence is one-to-one.

Corollary. *There is a one-to-one correspondence between bc-trees and connected graphs with complete blocks, such that in corresponding tree and graph the number of blue points of the one equals the number of points of the other.*

PROOF. It is proved in [1] that every connected graph with complete blocks is a block-graph. The corollary now follows from Theorem 2. It can also be proved directly by the method of proof of Theorem 3.

In Figure 2, we show the diagrams of all bc-trees with 4 blue points and the corresponding block-graphs with 4 points.

We now proceed to count the number of bc-trees. The methods and notation used are those of HARARY and PRINS [2]. We review briefly the main concepts: S_n is the symmetric group of degree n; $Z(S_n)$ is its cycle index, $Z(S_n, f(x, y))$ is obtained by substituting $f(x^k, y^k)$ for each variable t_k in $Z(S_n)$, and $Z(S_\infty)$ is symbolic notation for $\sum_0^\infty Z(S_n)$. We state the well-known identity given by PÓLYA:

$$Z(S_\infty, f(x, y)) = \exp \sum_1^\infty f(x^r, y^r)/x^r y^r$$

We shall enumerate bc-trees in the form of a generating series,

$$t(x, y) = \sum_{m=1,\, n=0}^{\infty} t_{m,n} x^m y^n$$

where $t_{m,n}$ is the number of bc-trees with m blue points and n coral points. Similarly let $T(x, y)$, $T_B(x, y)$, $T_C(x, y)$ be the generating series for rooted bc-trees, bc-trees rooted at a blue point, and bc-trees rooted at a coral point respectively.

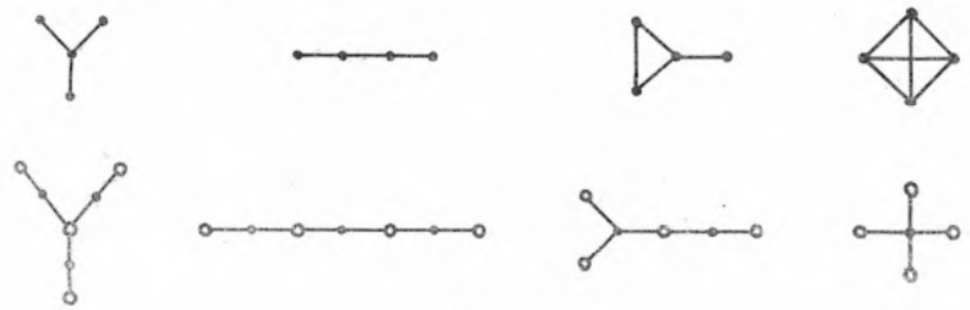

Fig. 2

Theorem 4.

$$T_B(x, y) = xZ(S_\infty, T_C(x, y))\cdot Z(S_\infty, yT_B(x, y)).$$

$$T_C(x, y) = y[Z(S_\infty, T_B(x, y)) - T_B(x, y) - 1].$$

$$T(x, y) = T_B(x, y) + T_C(x, y).$$

$$t(x, y) = T(x, y) - T_B(x, y)[T_C(x, y) + yT_B(x, y)].$$

PROOF. Consider a bc-tree rooted at a blue point v. This tree is completely determined by its branches at v. By removing the point v from such a branch, we obtain a bicolored tree R, rooted at a coral point u. If u is not an endpoint of R, then R is a bc-tree rooted at a coral point. If u is an endpoint of R, then $R-u$ is a bc-tree rooted at a blue point. The trees R are counted in the first instance by $T_C(x, y)$, in the second instance by $yT_B(x, y)$. As there may be arbitrarily many trees of each type at the root-point v, we have:

$$T_B(x, y) = xZ(S_\infty, T_C(x, y))Z(S_\infty, yT_B(x, y)).$$

Now consider a bc-tree rooted at a coral point. Again, this tree is completely determined by the branches at the root, and we notice that there are always at least two branches. Therefore:

$$T_C(x, y) = y \sum_{2}^{\infty} Z(S_i, T_B(x, y)) = y\,[Z(S_\infty, T_B(x, y)) - T_B(x, y) - 1].$$

We may now calculate $T_B(x, y)$ and $T_C(x, y)$ up to any desired power $x^m y^n$ by alternate consideration of the two generating functions. Since $T(x, y) = T_B(x, y) + T_C(x, y)$, we find explicitly:

$$T_B(x, y) = x + x^2 y + x^3(y + 2y^2) + x^4(y + 3y^2 + 4y^3) + x^5(y + 5y^2 + 10y^3 + 9y^4) + \dots$$

$$T(x, y) = x + x^2(2y) + x^3(2y + 3y^2) + x^4(2y + 5y^2 + 7y^3) + x^5(2y + 8y^2 + 17y^3 + 15y^4) + \dots$$

Finally, we find the generating series $t(x, y)$ for unrooted bc-trees by applying Otter's Theorem. As a bc-tree does not have a symmetry line (every line joins a blue point to a coral point), this theorem reduces in our case to:

$$t(x, y) = T(x, y) - L(x, y)$$

where $L(x, y)$ is the generating series for bc-trees rooted at a line. By an argument similar to the ones above, we obtain:

$$t(x, y) = T(x, y) - T_B(x, y)[T_C(x, y) + yT_B(x, y)].$$

Explicitly,

$$t(x, y) = x + x^2 y + x^3(y + y^2) + x^4(y + y^2 + 2y^3) + x^5(y + 2y^2 + 3y^3 + 3y^4) + \dots$$

In [3], NORMAN obtained the generating function for graphs in which every block is complete, i. e. for block-graphs, but had to use rather sophisticated methods to obtain this result. By applying Theorem 3 to Theorem 4, we obtain the same result in a more elementary way:

Corollary. *Let $\bar{b}_n$ and b_n be respectively the number of rooted and unrooted block-graphs with n points, and let*

$$\bar{B}(x) = \sum_1^\infty \bar{b}_n x^n \quad \textit{and} \quad B(x) = \sum_1^\infty b_n x^n$$

be their generating series. Then

$$\bar{B}(x) = T_B(x, 1)$$

$$B(x) = t(x, 1).$$

Explicitly:

$$\bar{B}(x) = x + x^2 + 3x^3 + 8x^4 + 25x^5 + \dots$$

$$B(x) = x + x^2 + 2x^3 + 4x^4 + 9x^5 + \dots$$

We note that the series $B(x)$ is the same as the counting series for ordinary rooted trees as for as given. This is purely accidental and does not hold for higher powers of x.

References

[1] F. HARARY, A characterization of block-graphs, *Canadian Math. Bull.* 6 (1963), 1—6.

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[3] R. Z. NORMAN, On the number of linear graphs with given blocks, *Doctoral Dissertation, University of Michigan,* 1954.

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