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On quasi-uniformly continuous functions and Lebesgue spaces

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Abstract. We show that every continuous function from a Lebesgue quasi-uniform space to a small-set symmetric quasi-uniform space is quasi-uniformly continuous. We also introduce a new notion of Lebesgue quasi-uniformity which involves the two topologies generated by a quasi-uniformity and its conjugate, namely a pair Lebesgue quasi-uniformity. We show that every bicontinuous function from a pair Lebesgue quasi-uniform space to a quasi-uniform space is quasi-uniformly continuous. Pair Lebesgue quasi-uniform spaces have several interesting properties. Thus, we observe that a topological space admits only pair Lebesgue quasi-uniformities if and only if it is hereditarily compact and quasi-sober and that a T_1 topological space admits a pair Lebesgue quasi-uniformity if and only if it is paracompact.

1. Introduction

A *quasi-uniformity* on a set X is a filter $\mathcal{U}$ on $X \times X$ such that (i) each member of $\mathcal{U}$ is a reflexive relation on X, and (ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$. (As usual $V^2 = \{(x,y) \in X \times X :$ there is $z \in X$ with $(x,z) \in V$ and $(z,y) \in V\}$.) The pair $(X,\mathcal{U})$ is called a *quasi-uniform space*. The topology $T(\mathcal{U}) = \{A \subseteq X :$ for each $x \in A$ there is $U \in \mathcal{U}$ with $U(x) \subseteq A\}$ is called the *topology induced by* $\mathcal{U}$ on X, where $U(x) = \{y \in X : (x,y) \in U\}$.

If $\mathcal{U}$ is a quasi-uniformity on X, then $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ is also a quasi-uniformity on X called *conjugate of* $\mathcal{U}$. As usual $\mathcal{U}^*$ will denote the coarsest uniformity finer than $\mathcal{U}$.

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A quasi-uniformity $\mathcal{U}$ on a set X is a *Lebesgue quasi-uniformity* provided that for each $T(\mathcal{U})$-open cover $\mathcal{G}$ of X there is $U \in \mathcal{U}$ such that the cover $\{U(x) : x \in X\}$ refines $\mathcal{G}$ [5]. The pair $(X,\mathcal{U})$ is called a *Lebesgue quasi-uniform space*.

The notion of a *small-set symmetric quasi-uniform space* was introduced by Fletcher and Hunsaker (see, for instance, [4]). It is proved in [13] that a quasi-uniform space $(X,\mathcal{U})$ is small-set symmetric if and only if $T(\mathcal{U}^{-1}) \subseteq T(\mathcal{U})$.

Terms and undefined concepts may be found in [5].

In Section 2 of this paper we shall show that every continuous function from a Lebesgue quasi-uniform space to a small-set symmetric quasi-uniform space is quasi-uniformly continuous.

In Section 3 we introduce and study the notion of a pair Lebesgue quasi-uniform space. We show that every bicontinuous function from a pair Lebesgue quasi-uniform space to a quasi-uniform space is quasi-uniformly continuous. Each pair Lebesgue quasi-uniformity $\mathcal{U}$ is bicomplete and both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are convergence complete. It is shown that a topological space admits only pair Lebesgue quasi-uniformities if and only if it is hereditarily compact and quasi-sober and that a T_1 topological space admits a pair Lebesgue quasi-uniformity if and only if it is paracompact. Some illustrative examples are also given.

2. Quasi-uniform continuity and Lebesgue quasi-uniform spaces

Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two quasi-uniform spaces. A function $f : X \to Y$ is *quasi-uniformly continuous* provided that for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $(f(x), f(y)) \in V$ whenever $(x,y) \in U$.

In [5, Cororally 1.45] it is proved that each continuous function from a Hausdorff compact quasi-uniform space to a uniform space is quasi-uniformly continuous (see also [15]). This result was recently strengthened by Künzi who proved in [10, Proposition 1] that each continuous function from a compact quasi-uniform space to a small-set symmetric quasi-uniform space is quasi-uniformly continuous (see [17] for a generalization of this result to locally quasi-uniform spaces).

Since each quasi-uniformity compatible with a compact space is a Lebesgue quasi-uniformity [5, Corollary 5.1], our next result generalizes Künzi's proposition cited above. It also generalizes [16, Proposition 9].

Proposition 2.1. *Let $(X,\mathcal{U})$ be a Lebesgue quasi-uniform space and $(Y,\mathcal{V})$ be a small-set symmetric quasi-uniform space. Then each continuous function $f:(X,T(\mathcal{U}))\to(Y,T(\mathcal{V}))$ is quasi-uniformly continuous.*

PROOF. Let $V\in\mathcal{V}$. Choose $W\in\mathcal{V}$ such that $W^2\subseteq V$. Since $T(\mathcal{V})=T(\mathcal{V}^*)$, for each $y\in Y$ there is $W_y\in\mathcal{V}$ such that $W_y(y)\subseteq W^*(y)$, where W^* denotes the entourage $W\cap W^{-1}$ of $\mathcal{V}^*$. Then $\mathcal{G}=\{f^{-1}(H_y):y\in Y\}$ is a $T(\mathcal{U})$-open cover of X where $H_y=T(\mathcal{V})$-int $W_y(y)$ for all $y\in Y$. Since $(X,\mathcal{U})$ is a Lebesgue quasi-uniform space there is $U\in\mathcal{U}$ such that $\{U(x):x\in X\}$ refines $\mathcal{G}$. Let (a,b) in U. There is $y\in Y$ such that $U(a)\subseteq f^{-1}(H_y)$, so that both $f(a)$ and $f(b)$ are in $W_y(y)$. Hence $(y,f(a))\in W^*$ and $(y,f(b))\in W^*$. We conclude that $(f(a),f(b))\in W^2\subseteq V$. Therefore f is quasi-uniformly continuous.

Remark 2.2. The above proof also shows that under the hypotheses of Proposition 2.1, f is also quasi-uniformly continuous from $(X,\mathcal{U})$ to $(Y,\mathcal{V}^{-1})$.

3. Quasi-uniform continuity and pair Lebesgue quasi-uniform spaces

Involving the two topologies generated by a quasi-uniformity and its conjugate, SALBANY proved in [19, Theorem 4.7] (see also [5, Theorem 1.21]) that if $(X,\mathcal{U})$ is a quasi-uniform space such that $(X,T(\mathcal{U}^*))$ is compact, then each bicontinuous function from $(X,\mathcal{U})$ to a quasi-uniform space $(Y,\mathcal{V})$ is quasi-uniformly continuous. An alternative proof of this result was recently presented by KOPPERMAN [7, Theorem 2.3]. (Let us recall that a function $f:(X,\mathcal{U})\to(Y,\mathcal{V})$ is bicontinuous provided that $f:(X,T(\mathcal{U}))\to(Y,T(\mathcal{V}))$ and $f:(X,T(\mathcal{U}^{-1}))\to(Y,T(\mathcal{V}^{-1}))$ are both continuous.)

In order to generalize Salbany's theorem we introduce in Definition 3.1 the notion of a pair Lebesgue quasi-uniform space.

Similarly to [2] and [18], a *pair open cover* of a quasi-uniform space $(X,\mathcal{U})$ is a collection of pairs $\{(G_\alpha,H_\alpha):\alpha\in\mathcal{A}\}$ such that each G_α is $T(\mathcal{U})$-open, each H_α is $T(\mathcal{U}^{-1})$-open and for each $x\in X$ there is $\alpha\in\mathcal{A}$ with $x\in G_\alpha\cap H_\alpha$.

Definition 3.1. Let $\mathcal{U}$ be a quasi-uniformity on a set X. We say that $\mathcal{U}$ is a *pair Lebesgue quasi-uniformity* if for each pair open cover $\{(G_\alpha, H_\alpha) : \alpha \in \mathcal{A}\}$ of $(X,\mathcal{U})$ there is $U \in \mathcal{U}$ such that the pair cover $\{(U(x), U^{-1}(x)) : x \in X\}$ refines $\{(G_\alpha, H_\alpha) : \alpha \in \mathcal{A}\}$ (i.e. for each $x \in X$ there is $\alpha \in \mathcal{A}$ such that $U(x) \subseteq G_\alpha$ and $U^{-1}(x) \subseteq H_\alpha$). In this case we say that $(X,\mathcal{U})$ is a *pair Lebesgue quasi-uniform space*.

The following result is the key in our study of pair Lebesgue quasi-uniformities.

Proposition 3.2. *Let $(X,\mathcal{U})$ be a quasi-uniform space such that $(X, T(\mathcal{U}^*))$ is a compact space. Then $(X,\mathcal{U})$ is a pair Lebesgue quasi-uniform space.*

Proof. Let $\{(G_\alpha, H_\alpha) : \alpha \in \mathcal{A}\}$ be a pair open cover of $(X,\mathcal{U})$. Then, for each $x \in X$ there is $V_x \in \mathcal{U}$ and $\alpha(x) \in \mathcal{A}$ such that $V_x^2(x) \subseteq G_{\alpha(x)}$ and $(V_x^{-1})^2(x) \subseteq H_{\alpha(x)}$. Since $T(\mathcal{U}^*)$ is a compact topology, the cover

$$\{T(\mathcal{U}^*)\text{-int}(V_x(x) \cap V_x^{-1}(x)) : x \in X\}$$

of X has a finite subcover $\{T(\mathcal{U}^*)\text{-int}(V_{x_k}(x_k) \cap V_{x_k}^{-1}(x_k)) : k = 1, 2, \dots, n\}$. Put $V = \bigcap\{V_{x_k} : k = 1, 2, \dots, n\}$. Then $V \in \mathcal{U}$ and we shall show that the pair cover $\{(V(x), V^{-1}(x)) : x \in X\}$ refines the pair open cover $\{(G_\alpha, H_\alpha) : \alpha \in \mathcal{A}\}$. Given $x \in X$ there is $k \in \{1, 2, \dots, n\}$ such that $x \in V_{x_k}(x_k) \cap V_{x_k}^{-1}(x_k)$. Thus $x \in G_{\alpha(x_k)} \cap H_{\alpha(x_k)}$. Let $y \in V(x)$, then $y \in V_{x_k}^2(x_k) \subseteq G_{\alpha(x_k)}$, so that $V(x) \subseteq G_{\alpha(x_k)}$. Similarly $V^{-1}(x) \subseteq H_{\alpha(x_k)}$. We conclude that $(X,\mathcal{U})$ is pair Lebesgue.

Example 3.3. Let X be the unit interval $I = [0, 1]$ and let d be the quasi-pseudometric defined on X by $d(x, y) = 0$ if $x \leq y$ and $d(x, y) = x - y$ if $y < x$. Then d is a totally bounded quasi-pseudometric which induces the Scott topology on the (continuous) complete partial order $(I, \leq)$ (see [20, Example 2.6]). Since $d^* = d \vee d^{-1}$is the usual metric on X it follows from Proposition 3.2 that $(X, \mathcal{U}(d))$ is a pair Lebesgue quasi-uniform space. (Note that $T(d)$ and $T(d^{-1})$ are the lower topology and the upper topology on I, respectively.)

Example 3.4. The Khalimsky line (used in image processing) consists of the integers $\mathbb{Z}$ with the topology generated by all sets of the form $\{2n - 1, 2n, 2n+1\}$, $n \in \mathbb{Z}$ (see [8], [9]). Clearly the quasi-pseudometric d defined on $\mathbb{Z}$ by $d(2n, 2n - 1) = d(2n, 2n + 1) = d(n, n) = 0$ for all $n \in \mathbb{Z}$ and $d(x, y) = 1$ otherwise, generates the Khalimsky line. It is easy to see that $\mathcal{U}(d)$ is a pair Lebesgue quasi-uniformity. (Note that d^* is the discrete metric on $\mathbb{Z}$).

Proposition 3.2 shows that, in fact, the next result is a generalization of Salbany's theorem cited above.

Proposition 3.5. *Let $(X,\mathcal{U})$ be a pair Lebesgue quasi-uniform space and $(Y,\mathcal{V})$ be a quasi-uniform space. Then each bicontinuous function $f:(X,T(\mathcal{U}),T(\mathcal{U}^{-1}))\to(Y,T(\mathcal{V}),T(\mathcal{V}^{-1}))$ is quasi-uniformly continuous.*

Proof. Let $V\in\mathcal{V}$. Choose $W\in\mathcal{V}$ such that $W^2\subseteq V$. For each $y\in Y$ let $G_y=T(\mathcal{V})$-int$W(y)$ and $H_y=T(\mathcal{V}^{-1})$-int$W^{-1}(y)$. Then $\{(f^{-1}(G_y),f^{-1}(H_y)):y\in Y\}$ is a pair open cover of $(X,\mathcal{U})$. Since $(X,\mathcal{U})$ is a pair Lebesgue quasi-uniform space there is $U\in\mathcal{U}$ such that the pair cover $\{(U(x),U^{-1}(x)):x\in X\}$ refines $\{(f^{-1}(G_y),f^{-1}(H_y)):y\in Y\}$. Let (a,b) in U. There is $y\in Y$ such that

$$U(a)\subseteq f^{-1}(G_y)\text{ and }\ U^{-1}(a)\subseteq f^{-1}(H_y).$$

Then $a\in f^{-1}(H_y)$, $b\in f^{-1}(G_y)$. Thus $(f(a),y)\in W$ and $(y,f(b))\in W$. We conclude that $(f(a),f(b))\in W^2\subseteq V$. Therefore f is quasi-uniformly continuous.

In the rest of the paper we shall give some properties of pair Lebesgue quasi-uniform spaces and characterize those (quasi-pseudometrizable) topological spaces for which every compatible (quasi-pseudometric) quasi-uniformity is pair Lebesgue.

Proposition 3.6. *Let $(X,\mathcal{U})$ be a pair Lebesgue quasi-uniform space. Then $(X,\mathcal{U}^*)$ is a Lebesgue uniform space.*

Proof. Let $\{G_\alpha:\alpha\in\mathcal{A}\}$ be a $T(\mathcal{U}^*)$-open cover of X. For each $x\in X$ there is $\alpha(x)\in\mathcal{A}$ and $U_x\in\mathcal{U}$ such that $x\in U_x(x)\cap U_x^{-1}(x)\subseteq G_{\alpha(x)}$. Then there exists $U\in\mathcal{U}$ such that the pair cover $\{(U(x),U^{-1}(x)):x\in X\}$ refines the pair cover $\{(U_x(x),U_x^{-1}(x)):x\in X\}$. Hence, for each $x\in X$ there is $y\in X$ with $U(x)\subseteq U_y(y)$ and $U^{-1}(x)\subseteq U_y^{-1}(y)$. Thus $U(x)\cap U^{-1}(x)\subseteq G_{\alpha(y)}$. We conclude that $\mathcal{U}^*$ is a Lebesgue uniformity.

A quasi-uniformity $\mathcal{U}$ is called *bicomplete* [5] if $\mathcal{U}^*$ is a complete uniformity. Since every Lebesgue uniformity is complete we have

Corollary 3.7. *Each pair Lebesgue quasi-uniformity is bicomplete.*

Proposition 3.8. *Let $(X,\mathcal{U})$ be a pair Lebesgue quasi-uniform space. Then both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are Lebesgue quasi-uniformities.*

Proof. We shall show that $\mathcal{U}$ is a Lebesgue quasi-uniformity. Let $\{G_\alpha:\alpha\in\mathcal{A}\}$ be a $T(\mathcal{U})$-open cover of X. For each $\alpha\in\mathcal{A}$ let $H_\alpha=X$. Then $\{(G_\alpha,H_\alpha):\alpha\in\mathcal{A}\}$ is a pair open cover of $(X,\mathcal{U})$, so that there is $U\in\mathcal{U}$ such that the pair open cover $\{(U(x),U^{-1}(x)):x\in X\}$ refines $\{(G_\alpha,H_\alpha):\alpha\in\mathcal{A}\}$. We conclude that $\{U(x):x\in X\}$ refines $\{G_\alpha:\alpha\in\mathcal{A}\}$. Similarly we see that $\mathcal{U}^{-1}$ is Lebesgue.

Since every Lebesgue quasi-uniformity is convergence complete [5, Proposition 5.7] we have

Corollary 3.9. *Let $(X,\mathcal{U})$ be a pair Lebesgue quasi-uniform space. Then both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are convergence complete quasi-uniformities.*

Example 3.10. Let X be the unit interval $I=[0,1]$ and let d be the quasi-pseudometric defined on X by $d(0,x)=d(x,1)=0$ for all $x\in X$, $d(x,0)=1$ for all $x\neq 0$, $d(x,x)=0$ for all $x\in X$ and $d(x,y)=x$ otherwise. Then both $T(d)$ and $T(d^{-1})$ are compact topologies, so that $\mathcal{U}(d)$ and $\mathcal{U}^{-1}(d)$ are Lebesgue quasi-uniformities. Furthermore $T(d^*)$ is the discrete topology on X and $\mathcal{U}(d^*)$ is not a Lebesgue uniformity. Therefore $\mathcal{U}(d)$ is not pair Lebesgue.

Note that the topology $T(d)$ of the above examples is not R_0. This fact is not accidental since every R_0 quasi-uniform space $(X,\mathcal{U})$ whit the property that $\mathcal{U}$ and $\mathcal{U}^{-1}$ are Lebesgue quasi-uniformities satisfies $T(\mathcal{U})=T(\mathcal{U}^{-1})=T(\mathcal{U}^*)$ [5, Corollary 5.2].

Let us recall that a nonempty subspace Y of a topological space (X,T) is *irreducible* [1, Chapter 2] if each pair of nonempty Y-open subsets has a nonempty intersection. (X,T) is called *quasi-sober* [6, p. 154] if each closed irreducible subset is of the form $T\text{-}\operatorname{cl} x$ for some $x\in X$.

Proposition 3.11. *A topological space admits only pair Lebesgue quasi-uniformities if and only if it is a hereditarily compact quasi-sober space.*

Proof. Let (X,T) be a space such that every compatible quasi-uniformity is pair Lebesgue. By Corollary 3.7, (X,T) admits only bicomplete quasi-uniformities, so that it is hereditarily compact and quasi-sober [12, Proposition 6]. Conversely, if (X,T) is hereditarily compact and quasi-sober then $T(\mathcal{P}^*)$ is a compact topology where $\mathcal{P}$ denotes the Pervin quasi-uniformity of (X,T) (see [3], [6]). We conclude that (X,T) admits a unique quasi-uniformity [11, Proposition 3] which is pair Lebesgue by Proposition 3.2.

A *quasi-pseudometric* d on a set X is called *pair Lebesgue* provided that the quasi-uniformity $\mathcal{U}(d)$ generated by d is pair Lebesgue.

Proposition 3.12. *A quasi-pseudometrizable topological space admits only pair Lebesgue quasi-pseudometrics if and only if it is a hereditarily compact quasi-sober space.*

Proof. Let (X,T) be a quasi-pseudometrizable space such that every compatible quasi-pseudometric is pair Lebesgue. By Corollary 3.7, (X,T) admits only bicomplete quasi-pseudometrics, so that it is hereditarily compact and quasi-sober [14, Theorem 2]. The converse follows from Proposition 3.11.

Problem 3.13. Characterize those topological spaces which admit a pair Lebesgue quasi-uniformity.

The preceding problem which remains open has a somewhat surprising solution in the case that the space is T_1.

Proposition 3.14. *A T_1 topological space admits a pair Lebesgue quasi-uniformity if and only if it is a (regular)paracompact space.*

Proof. Let (X, T) be a T_1 space admitting a pair Lebesgue quasi-uniformity $\mathcal{U}$. By Proposition 3.9, both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are Lebesgue quasi-uniformities. Therefore $T(\mathcal{U}) = T(\mathcal{U}^{-1}) = T(\mathcal{U}^*)$ [5, Corollary 5.2]. We conclude that $\mathcal{U}^*$ is a Lebesgue uniformity on X compatible with T. Consequently $\mathcal{U}^*$ is the fine uniformity of (X, T) and, thus, it is a (regular) paracompact space. The converse is clear.

Remark 3.15. In the proof of the preceding result we have used the well-known fact that if $\mathcal{U}$ is a Lebesgue uniformity on a set X, then $\mathcal{U}$ is exactly the fine uniformity of $(X, T(\mathcal{U}))$. The situation in the "pair Lebesgue" case is very different for topological spaces. In fact, it is not hard to prove that the fine quasi-uniformity of a T_1 topological space (X, T) is pair Lebesgue if and only if T is the discrete topology on X. However we have the following result of a bitopological nature.

Proposition 3.16. *Let $(X,\mathcal{U})$ be a pair Lebesgue quasi-uniform space and let $\mathcal{FN}$ denote the finest quasi-uniformity on X such that $T(\mathcal{FN}) = T(\mathcal{U})$ and $T(\mathcal{FN}^{-1}) = T(\mathcal{U}^{-1})$. Then $\mathcal{U} = \mathcal{FN}$.*

Proof. Clearly $\mathcal{U} \subseteq \mathcal{FN}$. Let $W \in \mathcal{FN}$, then there is a quasi-pseudometric d on X which $T(d) \subseteq T(\mathcal{U})$ and $T(d^{-1}) \subseteq T(\mathcal{U}^{-1})$ and there is an $r > 0$ such that $V_r \subseteq W$ where $V_r = \{(x, y) : d(x, y) < r\}$. Consider the pair open cover of $(X,\mathcal{U})$, $\mathcal{B} = \{(B_d(x, r/2), B_{d^{-1}}(x, r/2)) : x \in X\}$. Choose $U \in \mathcal{U}$ such that $\{(U(x), U^{-1}(x)) : x \in X\}$ refines $\mathcal{B}$. Given $(x, y) \in U$ there is $a \in X$ such that $U(x) \subseteq B_d(a, r/2)$ and $U^{-1}(x) \subseteq B_{d^{-1}}(a, r/2)$, so that $d(a, y) < r/2$ and $d(x, a) < r/2$. We conclude that $(x, y) \in V_r \subseteq W$, and thus $\mathcal{U}$=$\mathcal{FN}$.

Proposition 3.17. *A topological space admits a pair Lebesgue quasi-metric if and only if it is a metrizable space whose set of nonisolated points is compact.*

Proof. It is well-known that a metrizable space admits a Lebesgue metric if and only if the set of nonisolated points is compact. Let (X, T) be a space which admits a pair Lebesgue quasi-metric d. Similarly to the proof of Proposition 3.14, $T = T(d^*)$. Since, by Proposition 3.6, d^* is a Lebesgue metric, we conclude that (X, T) is a metrizable space whose sets of nonisolated points is compact. The converse is clear.

We finally give an example which answers some questions that may be asked in the light of the obtained results.

Example 3.18. Let $\mathbb{N}$ be the set of natural numbers. Define a quasi-metric d on $\mathbb{N}$ by $d(n, m) = 1/m$ if $n < m$, $d(n, m) = 1$ if $n > m$ and $d(n, n) = 0$ for all $n, m \in \mathbb{N}$. Thus $T(d)$ is the cofinite topology on $\mathbb{N}$, so that d is a Lebesgue quasi-metric. Since d^* is the discrete metric, it is a Lebesgue metric. However, $(\mathbb{N}, T(d))$ does not admits any pair Lebesgue quasi-uniformity as Proposition 3.14 shows.

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